AME 20214
Homework 5
Due: Thursday, 27 September 2012, in class

1. (50) Numerical methods are most important for equations without exact solutions. A famous example are the Lorenz equations; see `http://en.wikipedia.org/wiki/Lorenz_system`. The system arises from convective heat transfer; it is a *non-linear* system of ordinary differential equations, and can be written as

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x), \qquad x(0)=1,\\ \frac{dy}{dt} &= rx - y - xz, \qquad y(0)=1,\\ \frac{dz}{dt} &= -bz + xy, \qquad z(0)=1.\end{aligned}$$

   Write a `Fortran` code to approximate the solution to the Lorenz equations. Use the Euler method with $\Delta t = 0.001$ and generate the approximate solution and plots of $x(t)$, $y(t)$, $z(t)$, as well as a three-dimensional parametric plot of the curve formed in (x, y, z) space by the solution for $t \in [0, 25]$, under the following set of parameters:

   (a) $\sigma = 10$, $r = 10$, $b = 8/3$, and

   (b) $\sigma = 10$, $r = 28$, $b = 8/3$.

   You may find `MATLAB` functions `plot` and `plot3` useful for these plots. Do a little background reading on the Lorenz equations and give a short one-paragraph maximum interpretation of your results.

2. (50) Some functions $f(x)$ do not have integrals which can be written in terms of elementary functions. Their evaluation often relies upon numerical methods. For example, the so-called *trapezoidal method*, adds which states the integral I is well approximated by

$$I = \int_a^b f(x)dx \sim \frac{b-a}{2N}\left(f_1 + 2f_2 + 2f_3 + \cdots + 2f_{N-1} + 2f_N + f_{N+1}\right),$$

   is a common method. Here, we have divided the domain $x \in [a, b]$ into N equally spaced panels employing $N + 1$ grid points. One can in fact say

$$x_i = a + (b-a)\frac{i-1}{N}, \qquad i = 1, \dots, N+1.$$

   And we define

$$f_i = f(x_i), \qquad i = 1, \dots, N+1.$$

   Write a `Fortran` code which for arbitrary N, uses the trapezoidal rule to estimate the integral

$$I = \int_0^1 \frac{\sin(100x)}{x} dx.$$

   You will need a side calculation using l'Hôpital's rule to evaluate $f_1 = f(x = 0)$. A 50-digit precision estimate from `Mathematica` gives the highly accurate approximation

$$I \sim 1.5622254668890562933523451388045026772278249805411.$$

   Give an estimate of I for $N = 10$ and $N = 100$ using the trapezoidal rule. Then give a log-log plot of the error in your estimate as a function of N for many values of N spanning several orders of magnitude. See how accurate you can make your estimate.

Details of the trapezoidal method

The area of an individual trapezoid is

$$\text{average height} \times \text{width}.$$

For the i^{th} trapezoid in our domain, we have

$$\text{average height} = \frac{f_i + f_{i+1}}{2}.$$

The width of an individual trapezoid, assuming a uniform grid, is

$$\text{width} = \frac{b-a}{N}.$$

Thus,

$$\text{area single trapezoid} = \frac{b-a}{2N}(f_i + f_{i+1}).$$

When we add all of the areas, we get

$$I \sim \frac{b-a}{2N}((f_1 + f_2) + (f_2 + f_3) + (f_3 + f_4) + \cdots + (f_{N-1} + f_N) + (f_N + f_{N+1})).$$

We can rewrite this as

$$I \sim \frac{b-a}{2N}\left(f_1 + 2\sum_{i=2}^{N} f_i + f_{N+1}\right).$$