1. (10) Calorically perfect ideal air at 300 K, 100 kPa, 1000 m/s, is brought to rest isentropically. Determine its final temperature.

Solution
We have
\[ h_1 + \frac{v_1^2}{2} = h_0 + \frac{v_0^2}{2}. \]
But at the rest state \( v_0 = 0 \), so
\[ h_0 - h_1 = \frac{v_1^2}{2}. \]
\[ c_p(T_0 - T_1) = \frac{v_1^2}{2} \]
\[ T_0 = T_1 + \frac{v_1^2}{2c_p}. \]
For air \( c_p = 1004.5 \text{ J/kg/K} \), so
\[ T_0 = (300 \text{ K}) + \frac{(1000 \text{ m/s})^2}{(1004.5 \text{ J/kg/K})(2)}. \]
\[ T_0 = 797.76 \text{ K}. \]

2. (50) A small utility gasoline engine of 250 cm\(^3\) runs at 1500 RPM with a compression ratio of 8:1. The inlet state is 75 kPa, 17\(^\circ\)C, and the combustion adds 1500 kJ/kg to the charge. This engine runs a heat pump using R-410a with a high pressure of 4 MPa and an evaporator operating at 0\(^\circ\)C. Find the rate of heating the heat pump can deliver. Assume the gasoline engine is modeled well by an ideal Otto cycle with calorically imperfect air as the working fluid.

Solution
I'm going to be a little sloppy about interpolation in this problem. You can fix it if you like! We'll be close.
For the gasoline engine, we have
\[ P_{1G} = 75 \text{ kPa}, \]
\[ T_{1G} = 17 + 273 = 290 \text{ K}. \]

We can also take \( V_{1G} = 250 \text{ cm}^3 = 0.00025 \text{ m}^3 \). Now for the isentropic compression from 1 to 2, we have

\[ s_{1G} = s_{2G}. \]

Now

\[ s_{2G} - s_{1G} = 0 = s_{T2G}^0 - s_{T1G}^0 - R \ln \frac{P_2}{P_1}, \]

\[ s_{2G} - s_{1G} = 0 = s_{T2G}^0 - s_{T1G}^0 - R \ln \frac{T_2V_1}{T_1V_2}. \]

\[ 0 = s_{T2G}^0 - \left( 6.83521 \frac{kJ}{kg \text{ K}} \right) - \left( 0.287 \frac{kJ}{kg \text{ K}} \right) \ln \left( \frac{s_{T2}}{290 \text{ K}} \right). \]

We iterate a few times on \( T_2 \). \( T_2 = 660 \text{ K} \) is pretty close. When \( T_2 = 660 \text{ K} \), \( u_2 = 481.34 \text{ kJ/kg} \). Then

\[ u_{3G} = u_{2G} + 2q_3 = 481.34 + 1500 = 1981 \frac{kJ}{kg}. \]

At this internal energy, we’re pretty close to \( T_{3G} = 2300 \text{ K} \). Let’s say we are at this point, so \( u_{3G} = 1968.79 \text{ kJ/kg} \). Thus we really added 1487.45 \text{ kJ/kg} \). Now \( s_{T3G}^0 = 9.14189 \text{ kJ/kg/K} \). Now

\[ s_{4G} - s_{3G} = 0 = s_{T4G}^0 - s_{T3G}^0 - R \ln \frac{P_4}{P_3}, \]

\[ s_{4G} - s_{3G} = 0 = s_{T4G}^0 - s_{T3G}^0 - R \ln \frac{T_4V_2}{T_3V_1}. \]

\[ 0 = s_{T4G}^0 - \left( 9.14189 \frac{kJ}{kg \text{ K}} \right) - \left( 0.287 \frac{kJ}{kg \text{ K}} \right) \ln \left( \frac{1}{182300 \text{ K}} \right). \]

We iterate and find that \( T_{4G} \sim 1150 \text{ K} \). At this temperature \( u_{4G} = 889.21 \text{ kJ/kg} \).

Now

\[ w_G = (u_{3G} - u_{4G}) - (u_{2G} - u_{1G}) = 1968.79 - 889.21 - 481.34 + 207.19 = 805.43 \frac{kJ}{kg}. \]

Now

\[ m = \frac{P_{3G}V_{1G}}{RT_{1G}} = \frac{(75 \text{ kPa})(0.00025 \text{ m}^3)}{(0.287 \text{ kJ/kg/K})(290 \text{ K})} = 0.000225279 \text{ kg}. \]

The power is

\[ \dot{W} = mw_G(RPM)/60 = (0.000225279 \text{ kg})(805.43 \text{ kJ/kg})(1500)/60 = 4.54 \text{ kW}. \]

Now for the refrigerator, we have at the compressor entrance \( x_1 = 1 \) and \( T_1 = 0^\circ \text{C} \).

The saturation tables for R-410a give \( P_1 = 798.7 \text{ kPa} \), \( h_1 = 279.12 \text{ kJ/kg} \) and \( s_1 = 1.0368 \text{ kJ/kg/K} \). We have \( s_2 = s_1 = 1.0368 \text{ kJ/kg/K} \). We have \( P_2 = 4000 \text{ kPa} \).

Interpolating the superheated tables gives

\[ h_2 = 323.91 \frac{kJ}{kg}. \]

We then take \( x_3 = 0 \) with \( P_3 = P_2 = 4000 \text{ kPa} \). We interpolated the saturation tables to find

\[ h_3 = 171.786 \frac{kJ}{kg}. \]
The coefficient of performance for the heat pump is
\[
\beta' = \frac{\dot{q}_H}{\dot{w}_c} = \frac{h_2 - h_3}{h_2 - h_1} = \frac{323.91 - 171.786}{323.91 - 279.12} = 3.396.
\]

So if we put in 4.54 KW of work we can acquire
\[
\dot{Q}_h = \beta' \dot{W} = 3.396(4.54 kW) = 16.2 kW.
\]

3. (40) A fixed mass \( m \) of calorically perfect ideal gas with ratio of specific heats \( k = 7/5 \) undergoes a thermodynamic cycle. At the beginning of the cycle, the gas is known to have pressure \( P_1 \) and volume \( V_1 \). The cycle is composed of the following steps:

- 1 → 2: Polytropic compression to \( V_2 = V_1/5 \) via the path \( PV^2 = P_1V_1^2 \).
- 2 → 3: Isobaric heat addition to \( V_3 = 2V_2 \)
- 3 → 4: Isothermal expansion to \( V_4 = V_1 \)
- 4 → 1: Isochoric return to state 1.

(a) Sketch the process on \( P-V \) and \( T-s \) diagrams.
(b) Find the net work, net heat transfer, and thermal efficiency.

**Solution**

First from the ideal gas law, we have
\[
T_1 = \frac{P_1V_1}{mR}.
\]

For the polytropic compression we have
\[
P_2V_2^2 = P_1V_1^2.
\]

Since \( V_2 = V_1/5 \), we have
\[
P_2\left(\frac{V_1}{5}\right)^2 = P_1V_1^2.
\]

\[
P_2 = 25V_1.
\]

From the ideal gas law, we get
\[
T_2 = \frac{P_2V_2}{mR} = \frac{(25P_1)(V_1/5)}{mR} = \frac{5P_1V_1}{mR}.
\]

Now let’s get the work going from 1 to 2.
\[
\int_1^2 PdV.
\]
\[ W_2 = \int_1^2 \frac{P_1 V_1^2}{V^2} dV. \]

\[ W_2 = P_1 V_1^2 \left( \frac{1}{V_1} - \frac{1}{V_2} \right). \]

\[ W_2 = P_1 V_1^2 \left( \frac{1}{V_1} - \frac{5}{V_1} \right). \]

\[ W_2 = -4P_1 V_1. \]

Now let’s get the heat transfer going from 1 to 2. The first law tells us \( U_2 - U_1 = Q_2 - W_2 \), so

\[ Q_2 = U_2 - U_1 + W_2. \]

\[ Q_2 = mc_v(T_2 - T_1) + W_2. \]

\[ Q_2 = mc_v \left( \frac{5P_1 V_1}{mR} - \frac{P_1 V_1}{mR} \right) - 4P_1 V_1. \]

\[ Q_2 = c\frac{v}{R} (4P_1 V_1) - 4P_1 V_1. \]

\[ Q_2 = 4P_1 V_1 \left( \frac{c\frac{v}{R} - 1}{1} \right). \]

Now \( c_v/R = c_v/(c_p - c_v) = 1/(c_p/c_v - 1) = 1/(k - 1) \), so

\[ Q_2 = 4P_1 V_1 \left( \frac{1}{k - 1} - 1 \right). \]

\[ Q_2 = 4P_1 V_1 \left( \frac{2 - k}{k - 1} \right). \]

\[ Q_2 = 4P_1 V_1 \left( \frac{2 - 7/5}{7/5 - 1} \right) \]

\[ Q_2 = 6P_1 V_1. \]

Now \( P_3 = P_2 = 25P_1 \). And \( V_3 = 2V_2 = 2V_1/5 \). So

\[ T_3 = \frac{P_3 V_3}{mR} = \frac{25P_1 (2V_1/5)}{mR} = \frac{10P_1 V_1}{mR}. \]

\[ W_3 = \int_2^3 PdV = P_2 \int_2^3 dV = P_2 (V_3 - V_2) = 25P_1 \left( \frac{2}{5} V_1 - \frac{1}{3} V_1 \right) = 5P_1 V_1. \]

For the heat transfer, we have

\[ Q_3 = U_3 - U_2 + W_3. \]

\[ Q_3 = mc_v(T_3 - T_2) + W_3. \]

\[ Q_3 = mc_v \left( \frac{10P_1 V_1}{mR} - \frac{5P_1 V_1}{mR} \right) + 5P_1 V_1. \]

\[ Q_3 = \frac{c_v}{R} (5P_1 V_1) + 5P_1 V_1. \]

\[ Q_3 = 5P_1 V_1 \left( \frac{1}{k - 1} + 1 \right). \]

\[ Q_3 = 5P_1 V_1 \left( \frac{1}{7/5 - 1} + 1 \right). \]
\[ zQ_4 = \frac{35}{2} P_1 V_1. \]

Now for 3 to 4, we have \( T_4 = T_3 = 10P_1 V_1/(mR) \) and \( V_4 = V_1 \). We get

\[ P_4 = \frac{mRT_4}{V_4} = \frac{mR(10P_1 V_1)/(mR)}{V_1} = 10P_1. \]

The work is

\[ 3W_4 = \int_3^4 P dV = \int_3^4 \frac{mRT_3}{V} dV = mRT_3 \ln \frac{V_4}{V_3} = mR \left( \frac{10P_1 V_1}{mR} \right) \ln \frac{V_1}{V_3} = 10P_1 V_1 \ln \frac{5}{2}. \]

The heat transfer is

\[ 3Q_4 = U_4 - U_3 + 3W_4 = 10P_1 V_1 \ln \frac{5}{2} \]

For 4 to 1, we have

\[ 4W_1 = \int_1^4 P dV = 0. \]

Therefore

\[ 4Q_1 = U_1 - U_4 = mc_v(T_1 - T_4) = mc_v \left( \frac{P_1 V_1}{mR} - \frac{10P_1 V_1}{mR} \right) = C_v \left( -9P_1 V_1 \right) = \frac{-9}{k-1} P_1 V_1. \]

The net work is

\[ W_{net} = W_2 + 2W_3 + 3W_4 + 4W_1 = -4P_1 V_1 + 5P_1 V_1 + 10P_1 V_1 \ln \frac{5}{2} + 0. \]

\[ W_{net} = P_1 V_1 \left( 1 + 10 \ln \frac{5}{2} \right). \]

The heat transfer that we pay for is

\[ Q_{in} = 1Q_2 + 2Q_3 + 3Q_4, \]

\[ Q_{in} = 6P_1 V_1 + (35/2)P_1 V_1 + 10P_1 V_1 \ln \frac{5}{2}. \]

\[ Q_{in} = P_1 V_1 \left( 47/2 + 10 \ln \frac{5}{2} \right). \]

So the thermal efficiency is

\[ \eta = \frac{W_{net}}{Q_{in}}. \]

\[ \eta = \frac{1 + 10 \ln \frac{5}{2}}{\frac{47}{2} + 10 \ln \frac{5}{2}} = 0.31145. \]

Appropriate diagrams are sketched in Fig. 1.

The shapes of the \( P - V \) diagram is relatively obvious. The \( T - s \) diagram is less so. We have shown in class that the slope of an isobar is

\[ \frac{\partial T}{\partial s} \bigg|_P = \frac{T}{C_p} = \frac{T}{k c_v}. \]
and the slope of an isochore is
\[
\frac{\partial T}{\partial s} \bigg|_{v} = \frac{T}{c_v}.
\]
Since \( k > 1 \), the slope of the isochore is greater than the isobar. Now consider the slope for a general polytropic process for which \( P v^n = C \) Differentiate the polytropic equation:

\[
nv^{n-1}Pdv + v^n dP = 0.
\]

\[
nPdv + vdP = 0.
\]

Now differentiate the ideal gas law \( P v = RT \):

\[
Pdv + vdP = RdT.
\]

Eliminate \( vdP \):

\[
Pdv - nPdv = RdT.
\]

\[
(1-n)Pdv = RdT.
\]

\[
Pdv = \frac{R}{1-n}dT.
\]

Now consider the Gibbs law:

\[
Tds = du + Pdv.
\]

Eliminate \( du = c_v dT \) and \( Pdv = (R/(1-n))dT \):

\[
Tds = c_v dT + \frac{R}{1-n}dT.
\]

\[
Tds = \left( c_v + \frac{c_P - c_v}{1-n} \right) dT.
\]

\[
Tds = \left( \frac{c_v - nc_v + c_P - c_v}{1-n} \right) dT.
\]

\[
Tds = \left( \frac{c_P - nc_v}{1-n} \right) dT.
\]
\[ T_{ds} = \left( \frac{k - n}{1 - n} \right) \frac{dT}{c_v} \]

\[ \frac{\partial T}{\partial s} \bigg|_{polytropic} = \frac{1 - n}{k - n} \frac{T}{c_v} \]

So when \( n = 2 \) and \( k = 7/5 \), we get

\[ \frac{\partial T}{\partial s} \bigg|_{n=2} = \frac{5}{3} \frac{T}{c_v} \]

This is steeper than the isochore, which has

\[ \frac{\partial T}{\partial s} \bigg|_{v} = \frac{T}{c_v} \]

When \( k = 7/5 \), it is also steeper than the isobar which has

\[ \frac{\partial T}{\partial s} \bigg|_{p} = \frac{5}{7} \frac{T}{c_v} \].