AME 60611
Examination 2: SOLUTION
Prof. J. M. Powers
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1. (20) Given $x \in \mathbb{R}^1$, $f : \mathbb{R}^1 \to \mathbb{R}^1$,

$$f(x) = e^x, \qquad x \in [-1, 1],$$

find the first two terms in a Fourier-Legendre expansion of $f(x)$. The first two Legendre polynomials are $P_o(s) = 1$, $P_1(s) = s$, $s \in [-1, 1]$.

Solution

$$f(x) = e^x = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e}x + \ldots.$$

2. (20) If $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$, evaluate

$$I = \oint_C \mathbf{u}^T \cdot d\mathbf{r},$$

where C is a single loop around the closed contour described by the ellipse, $x^2 + 2y^2 = 1$.

Solution

Here we happen to have

$$\mathbf{u} = \nabla\phi,$$

where the potential ϕ is

$$\phi = \frac{1}{2}(x^2 + y^2).$$

So the integral becomes

$$I = \oint_C \nabla\phi \cdot d\mathbf{r}.$$

$$I = \oint_C d\phi = 0.$$

Green's theorem can also be used to achieve the same result.

3. (20) For $x \in \overline{\mathbb{L}}_2[0,1]$, $y \in \overline{\mathbb{L}}_2[0,1]$, $t \in \mathbb{R}^1$, find the angle between the two functions

$$x(t) = it,$$
$$y(t) = i.$$

Solution

$$||x||_2 = \frac{1}{\sqrt{3}}.$$
$$||y||_2 = 1.$$
$$<x, y> = \frac{1}{2}.$$
$$\cos\theta = \frac{\sqrt{3}}{2}.$$
$$\theta = \frac{\pi}{6}.$$

4. (20) For $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^2$, find the vector $\mathbf{x} \in \mathbb{R}^3$ of minimum $||\mathbf{x}||_2$ which minimizes $||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Solution

$$\mathbf{x} = \frac{11}{60} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

5. (20) Find the singular value decomposition of $\mathbf{A}$ where

$$\mathbf{A} = \begin{pmatrix} 3 & 4i & 0 \end{pmatrix}.$$

Solution

One decomposition is

$$\mathbf{A} = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3/5 & 4i/5 & 0 \\ 4i/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The decomposition is not unique.