AME 60611
Examination 2: Solution
J. M. Powers
26 November 2007

1. (20) Consider the lines in $\mathbb{E}^3$ given by

$$x = y = z$$

and

$$3x + y = y + 1 = z - 1.$$

It is straightforward to find the distance from a point on one line to a point on the other. Find the coordinates of the point on each line which minimizes this distance, and find the value of the distance.

Solution

There was lots of confusion on this problem, and this solution should be read carefully.

Let us first give a parametric description of each line. The first is simply $x = y = z = t$, which yields

$$x = t$$
$$y = t$$
$$z = t.$$

The second is $3x + y = y + 1 = z - 1 = s$, which yields

$$3x + y = s$$
$$y = s - 1$$
$$z = s + 1$$

Solving for x in the second set yields

$$x = \frac{s-y}{3} = \frac{s-s+1}{3} = \frac{1}{3}.$$

So in parametric form, the second line is given by

$$x = \frac{1}{3}$$
$$y = s - 1$$
$$z = s + 1$$

Now the square of the Euclidean distance from generic point on the first line to a generic point on the second is

$$\ell^2 = (t - 1/3)^2 + (t - (s-1))^2 + (t - (s+1))^2.$$

Now if ℓ is minimized, ℓ^2 is as well, so we will seek values of s and t which drive ℓ^2 to a minimum. At such minima, we must have $\partial\ell^2/\partial t = \partial\ell^2/\partial s = 0$. Forming the partial derivatives, we find

$$\frac{\partial \ell^2}{\partial t} = 2(t-1/3) + 2(t-s+1) + 2(t-s-1) = 0.$$

$$\frac{\partial \ell^2}{\partial s} = -2(t-s+1) - 2(t-s-1) = 0.$$

Expanding, we get

$$3t - 2s - \frac{1}{3} = 0$$
$$2t - 2s = 0$$

which has solution

$$s = t = \frac{1}{3}$$

So the square of the distance between these two points is

$$\ell^2 = (1/3-1/3)^2 + (1/3-1/3+1)^2 + (1/3-1/3-1)^2 = 2.$$

And the distance is thus

$$\ell = \sqrt{2}.$$

2. (20) Find $\mathbf{x}$ of minimum $||\mathbf{x}||_2$ which minimizes $||\mathbf{A}\cdot\mathbf{x} - \mathbf{b}||_2$ when

$$\mathbf{A} = \begin{pmatrix} 1+i & i \\ 2+2i & 2i \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solution

Most people did fine on this, although there were some calculation errors.

Even though the second row is twice the first in the matrix $\mathbf{A}$, the sub-determinant of the bottom four elements is $-2i$, and so the rank of $\mathbf{A}$ is two; that is, it is a full rank matrix. Therefore, this is an ordinary over-constrained system. Let us operate on both sides of the "equation" by the conjugate transpose of $\mathbf{A}$:

$$\mathbf{A}^H \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^H \cdot \mathbf{b}.$$

$$\begin{pmatrix} 1-i & 2-2i & 1 \\ -i & -2i & 0 \end{pmatrix} \begin{pmatrix} 1+i & i \\ 2+2i & 2i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1-i & 2-2i & 1 \\ -i & -2i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 5+5i \\ 5-5i & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1-i \\ -i \end{pmatrix}.$$

Use Cramer's rule to solve. The determinant of the coefficient matrix is $11(5) - 50 = 5$. So

$$x_1 = \frac{((1-i)(5) + (i)(5+5i))}{5} = 0.$$

$$x_2 = \frac{(11)(-i) - (1-i)(5-5i)}{5} = -\frac{i}{5}.$$

Thus, we have

$$\mathbf{x} = \begin{pmatrix} 0 \\ \frac{-i}{5} \end{pmatrix}.$$

This value of $\mathbf{x}$ lies entirely in the row space of $\mathbf{A}$. There is no non-trivial right null space; therefore, this is the $\mathbf{x}$ of minimum norm. The error itself in satisfying the original equation is

$$\mathbf{e} = \mathbf{A} \cdot \mathbf{x} - \mathbf{b} = \begin{pmatrix} 1+i & i \\ 2+2i & 2i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{i}{5} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ 0 \end{pmatrix}.$$

The magnitude of the error is

$$||\mathbf{e}||_2 = \sqrt{(-4/5)^2 + (2/5)^2} = \frac{2\sqrt{5}}{5}.$$

The norm of the vector $\mathbf{x}$ is

$$||\mathbf{x}||_2 = \sqrt{\begin{pmatrix} 0 & i/5 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{-i}{5} \end{pmatrix}} = \frac{1}{5}.$$

3. (20) In $\mathbb{R}^3$, a set of vectors is given as

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Determine if these form a basis in $\mathbb{R}^3$.

(b) Find the reciprocal basis.

Solution

Most people got this.

The vectors forms a basis if the only way to enforce

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

is to demand $c_1 = c_2 = c_3 = 0$. For our basis then, this becomes

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In matrix form, this is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of this matrix is unity, so it is not singular. Thus the only way to satisfy this equation is to demand that $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent, span the space, and thus form a basis in $\mathbb{R}^3$. Our matrix of basis vectors $\mathbf{V}$ is thus

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix of reciprocal basis vectors $\mathbf{V}^R$ is such that

$$\mathbf{V}^T \cdot \mathbf{V}^R = \mathbf{I}.$$

So

$$\mathbf{V}^R = \left(\mathbf{V}^{-1}\right)^T.$$

Forming $\mathbf{V}^{-1}$, we find

$$\mathbf{V}^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\mathbf{V}^R = \left(\mathbf{V}^{-1}\right)^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The reciprocal basis vectors are thus

$$v_1^R = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad v_2^R = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v_3^R = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

4. (20) Use a one-term Galerkin method with a polynomial basis function to estimate the solution to the differential equation

$$\frac{d^3y}{dx^3} + y = x, \qquad y(0) = 0,\ y(1) = 0,\ y'(0) = 0.$$

Solution

People had mixed performance on this. Many people did not get a good basis function. Others did not really know how to apply the method of weighted residuals. The calculations were difficult; a few people got it all right.

Assume, for the one term expansion, that the approximate solution is

$$y_a = c\phi(x).$$

where $\phi(x)$ is a polynomial which satisfies the boundary conditions. Let as assume a third order polynomial.

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Now $\phi(x)$ must satisfy the same boundary conditions as $y(x)$. So at $x = 0$, we have

$$\phi(0) = 0 = a_0 + a_1(0) + a_2 0^2 + a_3 0^3.$$

Thus $a_0 = 0$. The first derivative is then

$$\phi'(x) = a_1 + 2a_2 x + 3a_3 x^2.$$

At $x = 0$ one must satisfy the derivative boundary condition

$$\phi'(0) = 0 = a_1 + 2a_2(0) + 3a_3 0^2.$$

Thus $a_1 = 0$. So we have

$$\phi(x) = a_2 x^2 + a_3 x^3.$$

At $x = 1$, we must have

$$\phi(1) = 0 = a_2 + a_3.$$

Take $a_2 = 1$, thus $a_3 = -1$, and

$$\phi(x) = x^2(1 - x).$$

Now, we have

$$y_a = cx^2(1 - x).$$

The error with this approximation is

$$e = \frac{d^3 y_a}{dx^3} + y_a - x,$$

which evaluates to

$$e = -6c + cx^2(1 - x) - x$$

Now for the Galerkin method we need

$$<\phi, e> = 0.$$

Thus, we c such that

$$\int_0^1 \phi(x) e(x)\ dx = 0.$$

$$\int_0^1 x^2(1 - x)(-6c + cx^2(1 - x) - x)\ dx = 0.$$

$$\int_0^1 (-6cx^2 + (6c - 1)x^3 + (1 + c)x^4 - 2cx^5 + cx^6)\ dx = 0.$$

$$-2cx^3 + \left(\frac{3c}{2} - \frac{1}{4}\right) x^4 + \frac{1 + c}{5} x^5 - \frac{cx^6}{3} + \left.\frac{cx^6}{7}\right|_0^1 = 0.$$

$$-2c + \frac{3c}{2} - \frac{1}{4} + \frac{1 + c}{5} - \frac{c}{3} + \frac{c}{7} = 0.$$

Solve for c and get

$$c = -\frac{21}{206}.$$

So

$$y_a = -\frac{21}{206} x^2(1 - x).$$

The exact solution can be obtained from computer algebra, but is lengthy. It can, however, easily be plotted and compared with the Galerkin approximation. See Figure 1.

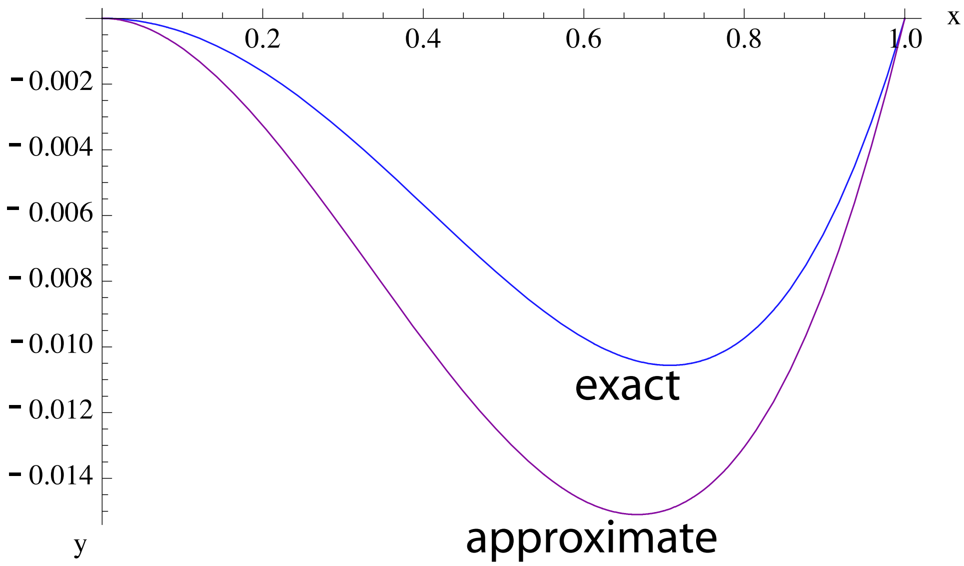

Figure 1: Plot of exact and Galerkin approximation solutions.

5. (20) Use Cartesian index notation to prove the identity

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla \cdot \nabla \mathbf{u}.$$

Solution

Most got this. Those who didn't, didn't get very far at all.

Consider the left side in Cartesian index notation:

$$\begin{aligned}
\epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}\frac{\partial}{\partial x_l}u_m &= \epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}u_m, \\
&= \epsilon_{kij}\epsilon_{klm}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}u_m, \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}u_m, \\
&= \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}u_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i \\
&= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i, \\
&= \nabla(\nabla \cdot \mathbf{u}) - \nabla \cdot \nabla \mathbf{u}.
\end{aligned}$$