From Morse theory (via Fukaya-Morse A_∞ category) to Feynman diagrams

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Plan

1. Bits of Morse theory
2. Morse trees: Fukaya-Morse A_∞ category
3. Connection to QFT,
   based on a joint work with O. Chekeres, A. Losev and D. Youmans, arXiv:2112.12756.

Morse theory: toy setting

$z = f(u,v)$ function on $\mathbb{R}^2$ with isolated nondegenerate extrema.

crit. point	Morse index
local maximum	2
saddle point	1
local minimum	0

Gradient trajectories: $\dot{x}(t) = -\mathrm{grad}\, f(x(t))$

(trajectory of a ball rolling downhill)

- A grad. trajectory stops only at a crit. point

Morse theory on a Riemannian manifold

Input:

- a) M Riemannian manifold, dim$=n$
- b) f a function on M with isolated nondegenerate crit. points

Morse lemma

Near a crit. point x_0, one can find loc. coordinates $(y_1, \dots, y_n)$ on M such that

$$f(x) = f(x_0) + y_1^2 + \cdots + y_k^2 - y_{k+1}^2 - \cdots - y_n^2$$

$$\begin{aligned} \#\,\text{minuses} &= \text{Morse index of } x_0 \\ \#\,\text{pluses} &= \text{Morse coindex of } x_0 \end{aligned}$$

E.g.:

loc. maximum of f	index n
loc. minimum	index 0

Gradient trajectories

- $v = -\mathrm{grad}\, f$ gradient vector field
- gradient trajectories= integral curves of v

$$\dot{x}(t) = -\mathrm{grad}\, f(x(t))$$

- gradient flow $\Phi_t \colon M \to M$ flow along v in time t.

Assume M *compact*. A grad. traj. through x runs asymptotically into some crit. pt. as $t \to +\infty$ and into some other one as $t \to -\infty$.

Stable and unstable manifolds. For $P \in \mathrm{crit}(f)$,

$$\begin{aligned} \mathrm{Stab}_P &= \{x \in M \mid \Phi_t(x) \underset{t\to+\infty}{\to} P\} \\ \mathrm{Unstab}_P &= \{x \in M \mid \Phi_t(x) \underset{t\to-\infty}{\to} P\} \end{aligned}$$

Stab_P	open disk in M	dim=$n - \mathrm{ind}(P)$
Unstab_P	open disk in M	dim=$\mathrm{ind}(P)$

Morse chain complex

Moduli space of grad. trajectories from P to Q:

- $\mathcal{M}(P,Q) = \dfrac{\mathrm{Unstab}_P \cap \mathrm{Stab}_Q}{x \sim \Phi_t(x)\ \forall t \in \mathbb{R}}$
- $\dim \mathcal{M} = \mathrm{ind}(P) - \mathrm{ind}(Q) - 1$
- One has rigid grad. trajectories $P \to Q$ if $\mathrm{ind}(P) = \mathrm{ind}(Q) + 1$.

Morse chain complex

$$MC_0(M,f) \xleftarrow{d_{\mathrm{Morse}}} MC_1(M,f) \xleftarrow{d_{\mathrm{Morse}}} MC_2(M,f) \xleftarrow{d_{\mathrm{Morse}}} \cdots \xleftarrow{d_{\mathrm{Morse}}} MC_n(M,f)$$

- $MC_k(M,f) = \mathrm{Span}_{\mathbb{Z}_2}(\mathrm{crit}_{\mathrm{ind}=k}(f))$
- $$\begin{array}{rccc} d_{\mathrm{Morse}}: & MC_k & \to & MC_{k-1} \\ & P & \mapsto & \sum_{Q \in \mathrm{crit}_{k-1}(f)} \#\mathcal{M}(P,Q) \cdot Q \end{array}$$

Lemma

$(d_{\mathrm{Morse}})^2 = 0$

Examples

Morse homology

$$MC_0(M,f) \xleftarrow{d_{\text{Morse}}} MC_1(M,f) \xleftarrow{d_{\text{Morse}}} MC_2(M,f) \xleftarrow{d_{\text{Morse}}} \cdots \xleftarrow{d_{\text{Morse}}} MC_n(M,f)$$

$$MH_k(M,f) := \frac{\{\alpha \in MC_k \mid d_{\text{Morse}}\alpha = 0\}}{\alpha \sim \alpha + d_{\text{Morse}}(\beta)\ \forall \beta \in MC_{k+1}}$$

Theorem

$MC_k(M,f) = H_k(M)$ for any Morse function f.

Idea: $MC_k(M,f) = C_k^{CW}(\Xi_f)$
Ξ_f – cell decomposition of M,
$\{\text{cells}\} = \{\text{Unstab}_P\}_{P\in\text{crit}(f)}$

Morse cochains:

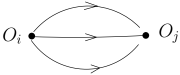
$$MC^0 \to MC^1 \to \cdots \to MC^n$$

with $MC^k(M,f) := MC_{n-k}(M,f) = \text{span}_{\mathbb{Z}_2}(\text{crit}_{\text{coind}=k}(f))$

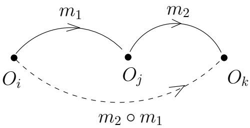
Categories (reminder)

Recall: A (usual) category has

- Objects O_i
- Morphisms $\mathrm{Mor}(O_i, O_j)$



- Composition of morphisms
  $\circ : \mathrm{Mor}(O_i, O_j) \times \mathrm{Mor}(O_j, O_k) \to \mathrm{Mor}(O_i, O_k)$



which is associative: $(m_3 \circ m_2) \circ m_1 = m_3 \circ (m_2 \circ m_1)$

$$O_i \xrightarrow{m_1} O_j \xrightarrow{m_2} O_k \xrightarrow{m_3} O_l$$

Examples: Vect, Top, Groups,...

A_∞ categories

An A_∞ category has:

- Objects O_i
- Morphisms $\mathrm{Mor}^\bullet(O_i, O_j)$ – a cochain complex with differential μ_1.
- Composition

$$\mu_2 \colon \mathrm{Mor}(O_i, O_j) \times \mathrm{Mor}(O_j, O_k) \to \mathrm{Mor}(O_i, O_k)$$

  – compatible with μ_1 and associative "up to homotopy":
  $\mu_2(\mu_2(x, y), z) - \mu_2(x, \mu_2(y, z)) =$
  $\mu_1(\mu_3(x, y, z)) - \mu_3(\mu_1(x), y, z) - \mu_3(x, \mu_1(y), z) - \mu_3(x, y, \mu_1(z))$
  where

$$\mu_3 \colon \mathrm{Mor}(O_i, O_j) \times \mathrm{Mor}(O_j, O_k) \times \mathrm{Mor}(O_k, O_l) \to \mathrm{Mor}(O_i, O_l)$$

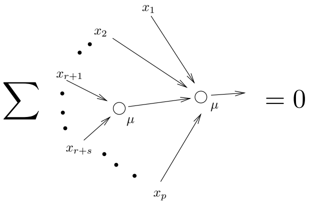
  – "higher composition"
- Higher compositions:
  $\mu_k \colon \mathrm{Mor}(O_{i_0}, O_{i_1}) \times \cdots \times \mathrm{Mor}(O_{i_{p-1}}, O_{i_p}) \to \mathrm{Mor}(O_{i_0}, O_{i_p})$

A_∞ relations

A_∞ **relations:** for $p \geq 1$, given a composable sequence of morphisms $O_{i_0} \xrightarrow{x_1} \cdots \xrightarrow{x_p} O_{i_p}$, one has

$$\sum_{r,s} \mu_{p-s}(x_1, \ldots, x_r, \mu_s(x_{r+1}, \ldots, x_{r+s}), x_{r+s+1}, \ldots, x_p) = 0$$

Or pictorially:



$p = 1$ | $(\mu_1)^2 = 0$

$p = 2$ | $\mu_1(\mu_2(x, y)) = \mu_2(\mu_1(x, y)) \pm \mu_2(x, \mu_1(y))$

$p = 3$ | homotopy associativity of μ_2

Fukaya-Morse A_∞ category $\mathbb{F}$

Reference: Fukaya '93

Fix M compact Riemannian manifold

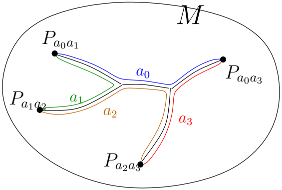
- Objects: functions $f_1, \dots, f_N$ on M.
- Morphisms:
  $\mathrm{Mor}(f_a, f_b) = \mathrm{span}_{\mathbb{Z}_2}(\mathrm{crit}(f_a - f_b)) = MC^\bullet(M, f_a - f_b)$,
  differential $\mu_1 = d_{\mathrm{Morse}}$.
- Higher compositions:
  $\mu_p \colon \mathrm{Mor}(f_{a_0}, f_{a_1}) \times \cdots \times \mathrm{Mor}(f_{a_{p-1}}, f_{a_p}) \to \mathrm{Mor}(f_{a_0}, f_{a_p})$

$$\mu_p(P_{a_0 a_1}, P_{a_1 a_2}, \dots, P_{a_{p-1} a_p}) = \sum_{P_{a_0 a_p}} \#\mathcal{M} \cdot P_{a_0 a_p}$$

$\mathcal{M}=$ "moduli space of Morse trees"

Morse trees

Points of $\mathcal{M}(P_{a_0a_1},\ldots,P_{a_{p-1}a_p},P_{a_pa_0})$ correspond to pictures in M:



- edges labeled by pairs (a,b) – grad. traj. of $f_a - f_b$.
- 1-valent vertex at P_{ab} emits an (a,b)-edge.
- 3-valent vertices $(a,b)+(b,c)\to(a,c)$.

$\mathcal{M}$ is a smooth manifold,

$$\dim\mathcal{M} = -\sum_{i=1}^{p}\mathrm{coind}(P_{a_{i-1}a_i}) + \mathrm{coind}P_{a_0a_p} + (p-2)$$

$$\#\mathcal{M} = \begin{cases} 0 & \text{if } \dim\mathcal{M}\neq 0 \\ \#\text{Morse trees} \pmod 2 & \text{if } \dim\mathcal{M} = 0 \end{cases}$$

Theorem

A_∞ relations hold.

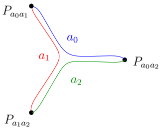
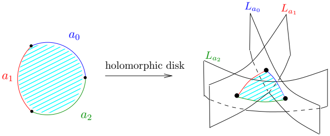
$\Rightarrow$ interesting quadratic relations among numbers of Morse trees

What kind of topological information about M does the A_∞ category $\mathbb{F}$ encode?

- μ_1 encodes $H^\bullet(M)$
- μ_2 encodes cup-product
- μ_3 encodes "Massey operation"
- $\mu_{>3}$ higher Massey operations

$\underset{\text{Quillen}-\text{Sullivan}}{\rightsquigarrow}$ $\mathbb{F}$ encodes "rational homotopy type" of M, $\mathbb{Q}\otimes\pi_k(M)$

Aside: Morse trees vs. holomorphic disks

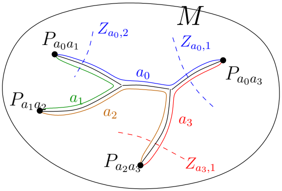
	Fukaya-Morse A_∞ category on M	Fukaya A_∞ category on T^*M
Ob:	$f_1, \ldots, f_N$	Lagrangians $L_a = \mathrm{graph}(\epsilon\, df_a)$
Mor:	$\mathrm{Mor}(F_a, F_b) = \mathrm{span}(\mathrm{crit}(f_a - f_b))$	$\mathrm{Mor}(L_a, L_b) =$ span(intersection points of L_a, L_b)
$\circ$:	$P_{a_{i-1}a_i} \in \mathrm{crit}(f_{a_{i-1}} - f_{a_i})$ $\mu(P_{a_0a_1}, P_{a_1a_2}, \ldots, P_{a_{p-1}a_p}) = \sum_{p_{a_0a_p} \in \mathrm{crit}(f_{a_0} - f_{a_p})} \#\mathcal{M}^{\mathrm{trees}} \cdot p_{a_0a_p}$ 	$p_{a_{i-1}a_i} \in L_{a_{i-1}} \cap L_{a_i}$ $\mu(p_{a_0a_1}, p_{a_1a_2}, \ldots, p_{a_{p-1}a_p}) = \sum_{p_{a_0a_p} \in L_{a_0} \cap L_{a_p}} \#\mathcal{M}^{\mathrm{hol.disks}} \cdot p_{a_0a_p}$ 

Enhancement of $\mathbb{F}$ by self-morphisms

Enhancement: $\mathrm{Mor}(f_a, f_a) = C^{\mathrm{sing}}_\bullet(M)$
Composition maps:

$$\mu\colon \mathrm{Mor}(f_{a_0}, f_{a_0})^{\otimes k_0} \otimes \mathrm{Mor}(f_{a_0}, f_{a_1}) \otimes \cdots \mathrm{Mor}(f_{a_{p-1}}, f_{a_p}) \otimes \mathrm{Mor}(f_{a_p}, f_{a_p})^{\otimes k_p} \to \mathrm{Mor}(f_{a_0}, f_{a_p})$$

$$\mu(\{Z_{a_0,\alpha}\}, P_{a_0 a_1}, \{Z_{a_1,\alpha}\}, \ldots, P_{a_{p-1}a_p}, \{Z_{a_p,\alpha}\}) = \sum_{P_{a_0 a_p}} \#\mathcal{M} \cdot P_{a_0 a_p}$$

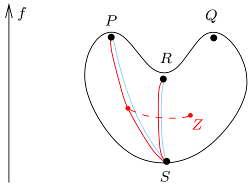


- differential: $\mu_1(Z) = \partial Z$
- $\mu_2(Z_1, Z_2) = Z_1 \cap Z_2$
- $\mu_n(Z_1, \ldots, Z_n) = 0,\ n \geq 3$

A_∞ relations – example

If $x_1, \ldots, x_p$ a composable sequence of morpisms:
$\mathrm{target}(x_i) = \mathrm{source}(x_{i+1})$, then
$\sum_{r,s} \mu(x_1, \ldots, x_r, \mu(x_{r+1}, \ldots, x_{r+s}), x_{r+s+1}, \ldots, x_p) = 0$

Example: "heart-shaped sphere" $M = S^2$, $N = 2$ functions, $f = f_1 - f_2$.



A_∞ relation: $f_1 \xrightarrow{P} f_2 \xrightarrow{Z} f_2$ composable sequence.

$$\boxed{\mu(d_{\mathrm{Morse}}(P), Z) + \mu(P, \partial Z) + d_{\mathrm{Morse}}\mu(P, Z) = 0}$$

Example: deformation of Morse differential by cycles

$N = 2$, $f = f_1 - f_2$. Fix $\{C_\alpha\}$ -cycles on M. $\{P_i\}$ – crit. points of f.
Generating function for compositions
$\mu\colon \mathrm{Mor}(f_1, f_2) \otimes \mathrm{Mor}(f_2, f_2)^{\otimes k} \to \mathrm{Mor}(f_1, f_2)$,

$$m_i^j(T) = \sum_{k\geq 1} \sum_{\alpha_1,\ldots,\alpha_k} \underbrace{\#\mathcal{M}(P_i, C_{\alpha_1}, \ldots, C_{\alpha_k}, P_j)}_{\#\text{grad traj } P_i \to P_j \text{ passing through cycles}} T_{\alpha_1} \cdots T_{\alpha_k}$$

T_α - generating parameters, $|T_\alpha| = 1 - \mathrm{codim}(C_\alpha)$.
A_∞ relations $\Rightarrow$ $\boxed{(d_{\mathrm{Morse}} + m(T))^2 = 0}$.

Explanation from HPT:

$$\begin{array}{c} K \circlearrowright \quad \Omega^\bullet(M), d + \sum_\alpha T_\alpha \delta_{C_\alpha} \\ p \downarrow \quad \uparrow i \\ MC^\bullet(M, f)\ , d_{\mathrm{Morse}} + m(T) \end{array}$$

Morse contraction

How to see Morse cochains as a direct summand in differential forms?

- $i\colon \underbrace{P}_{\text{crit. point}} \mapsto \delta_{\mathrm{Unstab}_P}$
- $p\colon \omega \mapsto \sum_P \left(\int_M \omega \wedge \delta_{\mathrm{Stab}_P} \right) \cdot P$
- $K = \int_0^\infty dt\ \iota_v\ e^{-t\mathcal{L}_v} : \quad \Omega^\bullet(M) \to \Omega^{\bullet-1}(M).$

  v – gradient vector field.

  Integral kernel: $\delta_Y \in \Omega_{\mathrm{distr}}(M \times M)$;

  $Y = \{(x,y) \mid x = \mathrm{Flow}_t(v) \circ y \text{ for some } t > 0\}$

"Physics": perturbed Gaussian integrals

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(S_0(\phi)+S_{\text{pert}}(\phi))} = \sum_{\text{graphs}\,\Gamma} \hbar^{E-V}\Phi_\Gamma$$

- S_0-quadratic in ϕ,
- S_{pert} – polynomial,
- Γ – Feynman graphs

BF theory

- Action: $S = \int_M \langle B, dA + \frac{1}{2}[A, A] \rangle$.
  (in Batalin-Vilkovisky formalism)
- Space of fields $\mathcal{F} = \Omega^\bullet(M, \mathfrak{g})[1] \oplus \Omega^\bullet(M, \mathfrak{g}^*)[n-2] \ \ni (A, B)$.
- $\mathfrak{g} = \mathrm{Mat}_{N\times N}$ coefficient Lie algebra.

$$\mathcal{F} = \left(\bigoplus_{a,b} \Omega_{ab}(M) \right) \oplus \left(\bigoplus_{a,b} \Omega_{ab}(M) \right)$$

Effective action for BF theory induced on Morse cochains

Let $\mathbb{M} = \bigoplus_{a,b} \mathbb{M}_{ab}$ with $\mathbb{M}_{ab} = \begin{cases} MC^\bullet(M, f_a - f_b) & \text{if } a \neq b \\ \Omega^\bullet(M) & \text{if } a = b \end{cases}$

Split $\mathcal{F} = \underbrace{\mathcal{F}'}_{\mathbb{M}\oplus\mathbb{M}} \oplus \mathcal{F}'' =$ "slow fields" $\oplus$ "fast fields"

Effective action induced on slow fields:

$$e^{\frac{i}{\hbar} S_{\text{eff}}(A',B')} = \int_{\mathcal{L}\subset\mathcal{F}''} \mathcal{D}A''\mathcal{D}B''\; e^{\frac{i}{\hbar} S_{\text{eff}}(A'+A'',B'+B'')}$$

$\mathcal{L} = \bigoplus_{a\neq b} \mathrm{im} K_{ab} \oplus \mathrm{im} K_{ab}$ – gauge-fixing Lagrangian in $\mathcal{F}''$.

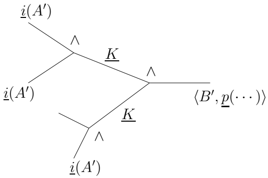
Put another way: we integrate out A_{ab}, B_{ab} fields with $a \neq b$, with "axial" gauge-fixing

$$\iota_{v_{ab}} A_{ab} = 0, \qquad \iota_{v_{ab}} B_{ab} = 0$$

– We induce S_{eff} on diagonal fields + "remainders" of off-diagonal fields.

Effective action via Feynman graphs

$$S_{\text{eff}}(A', B') = \sum_{\text{binary rooted trees } T}$$



$=$

$$= \sum_{p\geq 1} \sum_{a_0,\ldots,a_p=1}^{N} \langle B'_{a_p a_0}, \mu_p(A'_{a_0 a_1}, \ldots, A'_{a_{p-1} a_p})\rangle$$

Master equation

As a gauge theory, S_{eff} satisfies a PDE – the Batalin-Vilkovisky master equation

$$\{S_{\text{eff}}, S_{\text{eff}}\} = \sum_{a \neq b} \sum_{P_{ab}} \frac{\partial S_{\text{eff}}}{\partial A_{P_{ab}}} \frac{\partial S_{\text{eff}}}{\partial B_{P_{ba}}} + \sum_{a} \int_M \frac{\delta S_{\text{eff}}}{\delta A_{aa}} \frac{\delta S_{\text{eff}}}{\delta B_{aa}} = 0$$

$\Leftrightarrow$ A_∞ relations on compositions μ in $\mathbb{F}$.

Summary

- Gradient trajectories of a single Morse function remember the cohomology $H^\bullet(M)$.
- Morse trees (using several functions f_a) remember homological operations (Massey products).
- A_∞ category $\mathbb{F}$ has a "generating function" – the effective action for a topological gauge theory, computed in a special gauge. Morse trees become Feynman diagrams.
- Ways to prove A_∞ relations:
  - From homotopy transfer of algebraic structures $\Omega^\bullet(M) \otimes \mathrm{Mat}_{N\times N} \rightsquigarrow \mathbb{M}$.
  - Via effective action and BV master equation.
  - Via topological quantum mechanics on metric trees (from Stokes' theorem on the moduli space of trees).
  - By $2d \to 1d$ – from A_∞ relations for the Fukaya category of holomorphic disks, by $\epsilon \to 0$ limit.

Thank you!

References

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- M. Kontsevich, Y. Soibelman, "Homological mirror symmetry and torus fibrations." arXiv preprint math/0011041 (2000).
- O. Chekeres, A. Losev, P. Mnev, D. Youmans, "Two field-theoretic viewpoints on the Fukaya-Morse A_∞ category," arXiv:2112.12756.

Picture 1a: homotopy transfer

$$\underline{K} \circlearrowright \quad V = \Omega^\bullet(X) \otimes \mathrm{Mat}_{N\times N} = \bigoplus_{a,b=1}^{N} \Omega^\bullet_{ab}(X) \quad \text{– dg algebra}$$

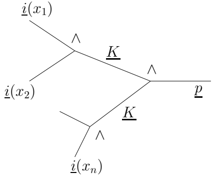
$$\underline{p} \downarrow \qquad \uparrow \underline{i}$$

$$\mathbb{M} = \bigoplus_{a,b} \mathbb{M}_{ab}, \qquad \mathbb{M}_{ab} = \begin{cases} MC(F_a - F_b), & a \neq b \\ \Omega^\bullet_{aa}, & a = b \end{cases}$$

$$\underline{i}, \underline{p}, \underline{K} = \begin{cases} \text{Morse contraction for } F_a - F_b, & a \neq b \\ \text{trivial } (\underline{i} = \underline{p} = \mathrm{id}, \underline{K} = 0), & a = b \end{cases}$$

Induced A_∞ algebra structure on $\mathbb{M}$:

$$m_n(x_1, \ldots, x_n) = \sum$$



$$\text{inputs: } x_i \in \underset{\text{with } b_i = a_{i+1}}{\mathbb{M}_{a_i b_i}} = \begin{cases} \text{Morse chain}, & a_i \neq b_i \\ \text{form/sing. chain } \delta_Z, & a_i = b_i \end{cases}$$

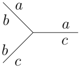
Picture 2: HTQM

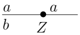
Topological quantum mechanics:

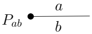
- Space of states: $\mathcal{H}_{ab} = \Omega^\bullet(X)$
(for a particle of (a,b)-type, $a \neq b$).
- BRST operator $Q = d$.
- Hamiltonian $H = \mathcal{L}_{v_{ab}} = [Q, G]_+$.
- $G = \iota_{v_{ab}}$.
- Evolution operator (superpropagator):
$U(t, dt) = e^{-t\,H - dt\,G} \quad \in \quad \Omega^\bullet(\mathbb{R}_+) \otimes \mathrm{End}(\mathcal{H}_{ab})$

HTQM on metric trees

HTQM on metric trees:

- 3-valent vertex 

$\sim$ $\mathcal{H}_{ab} \otimes \mathcal{H}_{bc} \xrightarrow{\wedge} \mathcal{H}_{ac}$
- 2-valent vertex 

$\sim$ operator $\mathcal{H}_{ab} \xrightarrow{\wedge \delta_Z} \mathcal{H}_{ab}$
- 1-valent vertex 

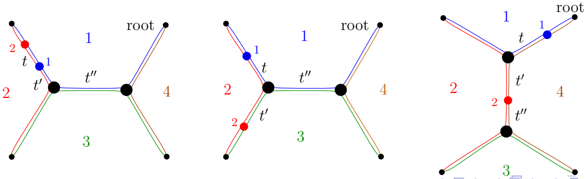
$\sim$ state $\delta_{\mathrm{Unstab}_{P_{ab}}}$
- (a, b)-edge of length t $\sim$ $U_{ab}(t, dt)$

Out of these building blocks, we build a form on the space of metric trees:

$$I \in \Omega^\bullet(MT_{N;k_1,\dots,k_N}) \otimes \mathrm{Hom}(\mathrm{Mor}_{1,1}^{\otimes k_1} \otimes \mathrm{Mor}_{1,2} \otimes \cdots \otimes \mathrm{Mor}_{N-1,N} \otimes \mathrm{Mor}_{N,N}^{\otimes k_N}, \mathrm{Mor}_{1,N})$$

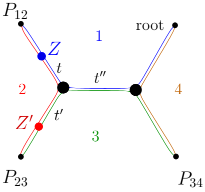
where $\mathrm{Mor}_{a,b} := \mathrm{Mor}(F_a, F_b)$.

Example: three top-cells in $MT_{4;1,1,0,0}$



Example

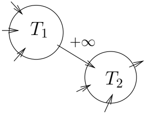
$$I\left(\quad \right) = \sum_{P_{14} \in \mathrm{crit}(F_1 - F_4)} \overline{I} \cdot [P_{14}]$$



$$\overline{I} = \int_X \delta_{\mathrm{Stab}_{P_{14}}} \wedge U_{13}(t'', dt'') \Big(U_{12}(t, dt)(\delta_Z \wedge \delta_{\mathrm{Unstab}_{P_{12}}}) \wedge \\ \wedge U_{23}(t', dt')(\delta_{Z'} \wedge \delta_{\mathrm{Unstab}_{P_{23}}}) \Big) \wedge \delta_{\mathrm{Unstab}_{P_{34}}}$$

Properties of I

1. $(d_{MT} + Q)I = 0$
2. Factorization on IR boundary of MT:

$I\left(\right.$ 

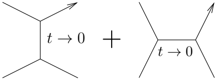
$\left.\right) = \left\langle I(T_2) \overset{\wedge}{,} I(T_1) \right\rangle$

3. Period $\displaystyle\int_{MT} I = m$ – the composition map in Fukaya-Morse category.

A_∞ relations from IR factorization of HTQM

$$(d_{MT}+Q)I=0 \qquad \Rightarrow \qquad \int_{\partial MT} I = -Q \underbrace{\int_{MT} I}_{m}$$

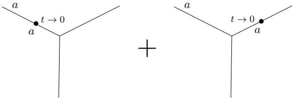
L.h.s. (contributions of boundary strata of MT):

a) 

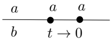
$= 0$

b) 

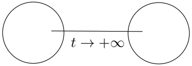
$= 0$

c) 

$= 0$

d) 

$\to$ terms $m(\cdots m(Z,Z')\cdots)$ in A_∞ relation.

e) 

$\to$ terms $m(\cdots m(\underbrace{\cdots}_{\geq 2 \text{ colors}})\cdots)$ in A_∞ relation.