Preliminaries

1 Introduction

The following is a brief introduction to some of the basic mathematics and physics that will be needed for a first course in fluid mechanics. You have probably seen all the material in earlier courses. It is here simply to remind you that it will be needed and to refresh your memory if you have forgotten some of the details. You would do well to consult the textbook you have used and locate the relevant places there as you read. The notation and presentation will undoubtedly be different though the concepts will be the same.

2 Vectors

A vector, as shown in Fig. 1, is a quantity with a magnitude and a direction. Here we will represent a vector by a bold-faced letter, like $\mathbf{a}$, while on the board it may be written as $\tilde{a}$, $\underline{a}$, or $\overrightarrow{a}$. A unit vector $\mathbf{n}$ is sometimes indicated as $\hat{n}$.

In three-dimensional space, $\mathbf{a}$ has three components. In Cartesian coordinates we can write it several different ways:

$$
\begin{aligned}
\mathbf{a} &= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} && (1)\\
&= a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z && (2)\\
&= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 && (3)
\end{aligned}
$$

The magnitude of the vector $\mathbf{a}$ is

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{4}$$

The dot or scalar product of $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a} \cdot \mathbf{b}$. It is a scalar with magnitude $|\mathbf{a}|\ |\mathbf{b}|\ \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

The cross product of $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a} \times \mathbf{b}$. As shown in Fig. 2 it is a vector with magnitude $|\mathbf{a}|\ |\mathbf{b}|\ \sin\theta$ and direction normal to both $\mathbf{a}$ and $\mathbf{b}$ in the sense of the right-hand rule.

3 Fields

Fields are quantities that vary with position in space. For example, a scalar field may be the temperature $T(x, y, z, t)$. A vector field may be the velocity $\mathbf{v}(x, y, z, t)$. A field may be unsteady if it depends on time t or unsteady if it does not depend on t.

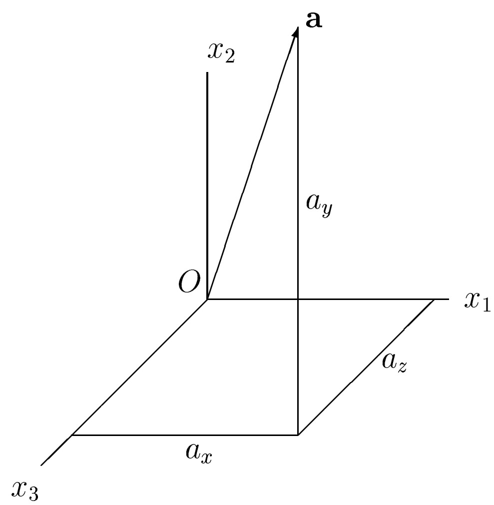

Figure 1: Vector **a** with Cartesian components a_x, a_y, a_z.

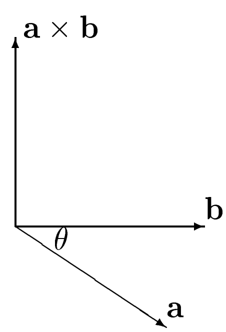

Figure 2: Cross product of **a** and **b**.

4 Calculus of a single variable

The derivative of $f(x)$ is written as $f'(x)$ or df/dx. A derivative operator $\mathcal{L}$ is linear if

$$\begin{aligned} \mathcal{L}(x+y) &= \mathcal{L}x + \mathcal{L}y && (5) \\ \mathcal{L}(\alpha x) &= \alpha \mathcal{L}x && (6) \end{aligned}$$

where $x = x(t)$, $y = y(t)$ and the derivative is with respect to t. A linear ordinary differential equation (ODE) can be written as

$$\mathcal{L}x = f(t) \tag{7}$$

A common equation is the linear nth order ordinary differential equation with constant coefficients

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_{n-1} \frac{dx}{dt} + a_n x = f(t) \tag{8}$$

where the operator is

$$\mathcal{L} = a_0 \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_{n-1} \frac{d}{dt} + a_n \tag{9}$$

The solution is

$$y(t) = y_h(t) + y_p(t) \tag{10}$$

where y_h is solution of the homogeneous equation and y_p is the particular solution.

Nonlinear equations are very common in fluid mechanics. Most have to be solved numerically. Exceptions include first-order equation that can be written in the separable form

$$f(x)\ dx = g(y)\ dy \tag{11}$$

where both sides are simply integrated.

Example 1

Solve the system of equations

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= -y \end{aligned}$$

5 Calculus of many variables

$f(x, y, z, t)$ can have 4 first derivatives: $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, $\partial f/\partial t$. Equations involving partial derivatives are called partial differential equations (PDE). Analytical solutions methods may be along one of the following lines:
(a) *Similarity solution:* The independent variables are grouped into a single variable $\eta = \eta(x, y, z, t)$. The PDE has a similarity solution if it and its boundary conditions are transformed into an ODE in η.
(b) *Separation of variables:* The dependent variable is written as products of functions, each one of which is a function of a single dependent variable. Thus, $f(x, y, z, t) = X(x)Y(y)Z(z)T(t)$. If the boundary conditions are such that $X(x)$, $Y(y)$, $Z(z)$ and $T(t)$ can be determined, the problem has a separation of variables solution.

6 Taylor series

Of functions of a single variable and many variables.

7 Integrals

In this section $f(x, y, z, t)$ is a scalar field and $\mathbf{f}(x, y, z, t)$ is a vector field.

7.1 Scalar integrand

The function $f(x, y, z, t)$ can be integrated with respect to a single variable, such as in

$$I(y, z, t) = \int_a^b f(x, y, z, t)\ dx \tag{12}$$

or with more than one variable as in

$$I(t) = \int_{a_z}^{b_z} \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x, y, z, t)\ dx\ dy\ dz \tag{13}$$

The scalar integrand may be in slightly more complicated form as in

$$I = \int_{\mathcal{A}} \mathbf{f} \cdot \mathbf{n}\ d\mathcal{A} \tag{14}$$

where $d\mathcal{A}$ an element of the surface $\mathcal{A}$.

Example 2

If $\rho(x,y,z)$ is the local density of a material of volume $\mathcal{V}$, show that its total mass is

$$M = \int_{\mathcal{V}} \rho(x,y,z)\ dx\ dy\ dz$$

and that the mean density is

$$\bar{\rho} = \frac{1}{M}\int_{\mathcal{V}} \rho(x,y,z)\ dx\ dy\ dz$$

7.1.1 Physical examples

7.2 Vectorial integrand

Examples are

$$\mathbf{I}(y,z,t) = \int_a^b \mathbf{f}(x,y,z,t)\ dx \tag{15}$$

and

$$\mathbf{I}(t) = \int_{a_z}^{b_z}\int_{a_y}^{b_y} \mathbf{f}(x,y,z,t)\ dx\ dy\ dz \tag{16}$$

One way to calculate integrals with vectorial integrands is to decompose the vector into its Cartesian components so that each component is an integral of a scalar integrand. For example, if

$$\mathbf{f} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} \tag{17}$$

then equation (15) becomes

$$\mathbf{I}(y,z,t) = \mathbf{i}\int_a^b f_x(x,y,z,t)\ dx + \mathbf{j}\int_a^b f_y(x,y,z,t)\ dx + \mathbf{k}\int_a^b f_z(x,y,z,t)\ dx \tag{18}$$

A more complicated form of the vectorial integrand is

$$\mathbf{I} = \int_{\mathcal{A}} (\mathbf{f}\cdot\mathbf{n})\mathbf{g}\ d\mathcal{A} \tag{19}$$

7.2.1 Physical examples

Force on a surface

$$\mathbf{F} = \int_{\mathcal{A}} p\ \mathbf{n}\ d\mathcal{A} \tag{20}$$

Momentum flux

$$\dot{\mathbf{J}} = \int_{\mathcal{A}} (\mathbf{J}\cdot\mathbf{n})\ \mathbf{v}\ d\mathcal{A} \tag{21}$$

8 Mechanics

Consult, for example, H.D. Young and R.A. Freedman, *University Physics*, 9th Ed., Addison-Wesley Publ. Co., Reading, MA, 1996.

8.1 Kinematics of a particle

If $\mathbf{r}(t)$ is the position vector of a particle, $\mathbf{v}(t)$ its velocity and $\mathbf{a}(t)$ its acceleration, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{22}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{23}$$

8.2 Newton's second law

The momentum

$$\mathbf{J} = m\mathbf{v} \tag{24}$$

of a particle of mass m and velocity $\mathbf{v}$ is a vector. Newton's second law for this particle is

$$\mathbf{F} = \frac{d\mathbf{J}}{dt} \tag{25}$$

For constant mass this becomes

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} \tag{26}$$

$$= m\mathbf{a} \tag{27}$$

8.3 Work

The work done by a force $\mathbf{F}$ is

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \tag{28}$$

where $\mathbf{r}$ is the position vector of the point of application of the force. The integral is along the curve $\mathcal{C}$ along which the point of application moves.

The kinetic energy of the particle

$$e = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} \tag{29}$$

is a scalar. From Newton's second law we can show that, for constant mass

$$\begin{aligned} \frac{de}{dt} &= m\mathbf{v} \cdot \mathbf{a} \\ &= \mathbf{v} \cdot \mathbf{F} \end{aligned} \tag{30}$$

This is the mechanical energy equation.

8.4 Moment

The moment of a force about a point O is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \tag{31}$$

where $\mathbf{F}$ is the force and $\mathbf{r}$ is the position of any point on the line of action of the force with respect to O.

9 Thermodynamics

The following, with slight changes in notation, is taken from M.J. Moran and H.N. Shapiro, *Fundamentals of Engineering Thermodynamics*, 3rd Ed., John Wiley & Sons, New York, 1995.

9.1 Properties

$$\begin{aligned} p &= \text{pressure} \\ s &= \text{specific entropy} \\ T &= \text{absolute temperature} \\ \hat{u} &= \text{specific internal energy} \\ \rho &= \text{density} \\ &= \frac{1}{\text{specific volume}} \\ h &= \text{specific enthalpy} \\ &= \hat{u} + \frac{p}{\rho} \\ c_v &= \text{specific heat at constant volume} \\ &= \left(\frac{\partial \hat{u}}{\partial T}\right)_v \\ c_p &= \text{specific heat at constant pressure} \end{aligned}$$

$$= \left(\frac{\partial h}{\partial T}\right)_p$$

9.2 Laws of thermodynamics

9.2.1 First law

9.2.2 Second law

9.3 Gibbs equation

$$T\ ds = d\hat{u} - \frac{p}{\rho^2}\ d\rho \quad (32)$$

$$T\ ds = dh - \frac{1}{\rho}\ dp \quad (33)$$

9.4 Incompressible fluid

$$c_p = c_v = c \quad (34)$$

For constant specific heat

$$\hat{u}_2 - \hat{u}_1 = c(T_2 - T_1) \quad (35)$$

$$h_2 - h_1 = c(T_2 - T_1) + \frac{1}{\rho}(p_2 - p_1) \quad (36)$$

$$s_2 - s_1 = c \ln \frac{T_2}{T_1} \quad (37)$$

9.5 Perfect gases

9.5.1 Thermally perfect gas

$$\begin{aligned} p &= \rho R T \\ &= \frac{M}{\mathcal{V}} R T \end{aligned}$$

9.5.2 Specific heats

$$\begin{aligned} d\hat{u} &= c_v\ dT \\ dh &= c_p\ dT \\ c_p - c_v &= R \end{aligned}$$

$$\begin{aligned}
\frac{c_p}{c_v} &= \gamma \\
c_v &= \frac{R}{\gamma - 1} \\
c_p &= \frac{\gamma R}{\gamma - 1}
\end{aligned}$$

9.5.3 Calorically perfect gas

(constant specific heats)

$$\begin{aligned}
\hat{u}_2 - \hat{u}_1 &= c_v(T_2 - T_1) \\
h_2 - h_1 &= c_p(T_2 - T_1)
\end{aligned}$$

9.5.4 Entropy change

$$\begin{aligned}
ds &= c_v \frac{dT}{T} - R\frac{d\rho}{\rho} \\
&= c_p \frac{dT}{T} - R\frac{dp}{p} \\
s_2 - s_1 &= c_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} \\
&= c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}
\end{aligned}$$

9.5.5 Isentropic process

$$\begin{aligned}
\frac{T_2}{T_1} &= \left(\frac{\rho_2}{\rho_1}\right)^{\gamma - 1} \\
\frac{T_2}{T_1} &= \left(\frac{p_2}{p_1}\right)^{(\gamma - 1)/\gamma} \\
\frac{p_2}{p_1} &= \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}
\end{aligned}$$

9.6 Mixtures of perfect gases

As an example consider moist air as a mixture of two ideal gases: water vapor and dry air. Each component has its own partial pressure and partial density, though the temperature is the same for both. The partial density of a component is the mass

	British	SI
Mass	slug	kilogram
Length	foot	meter
Time	second	second
Force	pound force	Newton
Pressure	pounds/ft^2	Pa = N/m^2
Energy	ft-pounds	J = N m

Table 1: Systems of units

of that component per unit volume of the mixture. The sum of the partial pressures and partial densities gives the total pressure and total density. You can use the ideal gas law for each component provided you use the partial density and pressure.

Saturated air contains all the water vapor it possibly can. Relative humidity is the ratio of water vapor density actually present in the air to the water vapor density that would be present if the air were saturated at the same temperature. If the relative humidity is 100%, the air is saturated and the water vapor partial pressure corresponding to the temperature is the saturated vapor pressure. Moist air at 50% relative humidity will have a density one-half of this.

10 Units

We will use a consistent set of units as in Table 1. Sometimes, the pound mass is used instead of slugs in the British system; the slug is 32.2 times larger than a pound mass.

Problems

1. Write the Taylor series expansion of the function e^x around $x = 1$. Plot the value of $e^{3/2}$ as a function of the number of terms that you use in this series. Show also the exact value on the graph.
2. Prove that if the speed of a particle is constant, its acceleration must be normal to its velocity.
3. Material occupying the volume $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$ has a local density $\rho(x, y, z) = 1 + x + 2y + 3z$. Find the mass of the material and its mean density.
4. Find the path of a particle starting from the origin and with velocity given by

$$\mathbf{v} = (1 + t)\mathbf{i} + (1 + t^2)\mathbf{j}$$

Find the acceleration of the particle. If the mass of the particle is m find the force on the particle.

5. Find the path of a particle starting from the origin and with velocity given by

$$\mathbf{v} = (1 + x^2)\mathbf{i} + (1 + y^2)\mathbf{j}$$

Find the acceleration of the particle. Why is it non-zero even though the velocity field does not explicitly depend on time? If the mass of the particle is m find the force on it.

6. An infinite set of point masses located at coordinates (i, j) at $t = 0$, where i and j are integers, are all moving at the same velocity $\mathbf{v}$. If the mass of each point mass is m, find the momentum entering and leaving a cube of size δ centered at the origin, averaged over unit time.

7. Find the force per unit depth on a surface if the pressure is given by

$$p = x + y$$

The surface is (a) the straight line connecting the origin with the point (1,1) and (b) the curve $y = x^2$ connecting the origin with the point (1,1).

8. Find the moment about the center of a horizontal beam due its weight. The beam extends from $x = -\ell$ to $x = \ell$ and its mass per unit length is $m(x)$.

9. Find the moment about the center of a horizontal square sheet due to its weight. The sheet is of side 2ℓ and its mass per unit area is $m(x, y)$.

10. Find the area of a sheet that is originally a square with corners at (0,0), (1,0), (1,1), (0,1) and uniform temperature T_0 after it is elevated to a known temperature $T(x, y) = T_0 + x + y^2$. α is the coefficient of linear expansion of the material.

11. Find the equation of a closed curve that was initially a circle of unit radius centered at the origin if a point on the curve at (x, y) moves to $(nx, y/n)$, where n is a constant.