AME 60611
Examination 1: Solution
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1. (25) Find the minimum distance between the ellipse described by $x^2 + 4y^2 = 1$ and the line $x + y = 2$.

Solution

Most people had foundational difficulties with this problem. It was not a Lagrange multiplier problem, as assumed by most. A few people were on the right track, which needed to account for the reality that there were two points to find, both with unknown coordinates.

The ellipse and the line are plotted in Fig. **??**.

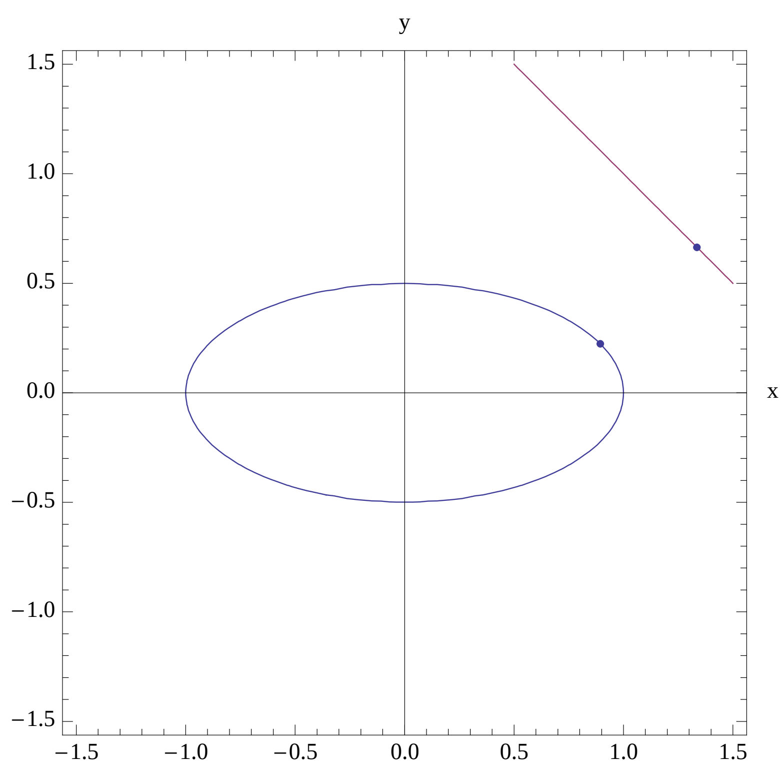

Figure 1: Plot relevant ellipse, line, and points of minimum distance.

Let us consider the point on the ellipse to be (x_1, y_1). Let us consider the point on the line to be (x_2, y_2). For the point on the ellipse we have

$$x_1^2 + 4y_1^2 = 1,$$

so

$$y_1 = \frac{\pm\sqrt{1-x_1^2}}{2}.$$

Examining Fig. **??**, we can infer that the "+" root is the appropriate root, so

$$y_1 = \frac{\sqrt{1-x_1^2}}{2}.$$

For the line, we have

$$y_2 = 2 - x_2.$$

Now take the distance between (x_1, y_1) and (x_2, y_2) to be ℓ. By the distance formula from analytic geometry, we have

$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Eliminating y_1 and y_2, we find

$$\ell^2 = (x_2 - x_1)^2 + \left(2 - x_2 - \frac{1}{2}\sqrt{1-x_1^2}\right)^2.$$

Now, if we minimize ℓ^2, we also minimize ℓ, so let us focus on choosing an appropriate x_1 and x_2 so as to minimize ℓ^2; this requires that the appropriate partial derivatives of ℓ^2 with respect to x_1 and x_2 be zero:

$$\left.\frac{\partial \ell^2}{\partial x_1}\right|_{x_2} = x_1\left(\frac{2 - x_2 - \frac{1}{2}\sqrt{1-x_1^2}}{\sqrt{1-x_1^2}}\right) - 2(x_2 - x_1) = 0.$$

$$\left.\frac{\partial \ell^2}{\partial x_2}\right|_{x_1} = -2\left(2 - x_2 - \frac{1}{2}\sqrt{1-x_1^2}\right) + 2(x_2 - x_1) = 0.$$

These provide two equations in two unknowns for x_1 and x_2. Solving the second for x_2, we find

$$x_2 = \frac{1}{4}\left(4 + 2x_1 - \sqrt{1-x_1^2}\right).$$

Use this to eliminate x_2 from the first, and then factor to obtain

$$\left(-4 + 2x_1 + \sqrt{1-x_1^2}\right)\left(-x_1 + 2\sqrt{1-x_1^2}\right) = 0.$$

Solving for x_1, we find

$$x_1 = \frac{2}{\sqrt{5}},\ \frac{8 \pm i\sqrt{11}}{5}.$$

We take the real root only, so $x_1 = 2/\sqrt{5}$. For this root, we then recover

$$x_2 = \frac{1}{20}\left(20 + 3\sqrt{5}\right).$$

We then solve for y_1 and y_2 and find the points on the ellipse and line, respectively, to be

$$(x_1, y_1) = \left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right).$$

$$(x_2, y_2) = \left(\frac{1}{20}\left(20 + 3\sqrt{5}\right), 2 - \frac{1}{20}\left(20 + 3\sqrt{5}\right)\right).$$

The corresponding distance between these two points is

$$\ell = \sqrt{\left(-\frac{2}{\sqrt{5}} + \frac{1}{20}\left(20 + 3\sqrt{5}\right)\right)^2 + \left(2 - \frac{1}{2\sqrt{5}} - \frac{1}{20}\left(20 + 3\sqrt{5}\right)\right)^2} = 0.623644...$$

2. (25) Find the appropriate Green's function solution for the differential equation

$$\frac{d^3y}{dx^3} = f(x), \qquad y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$$

Test your method if $f(x) = 1$.

Solution

Most people got a good start on this. Many got lost in the algebra. Many plowed through to a correct answer.

First consider $x < s$. The Green's function must satisfy the homogeneous version of the governing equation everywhere in this domain, so

$$\frac{d^3g}{dx^3} = 0,$$

$$\frac{d^2g}{dx^2} = C_1,$$

$$\frac{dg}{dx} = C_1x + C_2,$$

$$g(x,s) = \frac{1}{2}C_1x^2 + C_2x + C_3.$$

Now since this is valid for $x < s$, this portion of $g(x,s)$ must satisfy the boundary conditions at $x = 0$. By inspection, this requires that $C_1 = C_2 = C_3 = 0$, so we have

$$g(x,s) = 0, \qquad x < s.$$

For $x > s$, the equation for $g(x,s)$ must again satisfy the homogeneous portion of the governing equation. Thus we find again that

$$\frac{d^3g}{dx^3} = 0,$$

$$\frac{d^2g}{dx^2} = B_1,$$

$$\frac{dg}{dx} = B_1x + B_2,$$

$$g(x,s) = \frac{1}{2}B_1x^2 + B_2x + B_3, \qquad x > s.$$

Now, at $x = s$, we have continuity of g, dg/dx, and a jump condition for d^2g/dx^2. They are

$$0 = \frac{1}{2}B_1s^2 + B_2s + B_3,$$

$$0 = B_1s + B_2$$

$$1 = B_1.$$

Solving gives

$$B_1 = 1,$$

$$B_2 = -s,$$
$$B_3 = \frac{1}{2}s^2.$$

So
$$g(x,s) = \frac{1}{2}x^2 - sx + \frac{1}{2}s^2 = \frac{1}{2}(x-s)^2, \qquad x > s.$$

So the general solution for arbitrary $f(x)$ is

$$y(x) = \int_0^x f(s)g(x,s)ds + \int_x^\infty f(s)g(x,s)ds.$$
$$y(x) = \int_0^x f(s)\frac{1}{2}(x-s)^2 ds + \int_x^\infty f(s)0ds.$$
$$y(x) = \int_0^x f(s)\frac{1}{2}(x-s)^2 ds$$

Note that $y(0) = 0$ by construction. The first derivative, by Leibniz's rule, is

$$\frac{dy}{dx} = \int_0^x f(s)(x-s)ds.$$

Note that $y'(0) = 0$. The second derivative, by Leibniz's rule, is

$$\frac{d^2y}{dx^2} = \int_0^x f(s)ds.$$

Note the $y''(0) = 0$. The third derivative, by Leibniz's rule, is

$$\frac{d^3y}{dx^3} = f(x),$$

which is the original equation.

Let us check this when $f(x) = 1$.

$$y(x) = \int_0^x (1)\frac{1}{2}(x-s)^2 ds.$$
$$y(x) = -\frac{1}{2}\frac{1}{3}(x-s)^3\Big|_{s=0}^{s=x},$$
$$y(x) = \frac{x^3}{6}.$$

By inspection $y' = x^2/2$, $y'' = x$, $y''' = 1$, and all of the conditions are satisfied at $x = 0$.

3. (25) Find an exact solution for $y(x)$ if

$$x^2\frac{d^2y}{dx^2} + y = 0, \qquad y(1) = 0, \quad y'(1) = 1.$$

Solution

This is an Euler equation. Many people recognized this but did not follow through well. Some made a lot of progress on this.

Let us take the transformation

$$z = \ln x.$$

The inverse transformation is thus

$$x = e^z.$$

When $x = 1$, we get $z = 0$. We also have

$$\frac{dz}{dx} = \frac{1}{x} = e^{-z}.$$

Thus

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^{-z}\frac{dy}{dz}.$$

Thus

$$\frac{d}{dx} = e^{-z}\frac{d}{dz}.$$

Transforming the original ODE into z space, we find

$$e^{2z}e^{-z}\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right) + y = 0, \qquad y(0) = 0, \qquad \frac{dy}{dz}(0) = 1.$$

Expanding, we get

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + y = 0.$$

This is linear with constant coefficients. Take $y = Ae^{rz}$, which yields a characteristic polynomial

$$r^2 - r + 1 = 0.$$

This has roots

$$r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

So we can say

$$y(z) = e^{z/2}\left(C_1 \sin\left(\frac{\sqrt{3}z}{2}\right) + C_2 \cos\left(\frac{\sqrt{3}z}{2}\right)\right).$$

Now $y(z = 0) = 0$, so we find $C_2 = 0$. This leaves us with

$$y(z) = C_1 e^{z/2} \sin\left(\frac{\sqrt{3}z}{2}\right).$$

The derivative is

$$\frac{dy}{dz} = C_1 e^{z/2}\left(\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}z}{2}\right) + \frac{1}{2}\sin\left(\frac{\sqrt{3}z}{2}\right)\right)$$

Imposing the boundary condition, we get

$$1 = C_1\left(\frac{\sqrt{3}}{2}\right)$$

So

$$C_1 = \frac{2}{\sqrt{3}},$$

and

$$y(z) = \frac{2}{\sqrt{3}} e^{z/2} \sin\left(\frac{\sqrt{3}z}{2}\right).$$

In terms of x, we get

$$y(x) = \frac{2\sqrt{x}}{\sqrt{3}} \sin\left(\frac{\sqrt{3}\ln x}{2}\right).$$

The character of the solution is best revealed in the log-log plot of Fig. **??**. We plot

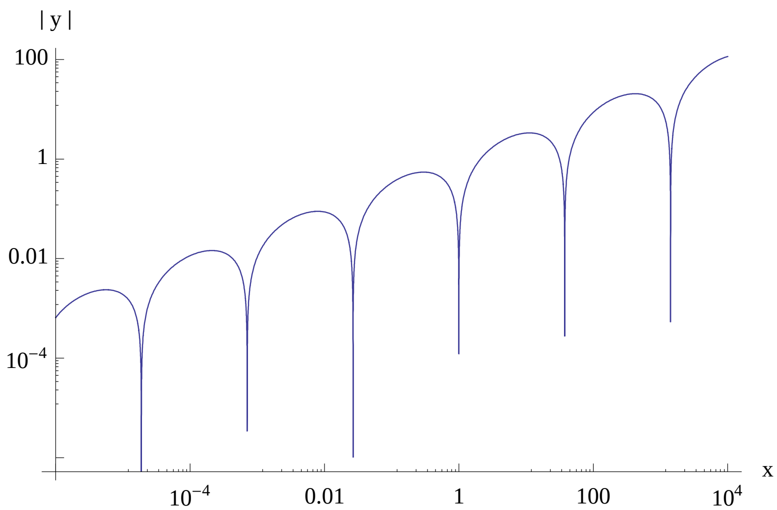

Figure 2: $|y(x)|$ for solution to $x^2y'' + y = 0, y(1) = 0, y'(1) = 1$.

$|y|$ because y itself is often negative, and this cannot be plotted on the log scale. The spikes indicated where zero crossings occur on a linear scale. We see there are many zero crossings. Moreover the amplitude is growing as $\sqrt{x}$. This is manifested as linear amplitude growth on the log scale.

4. (25) For $0 < \epsilon << 1$, find a uniformly valid approximate solution to $y(x)$ which satisfies the differential equation and boundary conditions

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy^2 = 0, \quad y(0) = 1, \; y(1) = 1.$$

Solution

Most people made a good try at this and a few got it fully correct.

First get the outer solution. Set $\epsilon = 0$ and solve

$$\frac{dy}{dx} + xy^2 = 0.$$

By separation of variables, we get

$$-\frac{dy}{y^2} = xdx.$$

Solving, we get

$$\frac{1}{y} = \frac{1}{2}x^2 + C,$$

$$y = \frac{2}{x^2 + 2C}.$$

Let's match the condition at $x = 1$ for the outer solution. If this proves not to work, we can try the other choice. But it will work, so there will be no need.

$$1 = \frac{2}{1^2 + 2C}.$$

$$2C + 1 = 2.$$

$$C = \frac{1}{2}.$$

Thus

$$y_{outer} = \frac{2}{x^2 + 1}.$$

Note that $y_{outer}(0) = 2$, which does not satisfy the boundary condition at $x = 0$. So we need an inner layer.

Let's try a new variable:

$$X = \frac{x}{\epsilon}.$$

This gives $d/dx = (1/\epsilon)d/dX$, and our differential equation transforms to

$$\frac{1}{\epsilon}\frac{d^2y}{dX^2} + \frac{1}{\epsilon}\frac{dy}{dX} + \epsilon X y^2 = 0.$$

$$\frac{d^2y}{dX^2} + \frac{dy}{dX} + \epsilon^2 X y^2 = 0.$$

At leading order this becomes

$$\frac{d^2y}{dX^2} + \frac{dy}{dX} = 0.$$

Assuming solutions of the type, $y = Ae^{rx}$, we find a characteristic polynomial of $r^2 + r = 0$, which has solutions $r = 0, -1$. So

$$y_{innner} = C_1 + C_2 e^{-X}.$$

Now at $x = 0$, we have

$$1 = C_1 + C_2 e^{-0},$$

so

$$C_2 = 1 - C_1.$$

Thus

$$y_{inner} = C_1 + (1 - C_1)e^{-X}.$$

Now as $x \to 0$, we have $y_{outer} \to 2$. And as $X \to \infty$, we have

$$y_{inner} \to C_1.$$

So for a proper matching, we require $C_1 = 2$. Adding the inner and outer solutions, then subtraction the common part of 2, we recover the uniformly valid solution

$$y \sim \frac{2}{x^2+1} - e^{-x/\epsilon}.$$

The full solution obtained by numerical integration, the outer solution, and the uniformly valid solution are plotted in Fig. **??**.

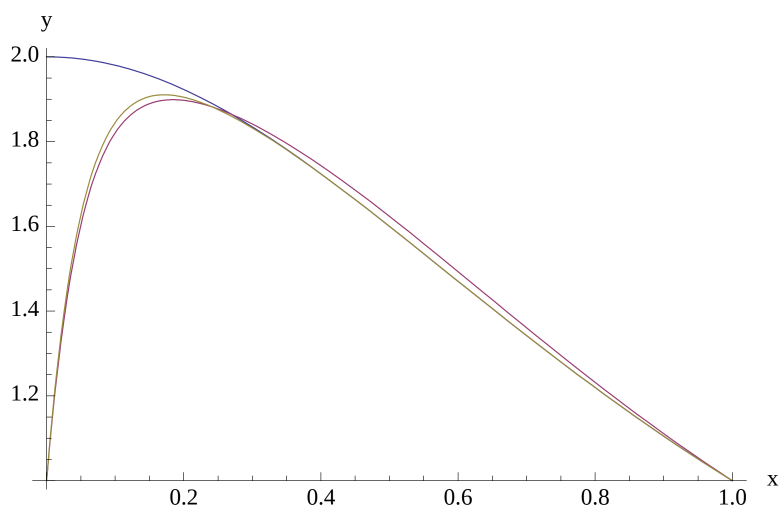

Figure 3: Full, outer, and uniformly valid asymptotic solution to $\epsilon y'' + y' + xy^2 = 0, y(0) = 1, y(1) = 1$, $\epsilon = 1/20$.