Perspective: A Method for Uniform Reporting of Grid Refinement Studies

Patrick J. Roache
Journal of Fluids Engineering
September 1994

Presented By: Steven Brus
AME40510: Introduction to Numerical Methods

May 4, 2011
Calculations must be verified to determine their accuracy

Verification→“solving governing equations right”

Accomplished through systematic grid refinement studies

Performance reports are often inconsistent and confusing

Paper purposes a Grid Convergence Index (GCI) to uniformly report results of grid refinements studies
Richardson Extrapolation

- GCI method relies on the theory of Richardson Extrapolation
- Used to give an error estimate for a solution computed with a certain grid
- Requires two solutions:
  - $f_1 \rightarrow$ fine grid solution with grid spacing $h_1$
  - $f_2 \rightarrow$ coarse grid solution with grid spacing $h_2$
- Assumed series representation for discrete solution $f$
  \[
  f = f_{\text{exact}} + g_1 h + g_2 h^2 + g_3 h^3 + \ldots \quad (1)
  \]
- Generalization for $p^{th}$ order methods and $r$ valued grid ratio ($h_2/h_1$)
  \[
  f_{\text{exact}} \approx f_1 + \frac{f_1 - f_2}{r^p - 1} \quad \text{(correction to fine grid)} \quad (2)
  \]
  \[
  f_{\text{exact}} \approx f_2 + \frac{(f_1 - f_2)r^p}{r^p - 1} \quad \text{(correction to coarse grid)} \quad (3)
  \]
GCI for Coarse and Fine Grid Solutions

Coarse Grid

\[ E_{\text{coarse grid}} = \frac{|\epsilon| r^p}{r^p - 1} \]  \hspace{1cm} \text{(4)}

\[ \text{GCI}_{\text{coarse grid}} = \frac{3|\epsilon| r^p}{r^p - 1} \]  \hspace{1cm} \text{(5)}

Fine Grid

\[ E_{\text{fine grid}} = \frac{|\epsilon|}{r^p - 1} \]  \hspace{1cm} \text{(6)}

\[ \text{GCI}_{\text{fine grid}} = \frac{3|\epsilon|}{r^p - 1} \]  \hspace{1cm} \text{(7)}

fractional deviation : \( \epsilon = \frac{f_2 - f_1}{f_1} \)  \hspace{1cm} \text{(8)}
For the GCI to be valid, both grids must be in the asymptotic range.

To ensure this, 3 grids must be computed:
- \( f_3 \) (coarse)
- \( f_2 \) (intermediate)
- \( f_1 \) (fine)

This yields 2 GCIs:
- \( f_1, f_2 \rightarrow \text{GCI}_{12} \)
- \( f_2, f_3 \rightarrow \text{GCI}_{23} \)

The asymptotic range has been achieved if

\[
\text{GCI}_{23} \approx r^p \text{GCI}_{12} \quad (9)
\]
**Example**

- Steady State Burger’s Equation (for $Re = 1000$)

\[
-u \frac{du}{dx} + \frac{1}{Re} \frac{d^2u}{dx^2} = 0 \quad u(0) = 1 \quad u(1) = 0
\]  

(10)

- Second order centered differences on a uniform grid, $p = 2$

- Evaluate $f = \frac{du}{dx}$ at $x = 1$, $r = 1.25$

  - Fine grid (2000 cells): $f_1 = -529.41$
  
  - Coarse grid (1600 cells): $f_2 = -544.48$

- Exact solution: $f_{exact} = -500$

\[
|\epsilon| = 100\% \times \frac{f_2 - f_1}{f_1} = 2.85\%
\]

(11)

\[
(r^p - 1) = (1.25^2 - 1) = 0.5625
\]

(12)
Example: Fine Grid

- Richardson Extrapolation error estimator

\[ E_{\text{fine grid}} = \frac{|\epsilon|}{r^p - 1} = \frac{2.85\%}{0.5625} = 5.07\% \] (13)

- Grid Convergence Index

\[ \text{GCI}_{\text{fine grid}} = \frac{3|\epsilon|}{r^p - 1} = \frac{3 \times 2.85\%}{0.5625} = 15.20\% \] (14)

- Exact fine grid error

\[ A = 100\% \times \frac{|f_1 - f_{\text{exact}}|}{f_{\text{exact}}} = 100\% \times \frac{|-529.41 + 500.0|}{500.0} = 5.88\% \] (15)
Example: Coarse Grid

- Richardson Extrapolation error estimator

\[
E_{\text{coarse grid}} = \frac{|\epsilon| r^p}{r^p - 1} = \frac{2.85\% \times 1.25^2}{0.5625} = 7.92\% \quad (16)
\]

- Grid Convergence Index

\[
\text{GCI}_{\text{fine grid}} = \frac{3|\epsilon| r^p}{r^p - 1} = \frac{3 \times 2.85\% \times 1.25^2}{0.5625} = 23.75\% \quad (17)
\]

- Exact coarse grid error

\[
A = 100\% \times \frac{|f_2 - f_{\text{exact}}|}{f_{\text{exact}}} = 100\% \times \frac{|-544.48 + 500.0|}{500.0} = 8.90\% \quad (18)
\]
GCI is a conservative error band whereas Richardson Extrapolation error estimator is not

Allows the results of grid refinement studies to be uniformly interpreted

Non-integer grid refinement can be used ($r = 1.1$ minimum)

Can be applied \textit{a posteriori} by editors and reviewers