AME 20214
Homework 5
Due: Thursday, 2 October 2014, in class

1. (50) Numerical methods are most important for equations without exact solutions. A famous example are the Lorenz equations; see `http://en.wikipedia.org/wiki/Lorenz_system`. The system arises from convective heat transfer; it is a *non-linear* system of ordinary differential equations, and can be written as

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x), \qquad x(0)=1, \\ \frac{dy}{dt} &= rx - y - xz, \qquad y(0)=1, \\ \frac{dz}{dt} &= -bz + xy, \qquad z(0)=1. \end{aligned}$$

Write a `Fortran` code to approximate the solution to the Lorenz equations. Use the Euler method with $\Delta t = 0.001$ and generate the approximate solution and plots of $x(t)$, $y(t)$, $z(t)$, as well as a three-dimensional parametric plot of the curve formed in (x, y, z) space by the solution for $t \in [0, 20]$, under the following set of parameters:

(a) $\sigma = 8$, $r = 4$, $b = 4$, and

(b) $\sigma = 8$, $r = 45$, $b = 4$.

You may find `MATLAB` functions `plot` and `plot3` useful for these plots. Be sure to write your `Fortran` output to a file and read this file into your `MATLAB` plotting program. Do a little background reading on the Lorenz equations and give a short one-paragraph maximum interpretation of your results.

2. (50) Some functions $f(x)$ do not have integrals which can be written in terms of elementary functions. Their evaluation often relies upon numerical methods. For example, the so-called *trapezoidal method* states the integral I is well approximated by

$$I = \int_a^b f(x)\, dx \sim \frac{b-a}{2N}\left(f_1 + 2f_2 + 2f_3 + \cdots + 2f_{N-1} + 2f_N + f_{N+1}\right).$$

Here, we have divided the domain $x \in [a, b]$ into N equally spaced panels employing $N + 1$ grid points. One can in fact say

$$x_i = a + (b-a)\frac{i-1}{N}, \qquad i = 1, \ldots, N+1.$$

And we define

$$f_i = f(x_i), \qquad i = 1, \ldots, N+1.$$

Write a `Fortran` code which for arbitrary N, uses the trapezoidal rule to estimate the integral

$$I = \int_0^1 \frac{1-\cos(100x)}{x}\, dx.$$

You will need a side calculation using l'Hôpital's rule to evaluate $f_1 = f(x = 0)$. A 50-digit precision estimate from `Mathematica` gives the highly accurate approximation

$$I \sim 5.18753467603223472078693855335647534409468865111 68.$$

Give an estimate of I for $N = 10^1$ and $N = 10^2$ using the trapezoidal rule. Then give a log-log plot of the error in your estimate as a function of N for many values of N spanning several orders of magnitude. See how accurate you can make your estimate.

Details of the trapezoidal method

The area of an individual trapezoid is

$$\text{average height} \times \text{width}.$$

For the i^{th} trapezoid in our domain, we have

$$\text{average height} = \frac{f_i + f_{i+1}}{2}.$$

The width of an individual trapezoid, assuming a uniform grid, is

$$\text{width} = \frac{b-a}{N}.$$

Thus,

$$\text{area single trapezoid} = \frac{b-a}{2N}(f_i + f_{i+1}).$$

When we add all of the areas, we get

$$I \sim \frac{b-a}{2N}((f_1 + f_2) + (f_2 + f_3) + (f_3 + f_4) + \cdots + (f_{N-1} + f_N) + (f_N + f_{N+1})).$$

We can rewrite this as

$$I \sim \frac{b-a}{2N}\left(f_1 + 2\sum_{i=2}^{N} f_i + f_{N+1}\right).$$

Æsthetics

For this homework, use the LaTeX processor to format both text and graphics. Only the briefest restatement of the problem is necessary. Pay particular attention to generating beautiful graphs and formatting them well. Include at least one properly formatted equation. There is a *three page maximum.*