AME 60611
Examination 2: **Solution**
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1. (25) Consider the curve in $\mathbb{R}^3$ defined parametrically by

$$\begin{aligned} x &= t, \\ y &= t, \\ z &= t^2. \end{aligned}$$

   (a) Find the length of the curve from $(0,0,0)$ to $(1,1,1)$. You need not numerically evaluate the resulting integral.
   (b) Find the unit tangent at the point $(1,1,1)$.

Solution

The point $(0,0,0)$ corresponds to $t=0$. The point $(1,1,1)$ corresponds to $t=1$.

By the Pythagorean theorem, we have for a differential element of arc length ds that

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

Scaling by dt, one gets

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Then making the substitutions and integrating, one gets

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{1^2 + 1^2 + (2t)^2}, \\ &= \sqrt{2+4t^2}, \\ ds &= \sqrt{2+4t^2}\; dt, \\ s &= \int_0^1 \sqrt{2+4t^2}\; dt, \\ s &= \frac{1}{2}\left(t\sqrt{2+4t^2} + \sinh^{-1}(\sqrt{2}t)\right)\Big|_0^1, \\ s &= \frac{1}{2}\left(\sqrt{6} + \sinh^{-1}(\sqrt{2})\right), \\ s &= 1.79785. \end{aligned}$$

The tangent vector is given by

$$\begin{aligned}
\mathbf{t} &= \frac{\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}, \\
&= \left.\frac{1\mathbf{i} + 1\mathbf{j} + 2t\mathbf{k}}{\sqrt{1^2 + 1^2 + (2t)^2}}\right|_{t=1}, \\
&= \frac{\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{\sqrt{6}}.
\end{aligned}$$

2. (25) Consider two functions in $\mathbb{L}_2[0,1]$: $v_1 = 1$, $v_2 = t^3$.

   (a) Determine if v_1 and v_2 are orthonormal.

   (b) Project the Heaviside function $H(t-1/2)$ onto the space spanned by v_1 and v_2; that is, find the constants α_1, α_2 that best approximate

$$\alpha_1 v_1 + \alpha_2 v_2 \sim H(t-1/2).$$

Solution

To test for orthogonality of v_1 and v_2, one can use the inner product, which is

$$\begin{aligned}
<v_1, v_2> &= \int_0^1 v_1(t) v_2(t)\, dt, \\
&= \int_0^1 (1) t^3\, dt, \\
&= \left.\frac{t^4}{4}\right|_0^1, \\
&= \frac{1}{4}.
\end{aligned}$$

The inner product is not zero, so the vectors are not orthogonal, so they cannot be orthonormal.

Next, let us project $H(t-1/2)$ onto the space spanned by v_1 and v_2. So we seek α_1 and α_2 such that

$$\alpha_1 v_1 + \alpha_2 v_2 \sim H(t-1/2).$$

Take two inner products, one with v_1 and the other with v_2:

$$\begin{aligned}
<v_1, \alpha_1 v_1 + \alpha_2 v_2> &= <v_1, H(t-1/2)>, \\
<v_2, \alpha_1 v_1 + \alpha_2 v_2> &= <v_2, H(t-1/2)>
\end{aligned}$$

Using the properties of the inner product, we find then that

$$\begin{aligned}
\alpha_1 <v_1, v_1> + \alpha_2 <v_1, v_2> &= <v_1, H(t-1/2)>, \\
\alpha_1 <v_2, v_1> + \alpha_2 <v_2, v_2> &= <v_2, H(t-1/2)>.
\end{aligned}$$

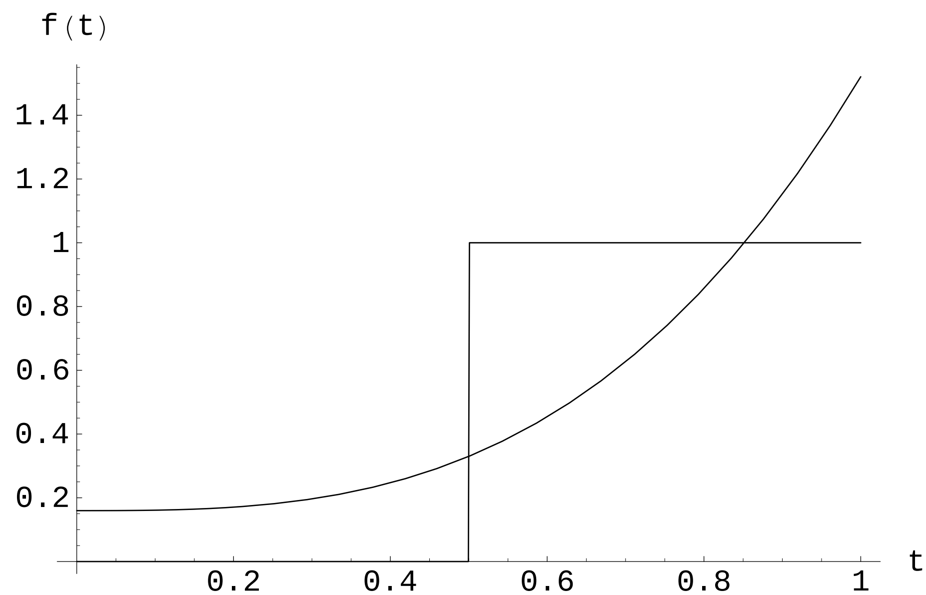

Figure 1: The function $H(t-1/2)$ and its projection onto the space spanned by the functions $v_1 = 1$ and $v_2 = t^3$.

In matrix form, this gives

$$\begin{pmatrix} <v_1, v_1> & <v_1, v_2> \\ <v_2, v_1> & <v_2, v_2> \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} <v_1, H(t-1/2)> \\ <v_2, H(t-1/2)> \end{pmatrix}$$

Now replace the inner product with its integral form to get

$$\begin{pmatrix} \int_0^1 v_1 v_1 \; dt & \int_0^1 v_1 v_2 \; dt \\ \int_0^1 v_2 v_1 \; dt & \int v_2 v_2 \; dt \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \int_{1/2}^1 v_1 \; dt \\ \int_{1/2}^1 v_2 \; dt> \end{pmatrix}$$

Now substitute for v_1 and v_2 to get

$$\begin{pmatrix} \int_0^1 (1)(1) \; dt & \int_0^1 (1)t^3 \; dt \\ \int_0^1 t^3(1) \; dt & \int t^3 t^3 \; dt \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \int_{1/2}^1 (1) \; dt \\ \int_{1/2}^1 t^3 \; dt> \end{pmatrix}$$

Evaluating each of the integrals, we find

$$\begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{15}{64} \end{pmatrix}$$

Solving the two equations in two unknowns gives

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{23}{144} \\ \frac{49}{36} \end{pmatrix}.$$

So the projection of $H(t-1/2)$ in the space spanned by 1 and t^3 is

$$H(t-1/2) \sim \frac{23}{144} + \frac{49}{36} \, t^3.$$

A plot of $H(t-1/2)$ and its projection is given in Figure 1.

3. (25) For $\mathbf{A} : \mathbb{C}^3 \to \mathbb{C}^2$, find the vector $\mathbf{x} \in \mathbb{C}^3$ of smallest $||\mathbf{x}||_2$ which minimizes the error norm $||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & i \\ 2 & 2 & 2i \end{pmatrix}.$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ 1+i \end{pmatrix}.$$

Solution

Consider

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{x} &\sim \mathbf{b}, \\
\mathbf{A}^H \cdot \mathbf{A} \cdot \mathbf{x} &= \mathbf{A}^H \cdot \mathbf{b}, \\
\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ -i & -2i \end{pmatrix} \begin{pmatrix} 1 & 1 & i \\ 2 & 2 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ -i & -2i \end{pmatrix} \begin{pmatrix} 0 \\ 1+i \end{pmatrix}, \\
\begin{pmatrix} 5 & 5 & 5i \\ 5 & 5 & 5i \\ -5i & -5i & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 2+2i \\ 2+2i \\ 2-2i \end{pmatrix}, \\
\begin{pmatrix} 5 & 5 & 5i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 2+2i \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Take as free variables $x_2 = s$, $x_3 = t$. Then, solving, one finds

$$x_1 = \frac{2}{5}(1+i) - s - it.$$

So one has

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}(1+i) \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}.$$

The vectors of which s and t are coefficients span the right null space of $\mathbf{A}$. The other vector has components in both the row space and right null space of $\mathbf{A}$. Let us find the part of that vector which lies in the row space. Thus, we solve for the constants α_1, α_2 and α_3 in

$$\begin{pmatrix} 1 & -1 & -i \\ 1 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}(1+i) \\ 0 \\ 0 \end{pmatrix}.$$

Solving, we find

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \frac{2}{15} \begin{pmatrix} 1+i \\ -1-i \\ -1+i \end{pmatrix}.$$

So that the solution vector is then expressed as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{2}{15}(1+i) \begin{pmatrix} 1 \\ 1 \\ -i \end{pmatrix} + \left(s - \frac{2}{15}(1+i)\right) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \left(t + \frac{2}{15}(-1+i)\right) \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}.$$

The vector $\mathbf{x}$ with smallest norm is found by removing the null space components. Doing so, we find

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{2}{15}(1+i)\begin{pmatrix} 1 \\ 1 \\ -i \end{pmatrix}.$$

4. (25) Consider

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}.$$

Cast the matrix $\mathbf{A}$ into Jordan canonical form; that is, find matrices $\mathbf{S}$ and $\mathbf{J}$ such that $\mathbf{A} = \mathbf{S} \cdot \mathbf{J} \cdot \mathbf{S}^{-1}$.

Solution

We first find the eigenvalues of $\mathbf{A}$:

$$\begin{aligned} (1-\lambda)(3-\lambda)+1 &= 0, \\ 3 - 4\lambda + \lambda^2 + 1 &= 0, \\ \lambda^2 - 4\lambda + 4 &= 0, \\ (\lambda - 2)^2 &= 0. \end{aligned}$$

So we have a repeated root

$$\lambda = 2, \qquad \lambda = 2.$$

First find the ordinary eigenvector associated with $\lambda = 2$.

$$\begin{aligned} \mathbf{A} \cdot \mathbf{e} &= \lambda \mathbf{e}, \\ \mathbf{A} \cdot \mathbf{e} &= \lambda \mathbf{I} \cdot \mathbf{e}, \\ (\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{e} &= \mathbf{0}, \\ \begin{pmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 1-2 & 1 \\ -1 & 3-2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

This gives e_2 as a free variable $e_2 = t$. Then we get

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For simplicity, take $t = 1$, so

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Seek now a generalized eigenvector $\mathbf{g}$ such that

$$\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})\cdot\mathbf{g} &= \mathbf{e},\\
\begin{pmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix}\cdot\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix},\\
\begin{pmatrix} 1-2 & 1 \\ -1 & 3-2 \end{pmatrix}\cdot\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix},\\
\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix},\\
\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix},
\end{aligned}$$

Take the free variable to be $g_2 = s$. This then yields $g_1 = -1 + s$. So,

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Choosing $s = 0$ still gives a non-trivial solution of

$$\mathbf{g} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Then we can construct the matrix $\mathbf{S}$ by placing $\mathbf{e}$ and $\mathbf{g}$ in its columns to get

$$\mathbf{S} \equiv \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now the inverse of $\mathbf{S}$ is easily shown to be

$$\mathbf{S}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

We can then get the matrix $\mathbf{J}$ by taking

$$\begin{aligned}
\mathbf{J} &= \mathbf{S}^{-1}\cdot\mathbf{A}\cdot\mathbf{S},\\
&= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},\\
&= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix},\\
&= \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.
\end{aligned}$$

It is also easy to verify that one recovers $\mathbf{A}$ when forming $\mathbf{S}\cdot\mathbf{J}\cdot\mathbf{S}^{-1}$:

$$\begin{aligned}
\mathbf{S}\cdot\mathbf{J}\cdot\mathbf{S}^{-1} &= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},\\
&= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ -2 & 2 \end{pmatrix},\\
&= \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}.
\end{aligned}$$