A Short Note on Error Analysis

Let us say we have a theory to predict experimental results based on our observations. Let us say our theory is intended to predict y, where y has a dependency on n independent variables $x_1, \ldots x_n$ and m parameters $\mu_1, \ldots \mu_m$.

$$y = F(x_1, x_2, \ldots, x_n; \mu_1, \mu_2, \ldots \mu_m), \tag{1}$$

for some function F, then the error $\triangle y_i$, in the final result y, due to the error $\triangle \mu_i$, in measurement of μ_i, is given by

$$\triangle y_i = \frac{\partial F}{\partial \mu_i} \triangle \mu_i. \tag{2}$$

Note that our theory is accounting for the variation in x_i, so as far as the theory goes, we need not compute its error. We may have independent estimates of our measurement error, however. The total error $\triangle y$ is then given by

$$\triangle y = \sqrt{\sum_{i=1}^{m} (\triangle y_i)^2}. \tag{3}$$

Here as an example we work out the error in the theoretical prediction of a free surface under rotation in Unit 1b. The equation for the free surface is given by

$$z = \frac{\omega^2 r^2}{2g}. \tag{4}$$

Here z is the height of the free surface, ω is the angular velocity of the surface, r is the local radial coordinate, and g is the acceleration due to gravity. Now we actually use the formula

$$\omega = \frac{2\pi N}{T}, \tag{5}$$

to determine ω, where N is the measured number of revolutions, and T is the measured time of revolution, so in terms of quantities which are directly measured, we have

$$z = \frac{2\pi^2 N^2}{gT^2} r^2. \tag{6}$$

Here, we take the dependent variable y to be z, there is only one independent variable, so $n = 1$, and $x_1 = r$, and the parameters μ_1, μ_2, and μ_3 are N, T, and g.

There will be an error $\triangle T$, in the measurement of time T. We also assume that there is an error $\triangle g$, in the acceleration due to gravity g, inside the laboratory, as well as an error $\triangle N$ in our measurement of the number of revolutions N. Taking the sum of the partial derivatives, we can estimate the total error as

$$\begin{aligned} \Delta z &= \frac{\partial z}{\partial T}\Delta T + \frac{\partial z}{\partial g}\Delta g + \frac{\partial z}{\partial N}\Delta N, & (7) \\ &= -\frac{4N^2\pi^2 r^2}{gT^3}\Delta T - \frac{2N^2\pi^2 r^2}{g^2T^2}\Delta g + \frac{4N\pi^2 r^2}{gT^2}\Delta N. & (8) \end{aligned}$$

Now we assume that the errors in T, g, and N, are known or can be estimated, and that they are random and uncorrelated. Thus, we might expect a total error to be of the order of the square root of the sum of the squares of the individual errors:

$$\Delta z = \frac{2N^2\pi^2 r^2}{gT^2}\sqrt{\left(\frac{2\Delta T}{T}\right)^2 + \left(\frac{\Delta g}{g}\right)^2 + \left(\frac{2\Delta N}{N}\right)^2}. \tag{9}$$

As an exercise one can verify Eq. (9) by deriving it. The theoretical prediction of the free surface is plotted in Fig. (1) using Eq. (4). Also plotted is the error in the theoretical prediction, by plotting $z \pm \triangle z$ *vs.* r. Here, we have actually taken data and estimated the random errors. Now, no figure is perfect. Some criticisms which could be leveled at this curve include

- One axis is dimensionless, while the other is dimensional.
- It is very busy, and hard to figure out everything.
- The least squares curve fit does not add a lot, and could be just included as an equation in the text.
- Negative r usually does not make sense, although it is rather visually effective here. Strictly speaking mathematically, it would be better to plot one result at for z versus r at $\theta = 0$, and another for z versus r at $\theta = \pi$.

If we neglect all errors except the one due to the stopwatch used for measuring time T, then the error $\triangle T$ is ± 0.01 s, which is the smallest fraction of time the stopwatch provided can measure. But we know that there are other errors involved such as human error in estimating when N rotations are over. One way to estimate the the error is to make M observations of time T taken for N rotations. Then one can find the mean and standard deviation as following

$$\overline{T} = \frac{T_1 + T_2 + \cdots + T_M}{M}, \tag{10}$$

$$\sigma = \frac{\sqrt{\sum_{i=1}^{M}(T_i - \overline{T})^2}}{M}, \tag{11}$$

where T_i is the time measured in the ith observation, $\overline{T}$ is the mean time, and σ is the standard deviation. Now one can quote the error in T in terms of the standard deviation. This tends to be valid for random errors such as human error. Hence, one can substitute $\triangle T = \pm\sigma$ and $T = \overline{T}$ in Eq. (9).

Another way to depict errors is using error bars as done for the data points in Fig. (1). There are several sources of errors when taking data in Unit 1b. One quantifiable error is the smallest length measured by the scale in vertical direction, which is ± 1 *mm*, and the other quantifiable error is the smallest length measured by the scale in the horizontal direction, which is ± 0.01 *in*. These errors are plotted as error bars in the two directions in the figure. Note that both z and r for the data points are relative lengths. For example in $z = h - h_o$, both h and h_o have a quantifiable error of $\pm 1\ mm$, hence, the quantifiable error in $h - h_o$ is $\pm\sqrt{2}\ mm$ following the error analysis on page 1. The following commands in matlab can be useful in producing plots and visualizing errors.

- polyfit
- polyval (not very robust)
- errorbar

Finally a review of the undergraduate text on measurements and error analysis is recommended.

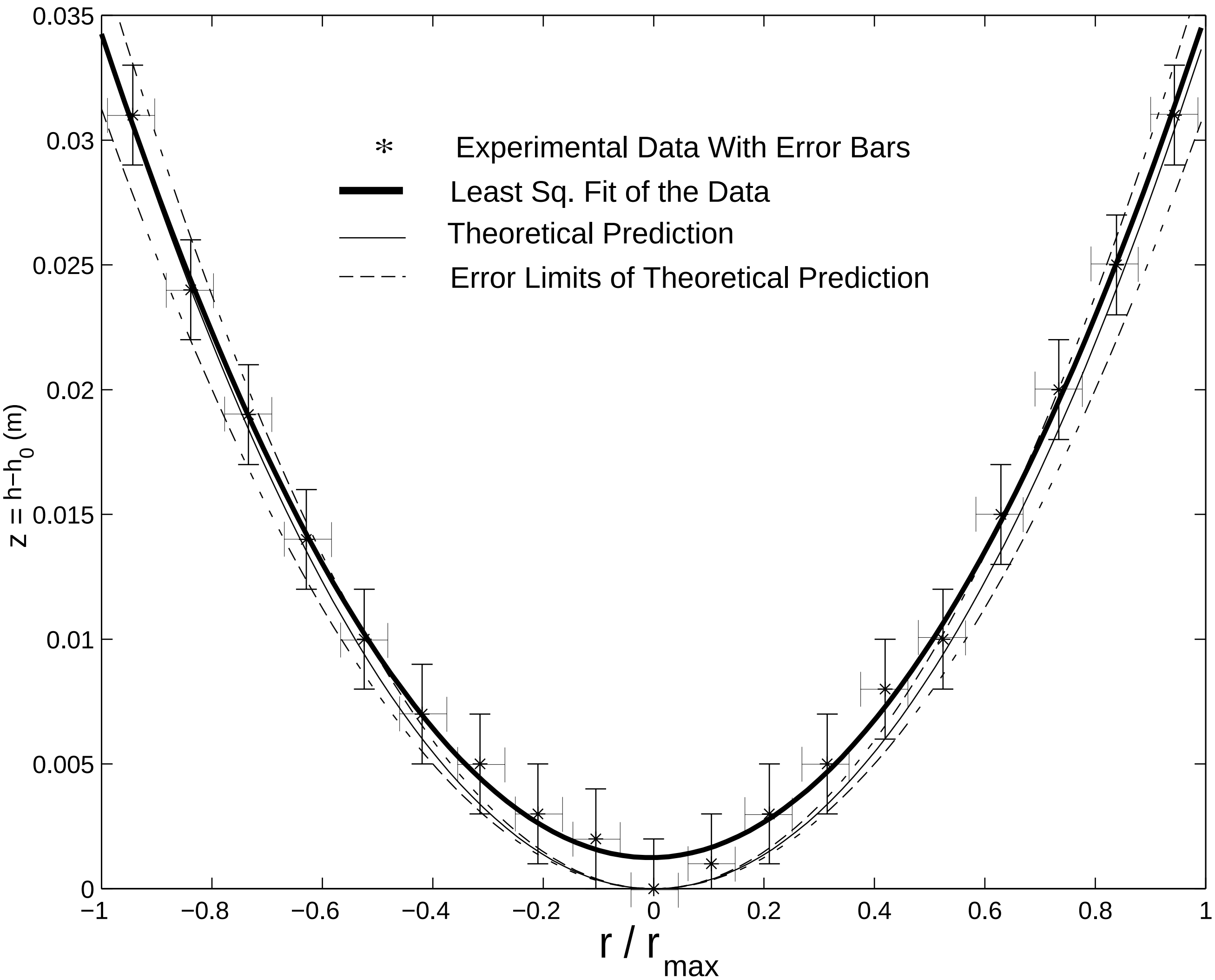

Figure 1: Free surface of water in rotating cylinder.