AME 561
Examination 2: **Solution**
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1. (15) Find the curvature of the parabola $y = x^2$ at the point $x = 1$.

Solution

Most people got this problem correct. $\kappa = \left.\frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}\right|_{x=1}$. So $\kappa = \frac{2}{(1+(2x)^2)^{3/2}} = \frac{2}{(1+(2(1))^2)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2\sqrt{5}}{25}$.

2. (15) Find the matrix $\mathbf{A}$ that operates on any vector in the $x-y$ plane so as to turn it through a counterclockwise angle θ about the z-axis without changing its length.

Solution

Most people got this one as well, but there was some confusion. There are a variety of ways one could approach this problem, and some are lengthy, albeit very general. Here we take an intuitive, non-rigorous approach, which can easily be checked to see that it does in fact work. Consider rotation of the unit vector in the x direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ about the z axis by an angle θ. This will yield the vector $\mathbf{c}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. Rotation of the unit vector in the y direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ through the same angle will yield $\mathbf{c}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. Combining these two operations gives us the necessary matrix

$$\mathbf{A} = \left(\mathbf{c}_1 \quad \mathbf{c}_2 \right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This matrix is obviously orthogonal since $||\mathbf{c}_1||_2 = ||\mathbf{c}_2||_2 = 1$, and $\mathbf{c}_1 \cdot \mathbf{c}_2 = 0$. Since $\overline{\mathbf{A}}^T \cdot \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, and because eigenvalues of $\mathbf{I}$ are $\lambda = 1, 1$, the spectral norm of $\mathbf{A}$ is 1, and therefore $||\mathbf{A} \cdot \mathbf{x}||_2 = ||\mathbf{x}||_2$, that is the length of the vector is not changed by matrix multiplication in this case.

3. (20) Given $x \in \mathbb{R}^1, f : \mathbb{R}^1 \to \mathbb{R}^1$,

$$f(x) = \frac{1}{x}, \qquad x \in [1,3],$$

find the first term in a Fourier-Laguerre expansion of $f(x)$. The set of orthonormal functions which arise from the Laguerre equation are

$$\varphi_n(s) = \left\{ e^{-s/2}, e^{-s/2}(1-s), \ldots, e^{-s/2}L_n(s) \right\}.$$

It is acceptable to express your answer in terms of a definite integral.

Solution

There was a good deal of confusion on this problem. A large fraction of people did not correctly transform the domain for the Laguerre polynomials, nor were they confident with the details of the expansion technique. The first job is to find any transformation which will take $x \in [1,3]$ into a transformed space $\hat{x} \in [0,\infty)$. Many transformations will work; one which certainly does is

$$\hat{x} = \frac{1-x}{x-3}.$$

Others are tempting but wrong, and one should check to see if the transformation chosen maps $x \in [1,3]$ into *positive* values of $\hat{x}$. One should also check to see if there are any interior singularities in the transformation selected. With the above transformation, we get an inverse transformation of

$$x = \frac{1+3\hat{x}}{1+\hat{x}}.$$

Now, our formula for a one-term Fourier-Laguerre coefficient is

$$A_0 = \int_0^\infty f(\hat{x})\phi_0(\hat{x}) \; d\hat{x}.$$

Now $\frac{1}{x} = \frac{1+\hat{x}}{1+3\hat{x}}$, so substituting this as well as the expression for φ_0 gives

$$A_0 = \int_0^\infty \frac{1+\hat{x}}{1+3\hat{x}} e^{-\hat{x}/2} \; d\hat{x}.$$

This integral is difficult. Mathematica shows that it evaluates to

$$A_0 = -\frac{2}{9}\left(-3 + e^{1/6}\mathrm{Ei}\left(-\frac{1}{6}\right)\right) = 1.02751.$$

So the final solution is $\frac{1}{x} \sim 1.02751 e^{-\hat{x}/2}$, or in terms of x,

$$\frac{1}{x} \sim 1.02751 \exp\left(\frac{x-1}{2x-6}\right).$$

4. (20) For $x \in [0,1] \in \mathbb{R}^1, y \in \mathbb{L}_2[0,1]$, consider

$$\frac{d^2y}{dx^2} + 8\sqrt{y} = x, \qquad y(0) = 0, \qquad y(1) = 0.$$

Use a one term collocation method to find an approximate solution.

Solution

Most people did acceptable work on this one. There were a few conceptual difficulties and some algebra blunders. Many choices were possible for trial functions. One that obviously works in that it satisfies the boundary conditions is

$$\phi = x(x-1).$$

So take

$$y_a = cx(x-1),$$

and find c to minimize the weighted error in a collocation technique. The error is

$$e(x) = \frac{d^2y_a}{dx^2} + 8\sqrt{y_a} - x,$$

so

$$e(x) = 2c + 8\sqrt{cx^2 - cx} - x.$$

Now we want $<\psi(x)e(x)> = 0$, so

$$\int_0^1 \psi(x)\left(2c + 8\sqrt{cx^2 - cx} - x\right)\, dx = 0.$$

For the collocation method let us arbitrarily choose the collocation point to be at $x = 1/2$ so that $\psi(x) = \delta(x - 1/2)$, so

$$\int_0^1 \delta(x - 1/2)\left(2c + 8\sqrt{cx^2 - cx} - x\right)\, dx = 0.$$

Evaluating, we get

$$2c + 8\sqrt{\frac{c}{4} - \frac{c}{2}} - \frac{1}{2} = 0.$$

Solving the resulting quadratic equation for c, we get

$$c = -\frac{7}{4} \pm \sqrt{3},$$

so we find *two* solutions for the non-linear equation

$$y(x) \sim \left(-\frac{7}{4} \pm \sqrt{3}\right) x(x-1).$$

5. (30) Consider

$$\mathbf{A} = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) Find $||\mathbf{A}||_2$.

Solution

Many people missed this one. Some people got the right answer for the wrong reason. The norm of an operator, such as $\mathbf{A}$, is, as defined in class

$$||\mathbf{A}|| = \sup_{\mathbf{x}\neq 0} \frac{||\mathbf{A}\cdot\mathbf{x}||}{||\mathbf{x}||}.$$

And the only norm we defined for a matrix was the so-called *spectral norm*, that is the square root of the maximum eigenvalue of $\mathbf{A}^H \cdot \mathbf{A}$, which can be shown to satisfy the definition of the norm of an operator. So

$$\mathbf{A}^H \cdot \mathbf{A} = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

The eigenvalues of the matrix are given by solutions to the characteristic polynomial

$$(1-\lambda)^2 - i(-i) = 0,$$

which are $\lambda = 0, 2$. Taking the positive square root of the maximum eigenvalue, we get

$$||\mathbf{A}|| = \sqrt{2}.$$

(b) Find the most general $\mathbf{x}$ which minimizes $||\mathbf{A}\cdot\mathbf{x} - \mathbf{b}||_2$.

Solution

Most people got this part correct. Considering the system of equations

$$\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

it is obvious that the second equation cannot be satisfied. So we can only expect a solution which minimizes the least squares error. Multiplying both sides by the Hermitian transpose we get

$$\begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Performing Gaussian elimination, we get

$$\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We then take $x_2 = t$, and solve the first equation to get $x_1 = 1 - it$, so the most general solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

The vector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is in the right null space of $\mathbf{A}$. The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ lies partially in the row space and partially in the right null space, but for this problem that is not important, since all that was requested was a general form. We see that

$$\mathbf{A} \cdot \mathbf{x} - \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

so

$$||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2 = 1.$$

(c) Of all the vectors which minimize $||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2$, find the vector $\mathbf{x}$ with minimum $||\mathbf{x}||_2$.

Solution

Most people missed this problem. It does not suffice to simply select $t = 0$ in the expression from the previous part, as what remains contains a component from the right null space. Since the row vector $\mathbf{r}$ and the right null space vector $\mathbf{n}$ are linearly independent (in fact orthogonal), they form a basis, and we can seek α and β such that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \mathbf{r} + \beta \mathbf{n}.$$

We know that $\mathbf{n} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. We must be careful with the row space vector. Living as a row vector it is $\begin{pmatrix} 1 & i \end{pmatrix}$. But when we cast it in column form, we must take the conjugate transpose so $\mathbf{r} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Therefore, we solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Solving, we find $\alpha = 1/2$, and $\beta = i/2$, so

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{i}{2} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Therefore the general solution can also be expressed as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \left(t + \frac{i}{2} \right) \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

We can suppress the right null space vector by choosing $t = -i/2$, giving

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

This has $||\mathbf{x}||_2 = \frac{\sqrt{2}}{2}$.