# Control Volume Derivations for Thermodynamics

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This document will give a summary of the necessary mathematical operations necessary to cast the conservation of mass and energy principles in a traditional control volume formulation. The analysis presented has been amalgamated from a variety of sources. Most directly, it is an specialization of my course notes for AME 538. [1] Basic mathematical foundations are covered well by Kaplan. [2] A very detailed a readable description, which has a stronger emphasis on fluid mechanics, is given in the undergraduate text of Whitaker. [3] A very rigorous treatment of the development of all equations presented here is included in the graduate text of Aris. [4] Popular mechanical engineering undergraduate fluids texts have closely related expositions. [5] [6] However, despite their detail, these texts have some minor flaws! The treatment given in the AME 327 text by Sonntag, Borgnakke, and Van Wylen [7] (SBVW) is not as detailed. This document will use a notation generally consistent with SBVW and show in detail how to arrive at its results.

# 1 Relevant mathematics

We will use several theorems which are developed in vector calculus. Here we give short motivations and presentations. The reader should consult a standard mathematics text for detailed derivations.

## 1.1 Fundamental theorem of calculus

The fundamental theorem of calculus is as follows

$$\int_{x=a}^{x=b} \phi(x)\ dx = \int_{x=a}^{x=b} \left(\frac{d\psi}{dx}\right)\ dx = \psi(b) - \psi(a). \tag{1}$$

It effectively says that to find the integral of a function $\phi(x)$, which is the area under the curve, it suffices to find a function $\psi$, whose derivative is $\phi$, i.e. $\frac{d\psi}{dx} = \phi(x)$, evaluate $\psi$ at each endpoint, and take the difference to find the area under the curve.

[1]J. M. Powers, 2003, Lecture Notes on Intermediate Fluid Mechanics, University of Notre Dame, $http://www.nd.edu/\sim powers/ame.538/notes.pdf$.

[2]W. Kaplan, 2003, *Advanced Calculus*, Fifth Edition, Addison-Wesley, New York.

[3]S. Whitaker, 1992, *Introduction to Fluid Mechanics*, Krieger, Malabar, Florida.

[4]R. Aris, 1962, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Dover, New York.

[5]F. M. White, 2002, *Fluid Mechanics*, Fifth Edition, McGraw-Hill, New York.

[6]R. W. Fox, A. T. McDonald, and P. J. Pritchard, 2003, *Introduction to Fluid Mechanics*, Sixth Edition, John Wiley, New York.

[7]R. E. Sonntag, C. Borgnakke, and G. J. Van Wylen, 2003, *Fundamentals of Thermodynamics*, Sixth Edition, John Wiley, New York.

## 1.2 Gauss's theorem

Gauss's [8] theorem is the analog of the fundamental theorem of calculus extended to volume integrals. Let us define the following quantities:

- $t \rightarrow$ time,
- $\mathbf{x} \rightarrow$ spatial coordinates,
- $V_a(t) \rightarrow$ arbitrary moving and deforming volume,
- $A_a(t) \rightarrow$ bounding surface of the arbitrary moving volume,
- $\mathbf{n} \rightarrow$ outer unit normal to moving surface,
- $\phi(\mathbf{x}, t) \rightarrow$ arbitrary scalar function of $\mathbf{x}$ and $t$

Gauss's theorem is as follows:

$$\int_{V_a(t)} \nabla\phi \; dV = \int_{A_a(t)} \phi\mathbf{n} \; dA \tag{2}$$

The surface integral is analogous to evaluating the function at the end points in the fundamental theorem of calculus.

Note if we take $\phi$ to be the scalar of unity (whose derivative must be zero), Gauss's theorem reduces to

$$\int_{V_a(t)} \nabla(1) \; dV = \int_{A_a(t)} (1)\mathbf{n} \; dA, \tag{3}$$

$$0 = \int_{A_a(t)} (1)\mathbf{n} \; dA, \tag{4}$$

$$\int_{A_a(t)} \mathbf{n} \; dA = 0. \tag{5}$$

That is the unit normal to the surface integrated over the surface, cancels to zero when the entire surface is included.

## 1.3 Divergence theorem

The extension of Gauss's theorem (2) to vector functions is the divergence theorem and is as follows:

$$\int_{V_a(t)} \nabla \cdot \boldsymbol{\alpha} \; dV = \int_{A_a(t)} \boldsymbol{\alpha} \cdot \mathbf{n} \; dA. \tag{6}$$

Here $\boldsymbol{\alpha}$ is an arbitrary vector function.

We will use the divergence theorem (6) extensively. It allows us to convert sometimes difficult volume integrals into easier interpreted surface integrals. It is often useful to use this theorem as a means of toggling back and forth from one form to another.

[8] Carl Friedrich Gauss, 1777-1855, Brunswick-born German mathematician, considered the founder of modern mathematics. Worked in astronomy, physics, crystallography, optics, biostatistics, and mechanics. Studied and taught at Göttingen.

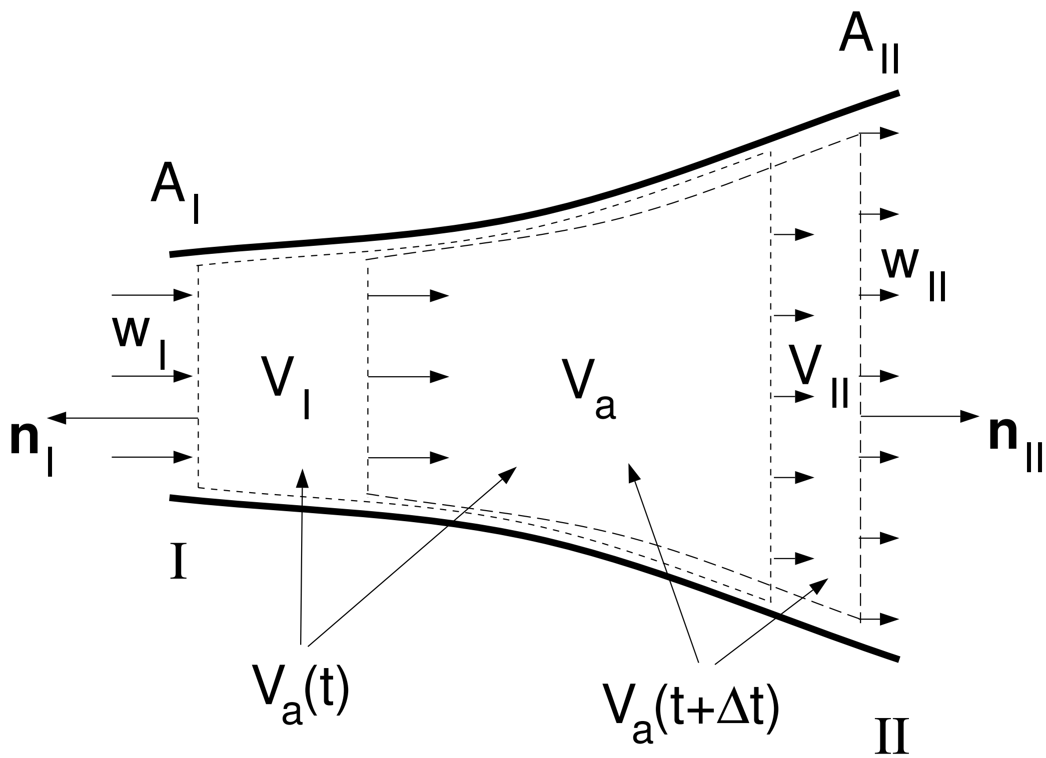


Figure 1: Sketch of the motion of an arbitrary volume $V_a(t)$. The boundaries of $V_a(t)$ move with velocity $\mathbf{w}$. The outer normal to $V_a(t)$ is $A_a(t)$. Here we focus on just two regions: $I$, where the volume is leaving material behind, and $II$, where the volume is sweeping up new material.

### 1.4 Leibniz's theorem

Leibniz's [9] theorem relates time derivatives of integral quantities to a form which distinguishes changes which are happening within the boundaries to changes due to fluxes through boundaries.

Let us consider the scenario sketched in Figure 1. Say we have some value of interest, $\Phi$, which results from an integration of a kernel function $\phi$ over $V_a(t)$, for instance

$$\Phi = \int_{V_a(t)} \phi \; dV. \tag{7}$$

We are often interested in the time derivative of $\Phi$, the calculation of which is complicated by the fact that the limits of integration are time-dependent. From the definition of the derivative, we find that

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_{V_a(t)} \phi \; dV = \lim_{\Delta t \to 0} \frac{\int_{V_a(t+\Delta t)} \phi(t+\Delta t) \; dV - \int_{V_a(t)} \phi(t) \; dV}{\Delta t}. \tag{8}$$

Now we have

$$V_a(t+\Delta t) = V_a(t) + V_{II}(\Delta t) - V_I(\Delta t). \tag{9}$$

[9]Gottfried Wilhelm von Leibniz, 1646-1716, Leipzig-born German philosopher and mathematician. Invented calculus independent of Newton and employed a superior notation to that of Newton.

Here $V_{II}(\Delta t)$ is the amount of new volume swept up in time increment $\Delta t$, and $V_I(\Delta t)$ is the amount of volume abandoned in time increment $\Delta t$. So we can break up the first integral into

$$\int_{V_a(t+\Delta t)} \phi(t+\Delta t)\, dV = \int_{V_a(t)} \phi(t+\Delta t)\, dV + \int_{V_{II}(\Delta t)} \phi(t+\Delta t)\, dV - \int_{V_I(\Delta t)} \phi(t+\Delta t)\, dV, \quad (10)$$

which gives us then

$$\frac{d}{dt}\int_{V_a(t)} \phi\, dV =$$
$$\lim_{\Delta t \to 0} \frac{\int_{V_a(t)} \phi(t+\Delta t)\, dV + \int_{V_{II}(\Delta t)} \phi(t+\Delta t)\, dV - \int_{V_I(\Delta t)} \phi(t+\Delta t)\, dV - \int_{V_a(t)} \phi(t)\, dV}{\Delta t}. \quad (11)$$

Rearranging (11) by combining terms with common limits of integration, we get

$$\frac{d}{dt}\int_{V_a(t)} \phi\, dV = \lim_{\Delta t \to 0} \frac{\int_{V_a(t)} \left(\phi(t+\Delta t) - \phi(t)\right)\, dV}{\Delta t}$$
$$+ \lim_{\Delta t \to 0} \frac{\int_{V_{II}(\Delta t)} \phi(t+\Delta t)\, dV - \int_{V_I(\Delta t)} \phi(t+\Delta t)\, dV}{\Delta t}. \quad (12)$$

Let us now further define

- $\mathbf{w} \to$ velocity vector of points on the moving surface $V_a(t)$,

Now the volume swept up by the moving volume in a given time increment $\Delta t$ is

$$dV_{II} = \underbrace{\mathbf{w}\cdot\mathbf{n}}_{\text{positive}}\, \Delta t\, dA_{II} = \underbrace{w_{II}\Delta t}_{\text{distance}}\, dA_{II}, \quad (13)$$

and the volume abandoned is

$$dV_I = \underbrace{\mathbf{w}\cdot\mathbf{n}}_{\text{negative}}\, \Delta t\, dA_I = -\underbrace{w_I \Delta t}_{\text{distance}}\, dA_I. \quad (14)$$

Substituting into our definition of the derivative, we get

$$\frac{d}{dt}\int_{V_a(t)} \phi\, dV = \lim_{\Delta t \to 0} \int_{V_a(t)} \frac{(\phi(t+\Delta t) - \phi(t))}{\Delta t}\, dV$$
$$+ \lim_{\Delta t \to 0} \frac{\int_{A_{II}(\Delta t)} \phi(t+\Delta t) w_{II} \Delta t\, dA_{II} + \int_{A_I(\Delta t)} \phi(t+\Delta t) w_I \Delta t\, dA_I}{\Delta t} \quad (15)$$

Now we note that

- We can use the definition of the partial derivative to simplify the first term on the right side of (15),
- The time increment $\Delta t$ cancels in the area integrals of (15), and
- $A_a(t) = A_I + A_{II}$,

so that

$$\underbrace{\frac{d}{dt}\int_{V_a(t)} \phi \; dV}_{\text{total time rate of change}} = \underbrace{\int_{V_a(t)} \frac{\partial \phi}{\partial t} \; dV}_{\text{intrinsic change within volume}} + \underbrace{\int_{A_a(t)} \phi \mathbf{w} \cdot \mathbf{n} \; dA}_{\text{net flux into volume}}. \tag{16}$$

This is the three-dimensional scalar version of *Leibniz's theorem*.

We can also apply the divergence theorem (6) to Leibniz's theorem (16) to convert the area integral into a volume integral to get

$$\frac{d}{dt}\int_{V_a(t)} \phi \; dV = \int_{V_a(t)} \left( \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{w}) \right) \; dV. \tag{17}$$

Say we have the very special case in which $\phi = 1$; then Leibniz's theorem (16) reduces to

$$\frac{d}{dt}\int_{V_a(t)} dV = \int_{V_a(t)} \frac{\partial}{\partial t}(1) \; dV + \int_{A_a(t)} (1)\mathbf{w} \cdot \mathbf{n} \; dA, \tag{18}$$

$$\frac{d}{dt}V_a(t) = \int_{A_a(t)} \mathbf{w} \cdot \mathbf{n} \; dA. \tag{19}$$

This simply says the total volume of the region, which we call $V_a(t)$, changes in response to net motion of the bounding surface.

Leibniz's theorem (16) reduces to a more familiar result in the one-dimensional limit. We can then say

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} \phi(x,t) \; dx = \int_{x=a(t)}^{x=b(t)} \frac{\partial \phi}{\partial t} \; dx + \frac{db}{dt}\phi(b(t),t) - \frac{da}{dt}\phi(a(t),t). \tag{20}$$

As in the fundamental theorem of calculus (1), for the one-dimensional case, we do not have to evaluate a surface integral; instead, we simply must consider the function at its endpoints. Here $\frac{db}{dt}$ and $\frac{da}{dt}$ are the velocities of the bounding surface and are equivalent to $\mathbf{w}$. The terms $\phi(b(t),t)$ and $\phi(a(t),t)$ are equivalent to evaluating $\phi$ on $A_a(t)$.

## 1.5 General Transport Theorem

Let $B$ be an arbitrary extensive thermodynamic property, and $\beta$ be the corresponding intensive thermodynamic property so that

$$dB = \beta dm. \tag{21}$$

The product of a differential amount of mass $dm$ with the intensive property $\beta$ give a differential amount of the extensive property. Since

$$dm = \rho dV, \tag{22}$$

where $\rho$ is the mass density and $dV$ is a differential amount of volume, we have

$$dB = \beta \rho dV. \tag{23}$$

If we take the arbitrary $\phi = \rho\beta$, the Leibniz theorem becomes our general transport theorem:

$$\frac{d}{dt}\int_{V_a(t)} \rho\beta \; dV = \int_{V_a(t)} \frac{\partial}{\partial t}(\rho\beta) \; dV + \int_{A_a(t)} \rho\beta \left(\mathbf{w}\cdot\mathbf{n}\right) \; dA. \tag{24}$$

Applying the divergence theorem to the general transport theorem, we find

$$\frac{d}{dt}\int_{V_a(t)} \rho\beta \; dV = \int_{V_a(t)} \left(\frac{\partial}{\partial t}(\rho\beta) + \nabla\cdot(\rho\beta\mathbf{w})\right) \; dV. \tag{25}$$

## 1.6 Reynolds transport theorem

We get the well known Reynolds [10] transport theorem if we force the arbitrary velocity of the moving volume to take on the velocity of a fluid particle, i.e. take

$$\mathbf{w} = \mathbf{v} \tag{26}$$

In this case, our arbitrary volume is no longer arbitrary. Instead, it always contains the same fluid particles. We call this volume a *material volume*, $V_m(t)$. The proper way to generalize laws of nature which were developed for point masses is to consider collections of fixed point masses, which will always reside within a material volume. That said, it is simple to specialize the general transport theorem to obtain the Reynolds transport theorem. Here we give two versions, the first using area integrals, and the second using volume integrals only:

$$\frac{d}{dt}\int_{V_m(t)} \rho\beta \; dV = \int_{V_m(t)} \frac{\partial}{\partial t}(\rho\beta) \; dV + \int_{A_m(t)} \rho\beta \left(\mathbf{v}\cdot\mathbf{n}\right) \; dA, \tag{27}$$

$$\frac{d}{dt}\int_{V_m(t)} \rho\beta \; dV = \int_{V_m(t)} \left(\frac{\partial}{\partial t}(\rho\beta) + \nabla\cdot(\rho\beta\mathbf{v})\right) \; dV. \tag{28}$$

## 1.7 Fixed (control) volumes

If we take our arbitrary volume to be fixed in space, it is most often known as a *control volume*. For such volumes

$$\mathbf{w} = \mathbf{0}. \tag{29}$$

Thus the arbitrary volume loses its time dependency, so that

$$V_a(t) = V, \qquad A_a(t) = A, \tag{30}$$

and the general transport theorem reduces to

$$\frac{d}{dt}\int_{V} \rho\beta \; dV = \int_{V} \frac{\partial}{\partial t}(\rho\beta) \; dV. \tag{31}$$

[10] Osborne Reynolds, 1842-1912, Belfast-born British engineer and physicist, educated in mathematics at Cambridge, first professor of engineering at Owens College, Manchester, did fundamental experimental work in fluid mechanics and heat transfer.

# 2 Conservation axioms

A fundamental goal of mechanics is to take the verbal notions which embody the basic axioms into usable mathematical expressions. First, we must list those axioms. The axioms themselves are simply principles which have been observed to have wide validity as long as length scales are sufficiently large to contain many molecules. Many of these axioms can be applied to molecules as well. The axioms cannot be proven. They are simply statements which have been useful in describing the universe.

A summary of the axioms in words is as follows

- *Mass conservation principle*: The time rate of change of mass of a material region is zero.
- *Linear momenta principle*: The time rate of change of the linear momenta of a material region is equal to the sum of forces acting on the region. This is Euler's generalization of Newton's second law of motion.
- *Angular momenta principle*: The time rate of change of the angular momenta of a material region is equal to the sum of the torques acting on the region. This was first formulated by Euler.
- *Energy conservation principle*: The time rate of change of energy within a material region is equal to the rate that energy is received by heat and work interactions. This is the first law of thermodynamics.
- *Entropy inequality*: The time rate of change of entropy within a material region is greater than or equal to the ratio of the rate of heat transferred to the region and the absolute temperature of the region. This is the second law of thermodynamics.

Here we shall systematically convert two of these axioms, the mass conservation principle and the energy conservation principle, into mathematical form.

## 2.1 Mass

Mass is an extensive property for which we have

$$B = m, \qquad \beta = 1. \tag{32}$$

The mass conservation axiom is simple to state mathematically. It is

$$\frac{d}{dt} m = 0. \tag{33}$$

A relevant material volume is sketched in Figure 2. We can define the mass enclosed within a material volume based upon the local value of density:

$$m = \int_{V_m(t)} \rho dV. \tag{34}$$

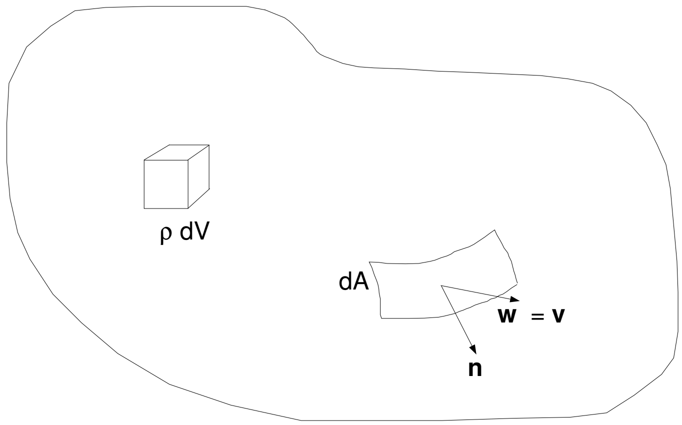


Figure 2: Sketch of finite material region $V_m(t)$, infinitesimal mass element $\rho dV$, and infinitesimal surface element $dA$ with unit normal $\mathbf{n}$, and general velocity $\mathbf{w}$ equal to fluid velocity $\mathbf{v}$.

So the mass conservation axiom is

$$\frac{d}{dt}\int_{V_m(t)} \rho dV = 0. \tag{35}$$

Invoking the Reynolds transport theorem (27), $\frac{d}{dt}\int_{V_m(t)}[\,]dV = \int_{V_m(t)} \frac{\partial}{\partial t}[\,]dV + \int_{A_m(t)} \mathbf{v}\cdot\mathbf{n}[\,]dA$, we get

$$\frac{d}{dt}\int_{V_m(t)} \rho \; dV = \int_{V_m(t)} \frac{\partial \rho}{\partial t} \; dV + \int_{A_m(t)} \rho \mathbf{v}\cdot\mathbf{n} \; dA = 0. \tag{36}$$

Now we invoke the divergence theorem, Eq. (6) $\int_{V(t)} \nabla\cdot[\,]dV = \int_{A(t)} \mathbf{n}\cdot[\,]dA$, to convert a surface integral to a volume integral to get the mass conservation axiom to read as

$$\int_{V_m(t)} \frac{\partial \rho}{\partial t} \; dV + \int_{V_m(t)} \nabla\cdot(\rho\mathbf{v}) \; dV = 0, \tag{37}$$

$$\int_{V_m(t)} \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) dV = 0. \tag{38}$$

Now, in an important step, we realize that the only way for this integral, which has absolutely arbitrary limits of integration, to always be zero, is for the integrand itself to always be zero. Hence, we have

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{39}$$

This is the very important differential form of the mass conservation principle.

We can get a very useful *control volume* formulation by integrating the mass conservation principle (39) over a *fixed* volume $V$:

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) \; dV = \int_V 0 \; dV. \tag{40}$$

Now the integral of 0 over a fixed domain must be zero. This is equivalent to saying $\int_a^b 0dx = 0$, where the area under the curve of 0 has to be zero. So we have

$$\int_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0. \tag{41}$$

Next apply the divergence theorem (6) to (41) to get

$$\int_V \frac{\partial \rho}{\partial t} \, dV + \int_A \rho \mathbf{v} \cdot \mathbf{n} \, dA = 0. \tag{42}$$

Applying now the result from (31) to (42), we see for the fixed volume that

$$\frac{d}{dt} \int_V \rho \, dV + \int_A \rho \mathbf{v} \cdot \mathbf{n} \, dA = 0. \tag{43}$$

We note now that at an inlet to a control volume that $\mathbf{v}$ points in an opposite direction to $\mathbf{n}$, so we have

$$\mathbf{v} \cdot \mathbf{n} < 0, \qquad \text{at inlets.} \tag{44}$$

At exits to a control volume $\mathbf{v}$ and $\mathbf{n}$ point in the same direction so that

$$\mathbf{v} \cdot \mathbf{n} > 0, \qquad \text{at exits.} \tag{45}$$

If now, we take the simplifying assumption that $\rho$ and $\mathbf{v}$ have no spatial variation across inlets and exits, we get for a control volume with one inlet and one exit that

$$\frac{d}{dt} \int_V \rho \, dV + \rho_e |\mathbf{v}_e| A_e - \rho_i |\mathbf{v}_i| A_i = 0. \tag{46}$$

Here the subcript $i$ denotes inlet, and the subscript $e$ denotes exit. Rearranging (46), we find

$$\frac{d}{dt} \int_V \rho \, dV = \rho_i |\mathbf{v}_i| A_i - \rho_e |\mathbf{v}_e| A_e. \tag{47}$$

We now define the mass in the control volume $m_{cv}$ as

$$m_{cv} = \int_V \rho \, dV. \tag{48}$$

Here (48) is equivalent to the equation on the top of p. 164 of SBVW. If we make the further simplifying assumption that $\rho$ does not vary within $V$, we find that

$$\underbrace{\frac{dm_{cv}}{dt}}_{\text{rate of change of mass}} = \underbrace{\rho_i |\mathbf{v}_i| A_i}_{\text{mass rate in}} - \underbrace{\rho_e |\mathbf{v}_e| A_e}_{\text{mass rate out}}. \tag{49}$$

Here $m_{cv}$ is the mass enclosed in the control volume. If there is no net rate of change of mass the control volume is in *steady state*, and we can say that the mass flow in must equal the mass flow out:

$$\rho_i |\mathbf{v}_i| A_i = \rho_e |\mathbf{v}_e| A_e. \tag{50}$$

We define the *mass flow rate* $\dot{m}$ as

$$\dot{m} = \rho|\mathbf{v}|A. \tag{51}$$

For steady flows with a single entrance and exit, we have

$$\dot{m} = \text{constant}. \tag{52}$$

For unsteady flows with a single entrance and exit, we can rewrite (49) as

$$\frac{dm_{cv}}{dt} = \dot{m}_i - \dot{m}_e. \tag{53}$$

For unsteady flow with many entrances and exits, we can generalize (49) as

$$\underbrace{\frac{dm_{cv}}{dt}}_{\text{rate of change of mass}} = \underbrace{\sum \rho_i|\mathbf{v}_i|A_i}_{\text{mass rate in}} - \underbrace{\sum \rho_e|\mathbf{v}_e|A_e}_{\text{mass rate out}}, \tag{54}$$

$$\underbrace{\frac{dm_{cv}}{dt}}_{\text{rate of change of mass}} = \underbrace{\sum \dot{m}_i}_{\text{mass rate in}} - \underbrace{\sum \dot{m}_e}_{\text{mass rate out}} \tag{55}$$

Note that (55) is fully equivalent to SBVW's Eq. (6.1), but that it actually takes a good deal of effort to get to this point with rigor! For steady state conditions with many entrances and exits we can say

$$\sum \rho_i|\mathbf{v}_i|A_i = \sum \rho_e|\mathbf{v}_e|A_e, \tag{56}$$

$$\sum \dot{m}_i = \sum \dot{m}_e. \tag{57}$$

Here (56) is the same as SBVW's (6.9).

## 2.2 Energy

For energy, we must consider the *total energy* which includes internal, kinetic, and potential. Our extensive property $B$ is thus

$$B = E = U + \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + mgz. \tag{58}$$

Here we have assumed the fluid resides in a gravitational potential field in which the gravitational potential energy varies linearly with height $z$. The corresponding intensive property $\beta$ is

$$\beta = e = u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz. \tag{59}$$

We recall the first law of thermodynamics, which states the the change of a material volume's total energy is equal to the heat transferred to the material volume less the work done by the material volume. Mathematically, this is stated as

$$dE = \delta Q - \delta W \tag{60}$$

We recall the total derivative is used for $dE$, since energy is a property and has an exact differential, while both heat transfer and work are not properties and do not have exact differentials. It is more convenient to express the first law as a rate equation, which we get by dividing (60) by $dt$ to get

$$\frac{dE}{dt} = \frac{\delta Q}{dt} - \frac{\delta W}{dt}. \tag{61}$$

Recall that the upper case letters denote extensive thermodynamic properties. For example, $E$ is total energy, inclusive of internal and kinetic and potential [11] , with SI units of $J$. Let us consider each term in the first law of thermodynamics in detail and then write the equation in final form.

### 2.2.1 Total energy term

For a fluid particle, the differential amount of total energy is

$$dE = \rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right) dV, \tag{62}$$

$$= \underbrace{\rho dV}_{\text{mass}} \underbrace{\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right)}_{\text{internal +kinetic +potential}} . \tag{63}$$

### 2.2.2 Work term

Let us partition the work into work $W_p$ done by a pressure force $\mathbf{F}_p$ and work done by other sources, which we shall call $W_{mv}$, where the subscript "$mv$" indicates "material volume."

$$W = W_p + W_{mv}. \tag{64}$$

Taking a time derivative, we get

$$\frac{\delta W}{dt} = \frac{\delta W_p}{dt} + \dot{W}_{mv}. \tag{65}$$

The work done by other sources is often called *shaft work* and represents inputs of such devices as compressors, pumps, and turbines. Its modeling is often not rigorous.

Recall that work is done when a force acts through a distance, and a work rate arises when a force acts through a distance at a particular rate in time (hence, a velocity is involved). Recall also that it is the dot product of the force vector with the position or velocity that gives the true work or work rate. In shorthand, we can say that the differential work done by the pressure force $\mathbf{F}_p$ is

$$\delta W_p = \mathbf{F}_p \cdot d\mathbf{x}, \tag{66}$$

$$\frac{\delta W_p}{dt} = \mathbf{F}_p \cdot \frac{d\mathbf{x}}{dt} = \mathbf{F}_p \cdot \mathbf{v}. \tag{67}$$

[11] Strictly speaking our derivation will only be valid for potentials which are time-independent. This is the case for ordinary gravitational potentials. The modifications for time-dependent potentials are straightforward, but require a more nuanced interpretation than space permits here.

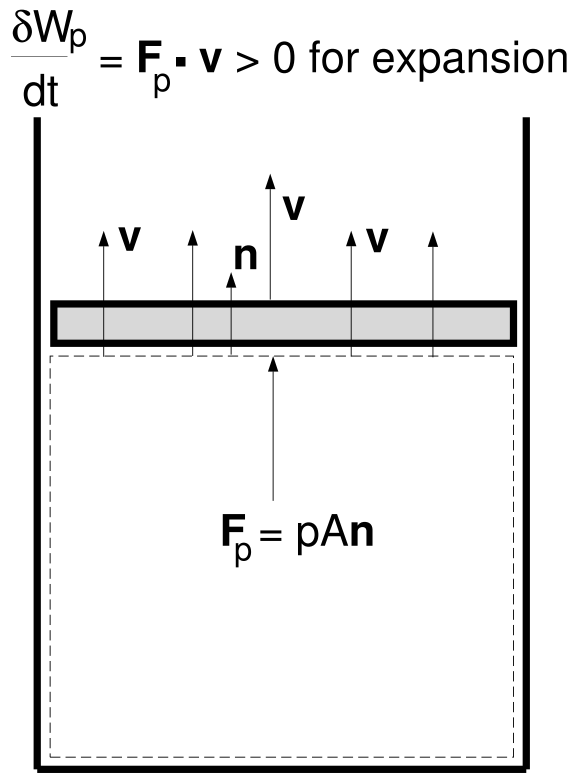


Figure 3: Sketch of fluid element doing work.

Here $W$ has the SI units of $J$, and $\mathbf{F}_p$ has the SI units of $N$. Now let us consider the work done by the pressure force. In a piston-cylinder arrangement in which a fluid exists with pressure $p$ within the cylinder and the piston is rising with velocity $\mathbf{v}$, the work rate done by the fluid is positive. We can think of the local stress vector in the fluid as pointing in the same direction as the fluid is moving at the piston surface, so that the dot product is positive. Now we can express the pressure force in terms of the pressure by

$$\mathbf{F}_p = pA\mathbf{n}. \tag{68}$$

Substituting (67) into (68), we get

$$\frac{\delta W_p}{dt} = pA\mathbf{n} \cdot \mathbf{v}. \tag{69}$$

It is noted that we have been a little loose distinguishing local areas from global areas. Better stated, we should say for a material volume that

$$\frac{\delta W_p}{dt} = \int_{A_m(t)} p\mathbf{n} \cdot \mathbf{v} \; dA. \tag{70}$$

This form allows for $p$ and $\mathbf{v}$ to vary with location.

This is summarized in the sketch of Figure 3.

### 2.2.3 Heat transfer term

If we were considering temperature fields with spatial dependency, we would define a heat flux vector. This approach is absolutely necessary to describe many real-world devices, and is the focus of a standard undergraduate course in heat transfer. Here we will take a very simplified assumption that the only heat fluxes are easily specified and are all absorbed into a lumped scalar term we will call $\dot{Q}_{mv}$. This term has units of $J/s = W$ in SI. So we have then

$$\frac{\delta Q}{dt} = \dot{Q}_{mv}. \tag{71}$$

### 2.2.4 The first law of thermodynamics

Putting the words of the first law into equation form, we get

$$\frac{d}{dt}\int_{V_m(t)} \rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right) dV = \frac{\delta Q}{dt} - \frac{\delta W}{dt}. \tag{72}$$

We next introduce our simplification of heat transfer (71) and partition of work (65) along with (70) into (72) to get

$$\frac{d}{dt}\int_{V_m(t)} \rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right) dV = \dot{Q}_{mv} - \left(\dot{W}_{mv} + \int_{A_m(t)} p\mathbf{n}\cdot\mathbf{v}\ dA\right). \tag{73}$$

Now we bring the pressure work integral to the right side of (73) to get

$$\frac{d}{dt}\int_{V_m(t)} \rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right) dV + \int_{A_m(t)} p\mathbf{n}\cdot\mathbf{v}\ dA = \dot{Q}_{mv} - \dot{W}_{mv}. \tag{74}$$

We next invoke the Reynolds transport theorem (27) into (74) to expand the derivative of the first integral so as to obtain

$$\int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right)\right) dV + \int_{A_m(t)} \left(\rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right)\right)\mathbf{v}\cdot\mathbf{n}\ dA$$
$$+ \int_{A_m(t)} p\mathbf{n}\cdot\mathbf{v}\ dA = \dot{Q}_{mv} - \dot{W}_{mv}. \tag{75}$$

We next note that the two area integrals have the same limits and can be combined to form

$$\int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right)\right) dV + \int_{A_m(t)} \left(\rho\left(u + \frac{p}{\rho} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right)\right)\mathbf{v}\cdot\mathbf{n}\ dA$$
$$= \dot{Q}_{mv} - \dot{W}_{mv}. \tag{76}$$

We recall now the definition of enthalpy $h$,

$$h = u + \frac{p}{\rho} \tag{77}$$

Invoking (77) into (76), we get

$$\int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho\left(u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right)\right) dV + \int_{A_m(t)} \left(\rho\left(h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz\right)\right)\mathbf{v}\cdot\mathbf{n}\ dA$$
$$= \dot{Q}_{mv} - \dot{W}_{mv}. \tag{78}$$

Next use the divergence theorem (6) to rewrite (78) as

$$\int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho\left(u+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right)dV + \int_{V_m(t)} \nabla\cdot\left(\rho\mathbf{v}\left(h+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right)\,dV = \dot{Q}_{mv} - \dot{W}_{mv}. \quad (79)$$

Now, for convenience, let us define the specific control volume heat transfer and work $q_{mv}$ and $w_{mv}$, each with SI units $J/kg$ such that

$$\dot{Q}_{mv} = \int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho q_{mv}\right)\,dV, \quad (80)$$

$$\dot{W}_{mv} = \int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho w_{mv}\right)\,dV, \quad (81)$$

so that by substituting (80) and (81) into (79), we get

$$\int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho\left(u+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right)dV + \int_{V_m(t)} \nabla\cdot\left(\rho\mathbf{v}\left(h+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right)\,dV = \int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho q_{mv}\right)\,dV - \int_{V_m(t)} \frac{\partial}{\partial t}\left(\rho w_{mv}\right)\,dV. \quad (82)$$

Now all terms in (82) have the same limits of integration, so they can be grouped to form

$$\int_{V_m(t)} \left(\frac{\partial}{\partial t}\left(\rho\left(u+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right) + \nabla\cdot\left(\rho\mathbf{v}\left(h+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right) - \frac{\partial}{\partial t}\left(\rho q_{mv}\right) + \frac{\partial}{\partial t}\left(\rho w_{mv}\right)\right)\,dV = 0. \quad (83)$$

As with the mass equation, since the integral is zero, in general we must expect the integrand to be zero, giving us

$$\frac{\partial}{\partial t}\left(\rho\left(u+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right) + \nabla\cdot\left(\rho\mathbf{v}\left(h+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right) - \frac{\partial}{\partial t}\left(\rho q_{mv}\right) + \frac{\partial}{\partial t}\left(\rho w_{mv}\right) = 0. \quad (84)$$

To get the standard control volume form of the equation, we then integrate (84) over a *fixed* control volume $V$ to get

$$\int_{V} \left(\frac{\partial}{\partial t}\left(\rho\left(u+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right) + \nabla\cdot\left(\rho\mathbf{v}\left(h+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+gz\right)\right) - \frac{\partial}{\partial t}\left(\rho q_{mv}\right) + \frac{\partial}{\partial t}\left(\rho w_{mv}\right)\right)\,dV = 0. \quad (85)$$

Now defining the control volume heat transfer rate and work rate, $\dot{Q}_{cv}$ and $\dot{W}_{cv}$,

$$\dot{Q}_{cv} = \int_{V} \frac{\partial}{\partial t}\left(\rho q_{mv}\right)\,dV, \quad (86)$$

$$\dot{W}_{cv} = \int_{V} \frac{\partial}{\partial t}\left(\rho w_{mv}\right)\,dV, \quad (87)$$

we employ (86) and (87) in (85) to get

$$\int_V \left( \frac{\partial}{\partial t} \left( \rho \left( u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz \right) \right) + \nabla \cdot \left( \rho \mathbf{v} \left( h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz \right) \right) \right) \, dV = \dot{Q}_{cv} - \dot{W}_{cv}. \tag{88}$$

Applying the divergence theorem (6) to (88) to convert a portion of the volume integral into an area integral, and (31) to bring the time derivative outside the integral for the fixed volume, we get

$$\frac{d}{dt}\int_V \rho \left( u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz \right) \, dV + \int_A \rho \mathbf{v}\cdot\mathbf{n} \left( h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz \right) \, dA = \dot{Q}_{cv} - \dot{W}_{cv}. \tag{89}$$

We now define the total energy in the control volume as

$$E_{cv} = \int_V \rho \left( u + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz \right) \, dV. \tag{90}$$

Next assume that all properties across entrances and exits are uniform so that the area integral in (88) reduces to

$$\int_A \rho \mathbf{v}\cdot\mathbf{n} \left( h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz \right) \, dA = \sum \dot{m}_e \left( h_e + \frac{1}{2}\mathbf{v}_e\cdot\mathbf{v}_e + gz_e \right) - \sum \dot{m}_i \left( h_i + \frac{1}{2}\mathbf{v}_i\cdot\mathbf{v}_i + gz_i \right). \tag{91}$$

Substituting (90) and (91) into (89), we get

$$\frac{dE_{cv}}{dt} + \sum \dot{m}_e \left( h_e + \frac{1}{2}\mathbf{v}_e\cdot\mathbf{v}_e + gz_e \right) - \sum \dot{m}_i \left( h_i + \frac{1}{2}\mathbf{v}_i\cdot\mathbf{v}_i + gz_i \right) = \dot{Q}_{cv} - \dot{W}_{cv}. \tag{92}$$

Rearranging (92), we get

$$\underbrace{\frac{dE_{cv}}{dt}}_{\text{rate of CV energy change}} = \underbrace{\dot{Q}_{cv}}_{\text{CV heat transfer rate}} - \underbrace{\dot{W}_{cv}}_{\text{CV shaft work rate}} + \underbrace{\sum \dot{m}_i \left( h_i + \frac{1}{2}\mathbf{v}_i\cdot\mathbf{v}_i + gz_i \right)}_{\text{total enthalpy rate in}} - \underbrace{\sum \dot{m}_e \left( h_e + \frac{1}{2}\mathbf{v}_e\cdot\mathbf{v}_e + gz_e \right)}_{\text{total enthalpy rate out}}. \tag{93}$$

Here (93) is equivalent to SBVW's Eq. (6.7). Note that the so-called *total enthalpy* is often defined as

$$h_{tot} = h + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + gz. \tag{94}$$

Employing (94) in (93), we find

$$\frac{dE_{cv}}{dt} = \dot{Q}_{cv} - \dot{W}_{cv} + \sum \dot{m}_i h_{tot,i} - \sum \dot{m}_e h_{tot,e}. \tag{95}$$

Here (95) is equivalent to SBVW's (6.8).

If there is a single entrance and exit, we lose the summation, so that (93) becomes

$$\frac{dE_{cv}}{dt} = \dot{Q}_{cv} - \dot{W}_{cv} + \dot{m}_i \left( h_i + \frac{1}{2}\mathbf{v}_i \cdot \mathbf{v}_i + g z_i \right) - \dot{m}_e \left( h_e + \frac{1}{2}\mathbf{v}_e \cdot \mathbf{v}_e + g z_e \right). \tag{96}$$

If the flow is steady, we have $\frac{dE_{cv}}{dt} = 0$ and $\dot{m}_i = \dot{m}_e = \dot{m}$, so the first law with a single entrance and exit becomes

$$0 = \dot{Q}_{cv} - \dot{W}_{cv} + \dot{m} \left( h_i - h_e + \frac{1}{2}(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_e \cdot \mathbf{v}_e) + g(z_i - z_e) \right). \tag{97}$$

Defining the specific control volume heat transfer and work as

$$q = \frac{\dot{Q}_{cv}}{\dot{m}}, \qquad w = \frac{\dot{W}_{cv}}{\dot{m}}, \tag{98}$$

and substituting (98) into (97), we get

$$0 = q - w + h_i - h_e + \frac{1}{2}(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_e \cdot \mathbf{v}_e) + g(z_i - z_e), \tag{99}$$

Now (99) can be rearranged to form SBVW's (6.14):

$$q + h_i + \frac{1}{2}\mathbf{v}_i \cdot \mathbf{v}_i + g z_i = w + h_e + \frac{1}{2}\mathbf{v}_e \cdot \mathbf{v}_e + g z_e. \tag{100}$$

If the flow is adiabatic, steady, has one entrance and one exit, and there is no shaft work, we find that the total enthalpy must remain constant:

$$h_i + \frac{1}{2}\mathbf{v}_i \cdot \mathbf{v}_i + g z_i = h_e + \frac{1}{2}\mathbf{v}_e \cdot \mathbf{v}_e + g z_e \tag{101}$$