AME 60611
Examination 1: Solution
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1. (20) Find all $y(x)$ which satisfy

$$\frac{d^4y}{dx^4} - y = x^2 + e^x.$$

Solution

A lot of people had problems with this. The homogeneous part was pretty easy, but many did not understand the fundamentals of this. The particular solution was mildly tricky, but not particularly tricky. Those who tried variation of parameters for the particular solution were led to a messy problem.

The equation is linear in y and has constant coefficients. First find the complementary functions, found by solving the homogeneous version of the equation:

$$\frac{d^4y}{dx^4} - y = 0.$$

Assume solutions of the form $y = Ce^{rx}$. With this assumption the homogeneous differential equation becomes

$$Cr^4e^{rx} - Ce^{rx} = 0.$$

Since we seek non-trivial solutions for which $C \neq 0$, and $e^{rx} \neq 0$, the equation reduces to the characteristic polynomial

$$r^4 - 1 = 0.$$

This yields

$$r^4 = 1,$$

which is quadratic in r^2, so that

$$\left(r^2\right)^2 = 1.$$

Solving for r^2, we get

$$r^2 = \pm 1.$$

Solving these two quadratic equations for r, we obtain the four roots

$$r = \pm 1, \qquad r = \pm i.$$

The four linearly independent complementary functions can be added to form a general solution to the homogeneous version of the differential equation:

$$y = C_1e^x + C_2e^{-x} + C_3e^{ix} + C_4e^{-ix}.$$

Euler's formula can be employed to remove i, thus arriving at the form

$$y = C_1e^x + C_2e^{-x} + C_3' \sin x + C_4' \cos x.$$

We next seek a particular solution. We have some concern because the forcing function e^x is itself a complementary function. Let us seek then solutions of the form

$$y = a_0 + a_1 x + a_2 x^2 + b_1 x e^x.$$

Taking derivatives, we get

$$y' = a_1 + 2a_2 x + b_1(xe^x + e^x).$$

$$y'' = 2a_2 + b_1(xe^x + 2e^x).$$

$$y''' = b_1(xe^x + 3e^x).$$

$$y'''' = b_1(xe^x + 4e^x).$$

Substituting into the differential equation, we find

$$\underbrace{b_1(xe^x + 4e^x)}_{=y''''} - \underbrace{(a_0 + a_1 x + a_2 x^2 + b_1 x e^x)}_{=y} = x^2 + e^x.$$

Rearranging, we get

$$e^x(4b_1 - 1) + xe^x(b_1 - b_1) - a_0 - a_1 x - x^2(a_2 + 1) = 0.$$

Since e^x, 1, x, and x^2 are linearly independent functions, the only way we can achieve the equality is to demand the coefficients of each function be zero, giving

$$b_1 = \frac{1}{4}, \qquad a_0 = 0, \qquad a_1 = 0, \qquad a_2 = -1.$$

So the particular solution is

$$y = -x^2 + \frac{1}{4} x e^x.$$

The total solution is obtained by adding the particular solution to linear combinations of the complementary functions:

$$y = C_1 e^x + C_2 e^{-x} + C_3' \sin x + C_4' \cos x - x^2 + \frac{xe^x}{4}.$$

2. (20) Use the method of strained coordinates to find the appropriate frequency modulation, valid at order ϵ, to achieve a secularity-free solution to the equation

$$\frac{d^2x}{dt^2} + x + \epsilon x^5 = 0, \qquad x(0) = 0, \left.\frac{dx}{dt}\right|_{t=0} = 1.$$

You have the identities

$$\sin^5\theta = \frac{5}{8}\sin\theta - \frac{5}{16}\sin 3\theta + \frac{1}{16}\sin 5\theta; \;\; \cos^5\theta = \frac{5}{8}\cos\theta + \frac{5}{16}\cos 3\theta + \frac{1}{16}\cos 5\theta.$$

Solution

A lot of people had foundational issues with this, and really did not make much progress towards a solution. A few made it harder than it was by using the method of multiple scales. I think one person got it right, and the small number of people who were on the right track got a lot of credit.

This is very similar to the Duffing equation studied in lecture. Recognizing that a regular expansion is likely to lead to secular terms, let us strain time at the outset

$$t = (1 + c_1\epsilon + c_2\epsilon^2 + \ldots)\tau.$$

This defines the strained time τ, but we do not yet know the value of c_1, c_2, etc.

Check for boundedness of the solution.

$$\dot{x}\ddot{x} + \dot{x}x + \epsilon\dot{x}x^5 = 0.$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\epsilon}{6}x^6\right) = 0,$$

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\epsilon}{6}x^6 = C.$$

The solution is bounded.

As in lecture,

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1}.$$

$$\frac{dx}{dt} \sim \frac{\frac{dx}{d\tau}}{1 + c_1\epsilon + \ldots}.$$

$$\frac{dx}{dt} \sim \frac{dx}{d\tau}(1 - c_1\epsilon + \ldots).$$

The second derivative then is

$$\frac{d^2x}{dt^2} \sim \frac{d^2x}{d\tau^2}(1 - c_1\epsilon + \ldots)^2.$$

$$\frac{d^2x}{dt^2} \sim \frac{d^2x}{d\tau^2}(1 - 2c_1\epsilon + \ldots).$$

Taking next the expansion

$$x \sim x_0 + \epsilon x_1 + \ldots,$$

our differential equation becomes

$$\underbrace{\left(\frac{d^2x_0}{d\tau^2} + \epsilon\frac{d^2x_1}{d\tau^2} + \ldots\right)(1 - 2c_1\epsilon + \ldots)}_{d^2x/dt^2} + \underbrace{(x_0 + \epsilon x_1 + \ldots)}_{=x} + \epsilon\underbrace{(x_0 + \ldots)^5}_{=x^5} = 0.$$

The initial conditions become

$$x_0(0) + \epsilon x_1(0) + \ldots = 0.$$

$$\left.\frac{dx_0}{d\tau}\right|_{t=0}(1 - 2c_1\epsilon + \ldots) + \epsilon\left.\frac{dx_1}{d\tau}\right|_{t=0}(1 - 2c_2\epsilon + \ldots) + \ldots = 1.$$

The leading order problem is given by

$$\frac{d^2x_0}{d\tau^2} + x_0 = 0, \qquad x_0(0) = 0, \qquad \left.\frac{dx_0}{d\tau}\right|_{t=0} = 1.$$

This has the obvious solution

$$x_0(\tau) = \sin\tau.$$

At $\mathcal{O}(\epsilon)$, we get the problem

$$\frac{d^2x_1}{d\tau^2} + x_1 = 2c_1\frac{d^2x_0}{d\tau^2} - x_0^5, \qquad x_1(0) = 0, \qquad \left.\frac{dx_1}{dt}\right|_{t=0} = 2c_1 \left.\frac{dx_0}{d\tau}\right|_{t=0}.$$

Substituting the known $x_0(\tau)$, the $\mathcal{O}(\epsilon)$ problem becomes

$$\frac{d^2x_1}{d\tau^2} + x_1 = -2c_1 \sin\tau - \sin^5\tau, \qquad x_1(0) = 0, \qquad \left.\frac{dx_1}{dt}\right|_{t=0} = 2c_1.$$

Using our expansion for $\sin^5\tau$, we get

$$\frac{d^2x_1}{d\tau^2} + x_1 = -2c_1 \sin\tau - \frac{5}{8}\sin\tau + \frac{5}{16}\sin 3\tau - \frac{1}{16}\sin 5\tau.$$

To avoid secularities, choose

$$c_1 = -\frac{5}{16}.$$

This gives the leading order frequency modulation, so the corrected solution is

$$x(t) = \sin\left(\left(1 + \frac{5}{16}\epsilon + \mathcal{O}(\epsilon^2)\right))t\right) + \mathcal{O}(\epsilon)$$

3. (20) Find $\left.\frac{\partial u}{\partial x}\right|_y$ if

$$\begin{aligned} x + 2y + \sin u \sin v &= 1, \\ u^2 + v^2 &= xy. \end{aligned}$$

Solution

There were few problems with this problem. Most people got full credit.

Take

$$f(x, y, u, v) = x + 2y + \sin u \sin v - 1 = 0,$$
$$g(x, y, u, v) = u^2 + v^2 - xy = 0.$$

Taking differentials, we find

$$df = dx + 2dy + \cos u \sin v du + \sin u \cos v dv = 0,$$

$$dg = -ydx - xdy + 2udu + 2vdv = 0.$$

We are holding y constant, so $dy = 0$ and

$$dx + \cos u \sin v du + \sin u \cos v dv = 0,$$

$$-ydx + 2udu + 2vdv = 0.$$

Dividing by dx, we get

$$1 + \cos u \sin v \left.\frac{\partial u}{\partial x}\right|_y + \sin u \cos v \left.\frac{\partial v}{\partial x}\right|_y = 0,$$

$$-y + 2u\left.\frac{\partial u}{\partial x}\right|_y + 2v\left.\frac{\partial v}{\partial x}\right|_y = 0.$$

In matrix form, this becomes

$$\begin{pmatrix} \cos u \sin v & \sin u \cos v \\ 2u & 2v \end{pmatrix} \begin{pmatrix} \left.\frac{\partial u}{\partial x}\right|_y \\ \left.\frac{\partial v}{\partial x}\right|_y \end{pmatrix} = \begin{pmatrix} -1 \\ y \end{pmatrix}.$$

Use Cramer's rule to solve for $\left.\frac{\partial u}{\partial x}\right|_y$ to get

$$\left.\frac{\partial u}{\partial x}\right|_y = \frac{-2v - y\sin u \cos v}{2v\cos u \sin v - 2u \sin u \cos v}.$$

4. (20) Find all $y(x)$ which satisfy

$$\left(\frac{d^2y}{dx^2}\right)^2 + x\frac{d^2y}{dx^2} = \frac{dy}{dx}, \quad y'(0) = 1,\ y(0) = 0.$$

A lot of people did not recognize that this was a Clairaut equation. If they did recognize it, many still had shaky solution methods. A few got it right.

Solution

Taking $v \equiv \frac{dy}{dx}$, this is a first order equation for v:

$$\left(\frac{dv}{dx}\right)^2 + x\frac{dv}{dx} = v, \qquad v(0) = 1.$$

Rearranging, we get

$$v = x\frac{dv}{dx} + \left(\frac{dv}{dx}\right)^2,$$

which is a Clairaut equation for $v(x)$. Letting then $\frac{dv}{dx} \equiv u(x)$, the equation reduces to

$$v = xu + u^2.$$

Differentiating with respect to x, we get

$$\frac{dv}{dx} = x\frac{du}{dx} + u + 2u\frac{du}{dx},$$

or

$$u = x\frac{du}{dx} + u + 2u\frac{du}{dx},$$

Rearranging, we get

$$\frac{du}{dx}(x + 2u) = 0.$$

Aiming first for the regular solution, we demand that

$$\frac{du}{dx} = 0.$$

This gives

$$u = C.$$

or

$$v = Cx + C^2.$$

Since $v(0) = 1$, we get

$$1 = C(0) + C^2.$$

So

$$C = \pm 1,$$

and

$$v = 1 \pm x.$$

Returning to y, we then get

$$\frac{dy}{dx} = 1 \pm x, \qquad y(0) = 0.$$

Integrating, we get

$$y(x) = x \pm \frac{x^2}{2} + C.$$

Applying the initial condition, we get

$$0 = C.$$

Thus, we find two solutions which satisfy the initial conditions and the ordinary differential equation:

$$y(x) = x\left(1 \pm \frac{x}{2}\right).$$

This illustrates the property that non-unique solutions can exist for non-linear differential equations.

5. (20) Consider the transformation from non-Cartesian coordinates (x^1, x^2) to Cartesian coordinates (ξ^1, ξ^2):

$$\begin{aligned} \xi^1 &= (x^1)^2, \\ \xi^2 &= x^1 + 2x^2. \end{aligned}$$

A vector field $\mathbf{u}$ has Cartesian representation $U^i = (2\xi^1, 3\xi^2)^T$. Find

(a) the metric tensor of the transformation, and

(b) an expression for the vector field components u^i in the non-Cartesian system, (x^1, x^2).

Solution

This problem was a mixed bag. There was no need to get the inverse transform, as many tried. Many people got the Jacobian right, but flunked simple matrix multiplication to get the metric tensor. A small number of people realized that one only need employ the definition of a contravariant vector to get the representation in the transformed coordinates.

First, get the Jacobian matrix:

$$\mathbf{J} = \frac{\partial \xi^i}{\partial x^j} = \begin{pmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} 2x^1 & 0 \\ 1 & 2 \end{pmatrix}$$

For the metric tensor, we get then

$$\mathbf{G} = \mathbf{J}^T \cdot \mathbf{J} = \begin{pmatrix} 2x^1 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2x^1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4\left(x^1\right)^2 + 1 & 2 \\ 2 & 4 \end{pmatrix}$$

From the definition of contravariance, $U^i = \frac{\partial \xi^i}{\partial x^l} u^l$,

$$U^1 = \frac{\partial \xi^1}{\partial x^1} u^1 + \frac{\partial \xi^1}{\partial x^2} u^2 = 2x^1 u^1.$$

$$U^2 = \frac{\partial \xi^2}{\partial x^1} u^1 + \frac{\partial \xi^2}{\partial x^2} u^2 = u^1 + 2u^2.$$

So

$$u^1 = \frac{U^1}{2x^1} = \frac{2\xi^1}{2x^1} = \frac{2(x^1)^2}{2x^1} = x^1,$$

$$u^2 = \frac{U^2}{2} - \frac{u^1}{2} = \frac{3\xi^2}{2} - \frac{x^1}{2} = \frac{3}{2}(x^1 + 2x^2) - \frac{x^1}{2} = x^1 + 3x^2.$$

Thus

$$u^i = (u^1, u^2)^T = (x^1, x^1 + 3x^2)^T.$$