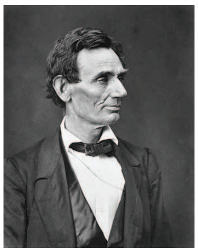
NAME:
AME 20231
Thermodynamics
Examination 1: SOLUTION
Prof. J. M. Powers
12 February 2010



"The advantageous use of Steam-power is, unquestionably, a modern discovery.
And yet, as much as two thousand years ago the power of steam was not only observed,
but an ingenius toy was actually made and put in motion by it, at Alexandria in Egypt."

Abraham Lincoln, 6 April 1858
Bloomington, Illinois

Happy 201st Birthday!

1. (20) Diatomic nitrogen, N_2, exists at $T = 65.9\ K$, $v = 0.4\ m^3/kg$. Find the pressure.

Solution

At such a low temperature, we must be concerned that the ideal gas model is invalid. A majority of students shared this concern, went to the appropriate table, and correctly interpolated. A significant minority did not realize that 65.9 K is a very cold temperature, and assumed the ideal gas law would apply. This was incorrect. Consultation with Table B.6 verified this concern to be real. At this T and v, N_2 is under the vapor dome. Thus the pressure is directly related to the temperature, and can easily be determined by linear interpolation:

$$P = (17.4\ kPa) + \frac{(38.6\ kPa) - (17.4\ kPa)}{(70\ K) - (65\ K)}((65.9\ K) - (65\ K)) = \boxed{21.216\ kPa.}$$

Note, use of the ideal gas law would have given

$$P = \frac{RT}{v} = \frac{\left(0.2968\ \frac{kJ}{kg\ K}\right)(65.9\ K)}{0.4\ \frac{m^3}{kg}} = 48.9\ kPa,$$

in error by a factor of over two.

2. (40) A mass, 10 kg, of H_2O initially at $T_1 = 30\ °C$, $v_1 = 0.001080\ m^3/kg$ is heated isochorically to state 2 where $T_2 = 140\ °C$. It then undergoes an isobaric process to state 3 where $T_3 = 250\ °C$.

   (a) Find the final specific volume.

(b) Accurately sketch the total process in the $P-v$, $T-v$, and $P-T$ planes. Label each state in your sketch giving numerical values for P, T, v. Include the vapor dome in its correct position.

(c) Find the work done in the total process.

Solution

Many people did very well on this problem. A few students attempted to use the ideal gas law in parts of this problem, and this was quite wrong. The diagrams were okay, but with several errors. The worst was putting a vapor dome in the $P-T$ plot. Orientation with the critical point was occasionally a problem. Also, many did not recognize where the critical specific volume was. Lastly, many did not recognize that under the dome an isotherm is also an isobar. There was also some confusion on specific work or total work.

State 1 is a saturated mixture of liquid and vapor. We could find x_1, but we do not really need it. We note $P_1 = 4.246\ kPa$. Since 1 to 2 is isochoric, we have $v_2 = v_1 = 0.001080\ m^3/kg$. And we are given $T_2 = 140\ °C$. Note that since $v_1 = v_2 < v_c$, isochoric heating may take us into the compressed liquid region. At this point there is an unfortunate ambiguity in the tables. Likely because of lack of significant digits, there are two possibilities, both admitted by the tables for state 2. This solution will focus on one of the possibilities. Examination of Table B.1.4 reveals that state 2 is a compressed liquid with

$$P_2 = 500\ kPa.$$

Now 2 to 3 is isobaric so $P_3 = P_2 = 500\ kPa$. And we are given $T_3 = 250\ °C$. Examination of the tables reveals that state 3 is superheated with

$$v_3 = \boxed{0.47436\ \frac{m^3}{kg}.}$$

The work is easy to calculate:

$$\begin{aligned}
{}_1W_3 &= {}_1W_2 + {}_2W_3, \\
&= \underbrace{m\int_1^2 P dv}_{=0} + m\int_2^3 P dv, \\
&= mP_2(v_3 - v_2), \\
&= (10\ kg)(500\ kPa)\left(\left(0.47436\ \frac{m^3}{kg}\right) - \left(0.001080\ \frac{m^3}{kg}\right)\right), \\
&= \boxed{2366.4\ kJ.}
\end{aligned}$$

Finally the appropriate sketches are given.

Alternatively, one could have state 2 as a saturated liquid at 140 $°C$. At this state $v_f = v_2 = 0.001080\ m^3/kg$. At this state, we have $P_2 = 361.3\ kPa$. The final v_3 needs small adjustment, as does the work, which is left as an exercise to the student.

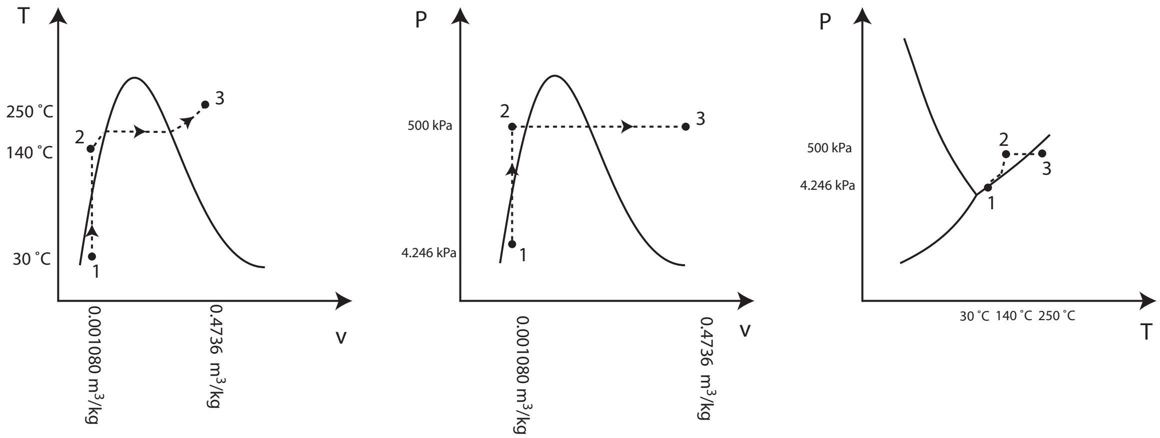


Figure 1: Process path in $T-v$, $P-v$, and $P-T$ planes (not to scale) for problem 2.

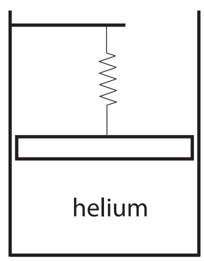


Figure 2: Piston-cylinder arrangement.

3. (40) A mass of 0.01 kg of helium at $P_1 = 100\ kPa$, $T_1 = 300\ K$ exists inside of the piston-cylinder arrangement of Fig. 2. The piston has a cross-sectional area of $A = 0.2\ m^2$. The helium is heated until $T_2 = 2000\ K$. The motion of the piston is resisted by a linear spring. The spring exerts no force at state 1, and has a spring constant of 1000 kN/m.

   (a) Find the final pressure.

   (b) Find the total work done.

Solution

Most people did well on this problem. The most common mistake was an incorrect approach to the quadratic equation which arose in the solution process. The next most common mistake was to neglect the ambient pressure in calculating the work. Occasionally in the force balance analysis, the ambient pressure was neglected incorrectly. Some people were confused about the difference between R and $\overline{R}$. A few people incorrectly assumed the process was isothermal. A few people incorrectly assumed this was an isochoric process. There were also unit errors in calculation of the work.

For helium, we have

$$R = \frac{\overline{R}}{M} = \frac{8.314 \ \frac{kJ}{kg \ K}}{4.003 \ \frac{kg}{kmol}} = 2.07694 \ \frac{kJ}{kg \ K}.$$

Now from the ideal gas law, we have

$$P_1 V_1 = mRT_1,$$

so

$$V_1 = \frac{mRT_1}{P_1} = \frac{(0.01 \ kg)\left(2.07695 \ \frac{kJ}{kg \ K}\right)(300 \ K)}{100 \ kPa} = 0.0623083 \ m^3.$$

Now for the piston, a force balance gives

$$PA = P_{atm}A + k(x - x_1),$$

$$PA = P_{atm}A + \frac{k}{A}(V - V_1),$$

$$P = P_{atm} + \frac{k}{A^2}(V - V_1).$$

When $V = V_1$, we have $P_1 = P_{atm}$, so

$$P = P_1 + \frac{k}{A^2}(V - V_1).$$

At state 2, we must have then

$$P_2 = P_1 + \frac{k}{A^2}(V_2 - V_1).$$

along with the ideal gas law

$$P_2 = \frac{mRT_2}{V_2}.$$

Substitute the ideal gas law into the force balance to eliminate P_2 and get

$$\frac{mRT_2}{V_2} = P_1 + \frac{k}{A^2}(V_2 - V_1).$$

$$mRT_2 = P_1 V_2 + \frac{k}{A^2}(V_2^2 - V_1 V_2).$$

This quadratic equation has two roots:

$$V_2 = \frac{-P_1 A^2 + kV_1 \pm \sqrt{4A^2 kmRT_2 + \left(-P_1 A^2 + kV_1\right)^2}}{2k}.$$

Substitution of numbers yields

$$V_2 = -0.0209609 \ m^3, \qquad \text{non-physical},$$

$$V_2 = 0.0792692 \ m^3, \qquad \text{physical}.$$

For the physical volume, we get

$$P_2 = \frac{mRT_2}{V_2} = \frac{(0.01 \ kg)\left(2.07695 \ \frac{kJ}{kg \ K}\right)(2000 \ K)}{0.0792692 \ m^3} = \boxed{524.023 \ kPa.}$$

Calculation of the work is straightforward:

$$
\begin{aligned}
{}_1W_2 &= \int_1^2 P dV, \\
&= \int_1^2 \left(P_1 + \frac{k}{A^2}(V - V_1) \right) dV, \\
&= \left(P_1 V + \frac{k}{2A^2}(V - V_1)^2 \right)\Bigg|_{V_1}^{V_2}, \\
&= P_1(V_2 - V_1) + \frac{k}{2A^2}(V_2 - V_1)^2, \\
&= (100\ kPa)\left(\left(0.0792692\ m^3\right) - \left(0.0623083\ m^3\right)\right) + \frac{1000\ \frac{kN}{m}}{(0.2\ m^2)^2}\left(\left(0.0792692\ m^3\right) - \left(0.0623083\ m^3\right)\right)^2, \\
&= \boxed{5.292\ kJ.}
\end{aligned}
$$

Finally an appropriate sketch of the $P - v$ plane is given.

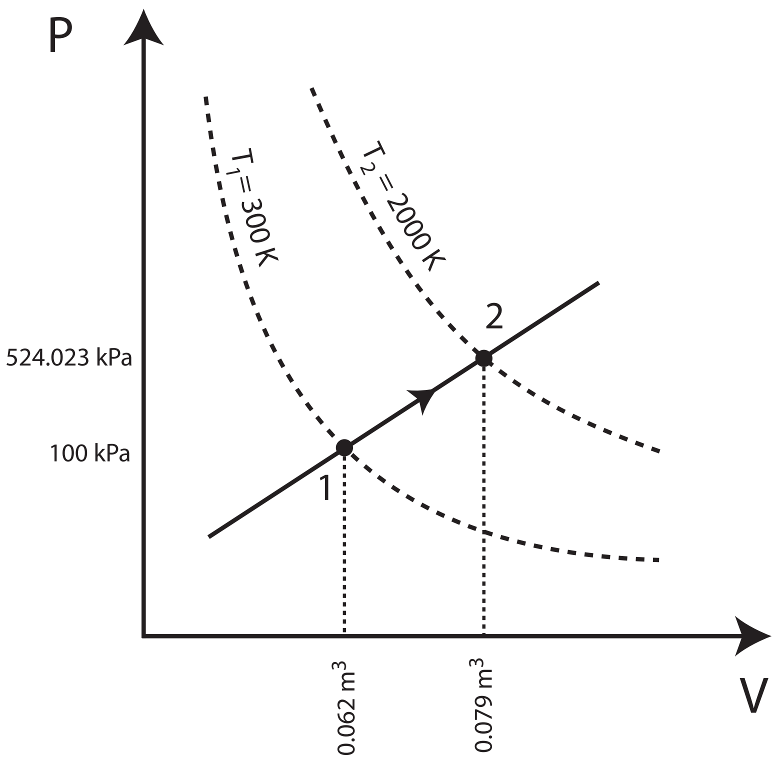


Figure 3: Process path in $P - v$ plane (not to scale).