AME 561 - Examination 1 - Fall 2002

1. Extremize $I=\int_0^1 (y')^2\,dx$ w/ $y(0)=0$ $y(1)=1$, $\int_0^1 y\,dx=1$

equivalent to $I=\int_0^1 (y')^2dx-\lambda\left[\int_0^1 y\,dx-1\right]=\int_0^1 (y')^2dx-\lambda\int_0^1 (y-1)\,dx$

$I=\int_0^1\left[y'^2-\lambda y-\lambda\right]dx=0$ $\quad F(y',y,x)=y'^2-\lambda y-\lambda \Rightarrow \frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$

$\rightarrow -\lambda-\frac{d}{dx}(2y')=0 \rightarrow \lambda+2\frac{d^2y}{dx^2}=0$; $y(0)=0$; $y(1)=1 \rightarrow y=-\frac{\lambda}{2}x^2+C_1x+C_2$

$y(0)=0 \Rightarrow C_2=0$ $\quad y(1)=1$ $\quad 1=-\frac{\lambda}{2}+C_1$ $\quad C_1=1+\lambda/2$ $\quad y=-\frac{\lambda}{2}x^2+(1+\lambda/2)\,x$

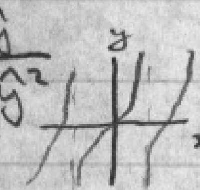
$\int_0^1 y\,dx=1 \Rightarrow \int_0^1 -\frac{\lambda}{2}x^2+(1+\lambda/2)x\,dx=1 \Rightarrow \left[-\frac{\lambda}{6}x^3+\frac{1}{2}\left(1+\frac{\lambda}{2}\right)x^2\right]_0^1=1$

$-\frac{\lambda}{6}+\frac{1}{2}\left(1+\frac{\lambda}{2}\right)=1 \Rightarrow \lambda=6 \Rightarrow \boxed{y=4x-3x^2}$; can be shown to minimize I

2. $\frac{d^2y}{dx^2}-2y\frac{dy}{dx}=0$ $\quad \left.\frac{dy}{dx}\right|_0=3$, $y(0)=0$; $u=\frac{dy}{dx}$ $\quad \frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx}=u\frac{du}{dy} \Rightarrow u\frac{du}{dy}-2yu=0$

$u\left(\frac{du}{dy}-2y\right)=0$ a. $u=0 \Rightarrow \frac{dy}{dx}=0$ Does not satisfy B.C. ∴ $u\neq 0$ ∴ $du/dy=2y$ & $u=y^2+c$

B.C. $3=0^2+c \Rightarrow c=3$ ∴ $dy/dx=y^2+3$ $\quad dy/(y^2+3)=x$ & $x+\hat{C}=\int_0^y \frac{d\hat{y}}{3+\hat{y}^2}$



$x=0$ $y=0 \Rightarrow C=0$ & $x=\int_0^y \frac{d\hat{y}}{\hat{y}^2+3}$ $\quad \hat{y}=\sqrt{3}\tan\theta$; $d\hat{y}=\sqrt{3}\sec^2\theta\,d\theta$

$x=\int_0^{\tan^{-1}(y/\sqrt{3})}\frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta+3}\,d\theta=\int_0^{\tan^{-1}(y/\sqrt{3})}\frac{1}{\sqrt{3}}\,d\theta=\frac{1}{\sqrt{3}}\tan^{-1}(y/\sqrt{3}) \Rightarrow \boxed{y=\sqrt{3}\tan(\sqrt{3}x)}$

3. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=\frac{e^x}{x}$; $r^2-2r+1=0$ $\quad (r-1)^2=0$ $\quad r=1, r=1$ $\quad y_H=Ae^x+Bxe^x$

variation of parameters for $y_P \Rightarrow y_1=e^x$, $y_2=xe^x$; $u_1'y_1+u_2'y_2=0$; $u_1'y_1'+u_2'y_2'=e^x/x$

$\Rightarrow u_1'e^x+u_2'xe^x=0$; $u_1'e^x+u_2'(xe^x+e^x)=e^x/x$ $\quad \begin{bmatrix} e^x & xe^x \\ e^x & (xe^x+e^x)\end{bmatrix}\begin{bmatrix}u_1' \\ u_2'\end{bmatrix}=\begin{bmatrix}0 \\ e^x/x\end{bmatrix}$

$u_1'=-1$; $u_2'=1/x \Rightarrow u_1=-x$, $u_2=\ln x$

$y_p=u_1y_1+u_2y_2=-xe^x+x\ln x\,e^x$ $\quad y=Ae^x+Bxe^x-xe^x+x\ln x\,e^x$

$\Rightarrow \boxed{y=e^x\left[A+x(B-1+\ln x)\right]}$

4. $\varepsilon\,d^2y/dx^2+dy/dx+x/y=0$ $\quad y(0)=1$, $y(1)=1$ $\quad$ outer: $dy/dx+x/y=0 \Rightarrow y\,dy+x\,dx=0$

$y^2/2+x^2/2=C$, try $y(1)=1$ $\quad 1/2+1/2=C$ $\quad C=1 \Rightarrow x^2+y^2=2$ & $y_{outer}=\pm\sqrt{2-x^2}$

$\tilde{x}=x/\varepsilon$ $\quad 1/\varepsilon\, d^2y/d\tilde{x}^2+1/\varepsilon\, dy/d\tilde{x}+\varepsilon\tilde{x}/y \Rightarrow d^2y/d\tilde{x}^2+dy/d\tilde{x}+\varepsilon^2\tilde{x}/y=0$

$d^2y/d\tilde{x}^2+dy/d\tilde{x}=0$ $\quad y=A+Be^{-\tilde{x}}$ $\quad y(0)=1 \rightarrow 1=A+B$ $\quad A=1-B$

$y_{inner}=1-B+Be^{-\tilde{x}}$ $\quad \lim_{\tilde{x}\to\infty} y_{inner}=1-B$ $\quad \lim_{x\to 0} y_{outer}=\pm\sqrt{2}$

$1-B=\pm\sqrt{2} \Rightarrow B=1\mp\sqrt{2}$ $\quad y_{comp}=1+(1\mp\sqrt{2})(e^{-\tilde{x}}-1)\pm\sqrt{2-x^2}\mp\sqrt{2}$

$y_{comp}=(1\mp\sqrt{2})e^{-\tilde{x}}\pm\sqrt{2-x^2} \Rightarrow \boxed{y\sim(1\mp\sqrt{2})\,e^{-x/\varepsilon}\pm\sqrt{2-x^2}}$