AME 60611
Examination 1: Solution
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1. (25) Consider the curve $\mathcal{C}$ defined by the intersection of two surfaces: 1) the unit sphere

$$x^2 + y^2 + z^2 = 1,$$

and 2) the plane

$$x + y + z = 1.$$

Find the minimum value of y on $\mathcal{C}$ and the values of x and z on $\mathcal{C}$ where y takes on its minimum value.

Solution

Most people tried to do this with Lagrange multipliers. With some special efforts, this is possible, but most people did not perform those special efforts, and got diverted onto the wrong track. One student, if fact, was able to pull this off. Kudos! I think a better way was to take a more direct approach, and simply eliminate either z or x, get a single equation, and minimize y using the approach of freshman calculus. That is the approach I take here.

The two surfaces are shown in Figure 1. in Figure 2. Eliminating z, one gets

$$x^2 + y^2 + (1 - x - y)^2 = 1.$$

Expanding, one finds

$$x^2 + y^2 - x + xy - y = 0.$$

Solving for $y(x)$ via the quadratic equation, one gets

$$y = \frac{1 - x \pm \sqrt{1 + 2x - 3x^2}}{2}.$$

The critical points are found at values of x for which $dy/dx = 0$. Taking the derivative, one finds

$$\frac{dy}{dx} = \frac{1}{2}\left(-1 \pm \frac{1 - 3x}{\sqrt{1 + 2x - 3x^2}}\right).$$

Setting the derivative to 0 for the critical points, one finds

$$\begin{aligned}
0 &= \frac{1}{2}\left(-1 \pm \frac{1 - 3x}{\sqrt{1 + 2x - 3x^2}}\right), \\
1 &= \pm\frac{1 - 3x}{\sqrt{1 + 2x - 3x^2}}, \\
\sqrt{1 + 2x - 3x^2} &= \pm(1 - 3x), \\
1 + 2x - 3x^2 &= 1 - 6x + 9x^2, \\
8x - 12x^2 &= 0, \\
4x(2 - 3x) &= 0.
\end{aligned}$$

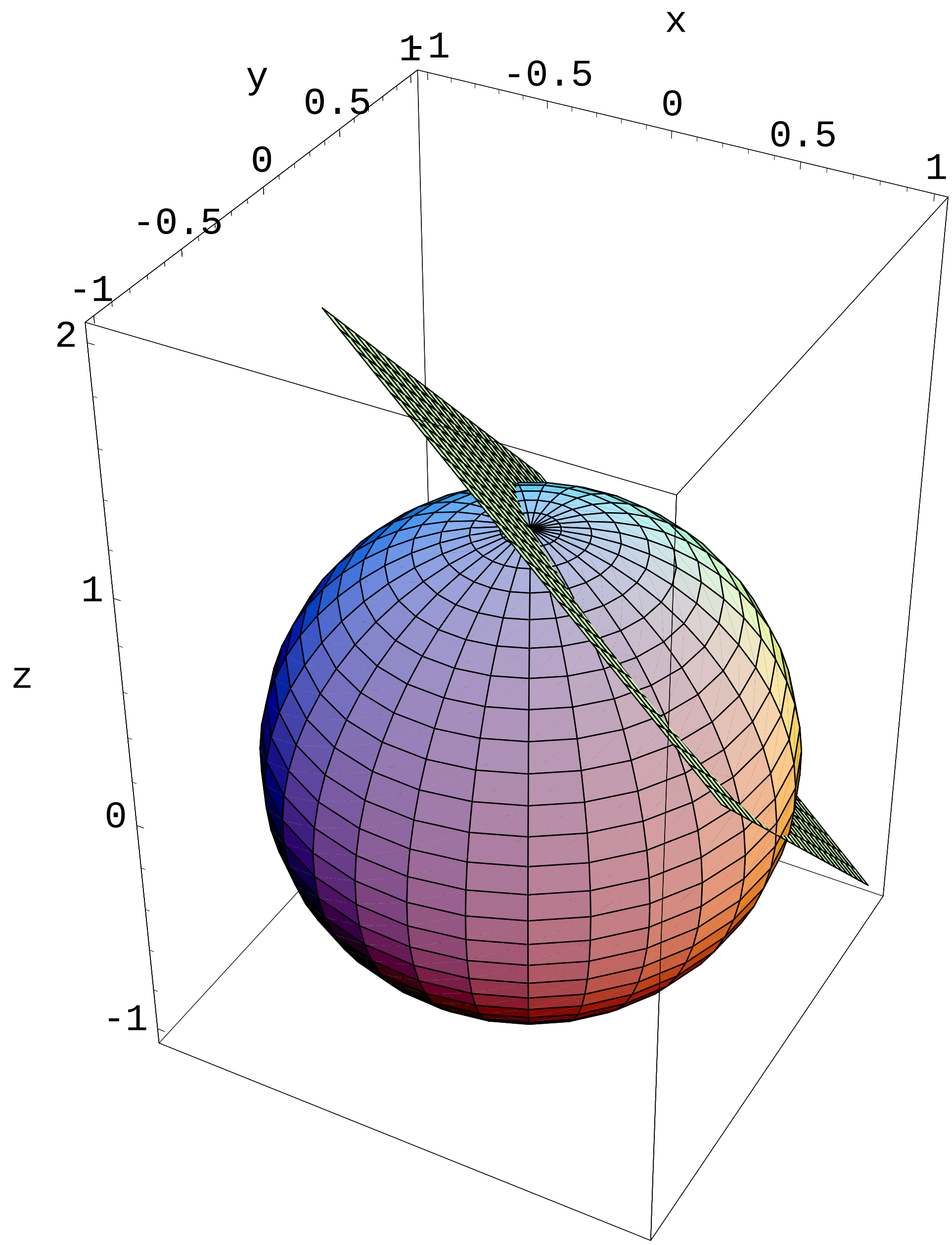

Figure 1: Plot of two intersecting surfaces given by the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 1$.

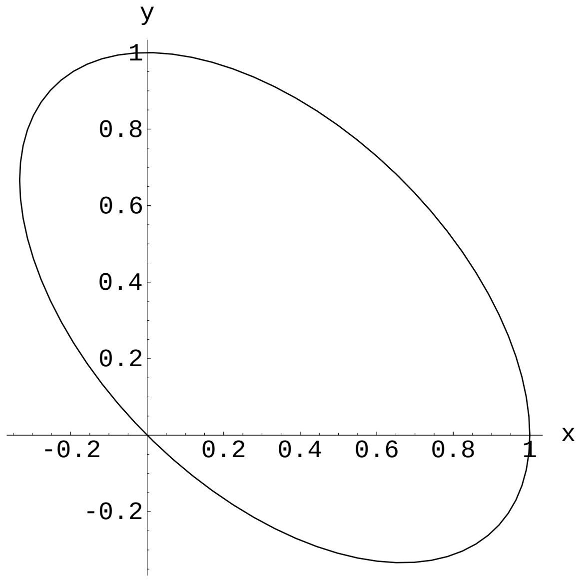

Figure 2: Plot of $\mathcal{C} : x^2 + y^2 - x + xy - y = 0$, which is the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 1$.

This yields

$$x = 0, \qquad x = \frac{2}{3}.$$

When $x = 0$, $y = 0$ or $y = 1$. When $x = 2/3$, $y = -1/3$ or $y = 2/3$. One could do a second derivative test, which would reveal what seems obvious, that y takes on a mininum value at $x = 2/3$. For this point, we find

$$z = 1 - x - y = 1 - \frac{2}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}.$$

So y is minimized on $\mathcal{C}$ at the point $(x, y, z) = (2/3, -1/3, 2/3)$. This is obvious when one plots the curve $\mathcal{C}$, as done in Figure 2.

The same solution can be achieved with the alternative *Lagrange multipliers approach.* With this approach, one must remember that y is being minimized. One could then take several approaches. Let us pose the function to be minimized as

$$y = 1 - x - z.$$

The constraint must not contain y. So take the constraint as

$$0 = 1 - x^2 - z^2 - (1 - x - z)^2.$$

The constraint can be rewritten as

$$0 = x - x^2 + z - xz - z^2.$$

So, we can take the Lagrange multiplier formulation as

$$y = 1 - x - z + \lambda(x - x^2 + z - xz - z^2).$$

Now take the appropriate partial derivatives

$$\begin{aligned}\frac{\partial y}{\partial x} &= -1+\lambda(1-2x-z)=0,\\ \frac{\partial y}{\partial z} &= -1+\lambda(1-x-2z)=0.\end{aligned}$$

Combine these with the constraint to form three equations in three unknowns, x, z, and λ.

$$\begin{aligned}-1+\lambda(1-2x-z) &= 0,\\ -1+\lambda(1-x-2z) &= 0,\\ x-x^2+z-xz-z^2 &= 0.\end{aligned}$$

Leaving out the solution details, there are two roots for these equations:

$$(x,z,\lambda)=\left(\frac{2}{3},\frac{2}{3},-1\right),\quad (0,0,1).$$

For the first root, we get $y=-1/3$. For the second root, we get $y=1$. Obviously, the first root gives the minimum y. So, the solution is the point

$$(x,y,z)=\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right).$$

The other root corresponds to a maximum value of y on $\mathcal{C}$.

2. (25) Consider

$$x\frac{dy}{dx}-y^2+y=0,\qquad y(0)=-1.$$

Determine a solution if a solution exists. If it exists, determine whether it is unique.

Solution

This equation is singular at $x=0$, so we expect some potential troubles, especially since the initial condition is specified at $x=0$. There are two straightforward ways to deal with this problem: 1) as a Bernoulli equation, whose approach I outline in detail in the following paragraphs, or 2) separation of variables, in which one gets

$$\frac{dy}{y^2-y}=\frac{dx}{x},$$

followed by a partial fraction expansion of $1/(y^2-y)$ and integration of what remains.

Here is the approach treating the equation as a Bernoulli equation. Rearranging the equation, we get

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{x}y^2.$$

This is a Bernoulli equation. Define then,

$$\begin{aligned} u &\equiv \frac{1}{y}, \\ y &= \frac{1}{u}, \\ \frac{dy}{dx} &= -\frac{1}{u^2}\frac{du}{dx}. \end{aligned}$$

Replacing y with u in the ODE, we get

$$\begin{aligned} -\frac{1}{u^2}\frac{du}{dx} + \frac{1}{x}\frac{1}{u} = \frac{1}{x}\frac{1}{u^2}, \\ \frac{du}{dx} - \frac{1}{x}u &= -\frac{1}{x}. \end{aligned}$$

The integrating factor is

$$\exp\left(\int\left(\frac{-1}{x}\right)dx\right) = \exp\left(-\ln x\right) = \frac{1}{x}.$$

Multiplying the ODE by the integrating factor, we get

$$\begin{aligned} \frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u &= -\frac{1}{x^2}, \\ \frac{d}{dx}\left(\frac{u}{x}\right) &= -\frac{1}{x^2}, \\ \frac{u}{x} &= \frac{1}{x} + C, \\ u &= 1 + Cx, \\ \frac{1}{y} &= 1 + Cx, \\ y &= \frac{1}{1 + Cx} \end{aligned}$$

Now, when $x = 0$, $y = 1$, so the initial condition is not satisfied. Thus, a solution does not exist.

In fact a solution exists *only* for the initial condition $y(0) = 1$. However, in that case, the solution is *not unique*, since for all $C \in \mathbb{R}^1$, the differential equation and initial condition are satisfied.

3. (25) Use the Green's function method to find the general solution on the domain $x \in [0, \infty)$ to

$$\frac{dy}{dx} + y = f(x), \qquad y(0) = 1.$$

It can help to transform y to a new dependent variable to render the boundary condition to be homogeneous.

Solution

There was general mis-understanding at all levels on this problem, and it would be worthwhile to review carefully just what constitutes this method.

We need homogeneous boundary conditions, so take

$$z = y - 1.$$

Despite the hint, most students did not understand what was suggested here, and missed this essential step for the Green's function method.

The transformed problem is

$$\frac{dz}{dx} + (z+1) = f(x), \qquad z(0) = 0.$$

Take now

$$h(x) = f(x) - 1,$$

so that

$$\frac{dz}{dx} + z = h(x), \qquad z(0) = 0.$$

Here the operator $\mathbf{L}$ is

$$\mathbf{L} = \frac{d}{dx} + 1.$$

Consider first $x < s$. Solving for $\mathbf{L}g = 0$, we get

$$\begin{aligned} \frac{dg}{dx} + g &= 0, \\ g &= Ae^{-x} \end{aligned}$$

Many students chose the *incorrect* path (or some permutation) of ignoring terms and solving $dg/dx = 0$; $g = C$. This does not work.

Now g must satisfy the boundary conditions on x so

$$g(0) = 0 = Ae^0.$$

Therefore $A = 0$, and

$$g = 0, \qquad x < s.$$

Now for $x > s$, we have

$$\begin{aligned} \frac{dg}{dx} + g &= 0, \\ g &= Be^{-x}. \end{aligned}$$

The highest order derivative is unity, so g itself must suffer a jump at $x = s$. Most students missed this important point. The jump for an equation of order n occurs at the $n-1$ level. So for this case there is no jump in the first derivative, as there is for second order ODEs; the jump is on g itself. Since the leading coefficient on dy/dx is unity, the jump on g is also unity:

$$\begin{aligned} g(s+\epsilon) - g(s-\epsilon) &= 1, \\ Be^{-s+\epsilon} - 0 &= 1, \\ B &= e^{s-\epsilon}, \\ \lim_{\epsilon \to 0} B &= e^s. \end{aligned}$$

Therefore one gets

$$g(x,s) = e^{s-x}, \qquad x > s.$$

So the general solution is

$$\begin{aligned}
z &= \int_0^x g(x,s)h(s)\ ds + \int_x^\infty g(x,s)h(s)\ ds,\\
&= \int_0^x e^{s-x}\ h(s)\ ds + \int_x^\infty 0\ h(s)\ ds,\\
&= e^{-x}\int_0^x e^s h(s)\ ds,\\
y-1 &= e^{-x}\int_0^x e^s(f(s)-1)\ ds,\\
y(x) &= 1+e^{-x}\int_0^x e^s(f(s)-1)\ ds.
\end{aligned}$$

This can be simplified to form

$$y(x) = e^{-x}\left(1+\int_0^x e^s f(s)\ ds\right).$$

Let us test our solution in the case where $f(x)=2$. Then

$$\begin{aligned}
y(x) &= e^{-x}\left(1+\int_0^x 2e^s\ ds\right),\\
&= e^{-x}\left(1+2\ e^s\ |_0^x\right),\\
&= e^{-x}\left(1+2\left(e^x-1\right)\right),\\
&= 2-e^{-x}.
\end{aligned}$$

The boundary condition is satisfied:

$$y(0) = 2-e^0 = 1.$$

The differential equation, $dy/dx+y=2$ is satisfied as it reduces to

$$e^{-x}+2-e^{-x}=2.$$

4. (25) If $0<\epsilon<<1$, $x\in[0,1]$, find an appropriate $O(1)$ and $O(\epsilon)$ solution for

$$x\frac{dy}{dx}-\epsilon y = 0, \qquad y(1)=1.$$

Compare to the exact solution.

Solution

Many students got this right but there were some troubling undercurrents which were common to many exams.

First try a regular expansion

$$y = y_0+\epsilon y_1+\epsilon^2 y_2+\dots.$$

Substituting into the ODE and IC, we get

$$x\frac{d}{dx}\left(y_0+\epsilon y_1+\ldots\right)-\epsilon\left(y_0+\ldots\right)=0,\qquad y_0(1)+\epsilon y_1(1)+\ldots=1.$$

At leading order we get then

$$x\frac{dy_0}{dx}=0,\qquad y_0(1)=1.$$

For $x\neq 0$, the unique solution is

$$y_0=1.$$

A disturbingly large number of students decided to take some permutation of the *incorrect* step of $dy_0=(0/x)dx$; $dy_0=dx$, $y_0=x+C$. This leads one far afield.

At $O(\epsilon)$, one gets

$$\begin{aligned}
x\frac{dy_1}{dx} &= y_0, \qquad y_1(1)=0,\\
x\frac{dy_1}{dx} &= 1,\\
\frac{dy_1}{dx} &= \frac{1}{x},\\
y_1 &= \ln x+C,\\
0 &= \ln(1)+C,\\
0 &= C,\\
y_1 &= \ln x,\\
y &\sim 1+\epsilon\ln x+\ldots.
\end{aligned}$$

Obviously, this solution encounters problems as $x\to 0$. In fact the first term is as large as the second when

$$\epsilon\ln x\sim 1,$$
$$x\sim e^{-1/\epsilon}.$$

Try then the stretching

$$X=\frac{x}{e^{-1/\epsilon}}.$$

This gives

$$x=e^{-1/\epsilon}X,\qquad dx=e^{-1/\epsilon}dX.$$

The ODE becomes then

$$\begin{aligned}
e^{-1/\epsilon}Xe^{1/\epsilon}\frac{dy}{dX}-\epsilon y &= 0, \qquad y(1)=1,\\
X\frac{dy}{dX}-\epsilon y &= 0.
\end{aligned}$$

This is unchanged from the orginal, so the stretching does no good!

Let's try to get an exact solution. The equation is first order linear, with an integrating factor of $\exp(\int -\epsilon/x\ dx)=\exp(-\epsilon\ln x)=1/x^{\epsilon}$, and can be solved with standard methods:

$$\begin{aligned}
\frac{dy}{dx}-\frac{\epsilon}{x}y &= 0,\\
\frac{1}{x^{\epsilon}}\frac{dy}{dx}-\frac{\epsilon}{x^{1+\epsilon}}y &= 0,\\
\frac{d}{dx}\left(\frac{y}{x^{\epsilon}}\right) &= 0,
\end{aligned}$$

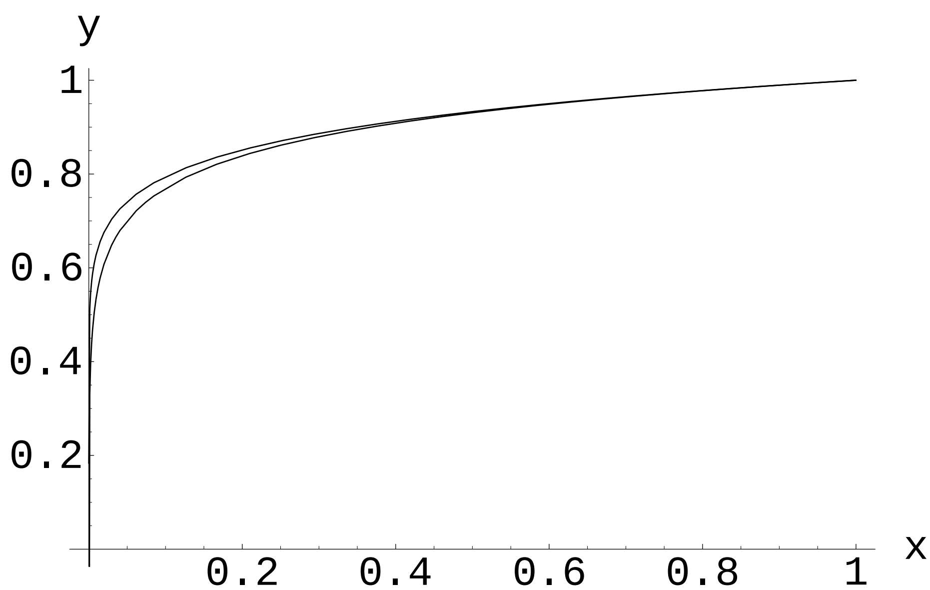

Figure 3: Exact solution, $y = x^\epsilon$, and two-term asymptotic solution $y \sim 1 + \epsilon \ln x$ for $\epsilon = 0.1$.

$$\begin{aligned} \frac{y}{x^\epsilon} &= C, \\ y &= Cx^\epsilon, \\ 1 &= C1^\epsilon, \\ 1 &= C, \\ y &= x^\epsilon. \end{aligned}$$

To get a Taylor series of the *exact* solution, it is helpful to re-express it as

$$y = e^{\epsilon \ln x}.$$

Now the Taylor series expansion of the exact solution about $\epsilon = 0$ yields

$$y = 1 + \epsilon \ln x + \frac{1}{2} (\epsilon \ln x)^2 + \ldots + \frac{1}{n!} (\epsilon \ln x)^n .$$

The ratio test tells us about the convergence of the series and gives the ratio of the n-term to the $n-1$-term, r, as

$$r = \frac{\frac{1}{n!} (\epsilon \ln x)^n}{\frac{1}{(n-1)!} (\epsilon \ln x)^{n-1}} = \frac{\epsilon}{n} \ln x.$$

For any fixed values of ϵ and x, other than zero, the ratio of terms goes to zero as $n \to \infty$, thus the series is convergent for $x \neq 0$. So in fact, there is nothing wrong with the outer solution that was found earlier, except for the singularity at $x = 0$. Note that the exact solution gives $y(0) = 0$, and is well behaved for $x \in [0, 1]$.

The exact solution and the two term asymptotic solution are shown in Figure 3.