AME 60611
Examination 1: Solution
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1. (25) If

$$u^2 + v^2 + x^2 + y^2 = 1, \qquad u + 2v^3 + 3x^3 + 4y^4 = 0,$$

find $\left.\frac{\partial u}{\partial x}\right|_y$.

*Solution*

Let

$$f(u, v, x, y) = u^2 + v^2 + x^2 + y^2 - 1 = 0,$$
$$g(u, v, x, y) = u + 2v^3 + 3x^3 + 4y^4 = 0.$$

Then, differentiating gives

$$\begin{aligned} df &= 2u\ du + 2v\ dv + 2x\ dx + 2y\ dy = 0, \\ dg &= du + 6v^2\ dv + 9x^2\ dx + 16y^3\ dy = 0. \end{aligned}$$

Now, we consider $dy = 0$, so

$$\begin{aligned} df &= 2u\ du + 2v\ dv + 2x\ dx = 0, \\ dg &= du + 6v^2\ dv + 9x^2\ dx = 0. \end{aligned}$$

Divide by $dx$ to get

$$\begin{aligned} 2u\ \left.\frac{\partial u}{\partial x}\right|_y + 2v\ \left.\frac{\partial v}{\partial x}\right|_y + 2x &= 0, \\ \left.\frac{\partial u}{\partial x}\right|_y + 6v^2\ \left.\frac{\partial v}{\partial x}\right|_y + 9x^2 &= 0. \end{aligned}$$

In matrix form, we get

$$\begin{pmatrix} 2u & 2v \\ 1 & 6v^2 \end{pmatrix} \begin{pmatrix} \left.\frac{\partial u}{\partial x}\right|_y \\ \left.\frac{\partial v}{\partial x}\right|_y \end{pmatrix} = \begin{pmatrix} -2x \\ -9x^2 \end{pmatrix}.$$

Solve for $\left.\frac{\partial u}{\partial x}\right|_y$ via Cramer's rule:

$$\left.\frac{\partial u}{\partial x}\right|_y = \frac{\begin{vmatrix} -2x & 2v \\ -9x^2 & 6v^2 \end{vmatrix}}{\begin{vmatrix} 2u & 2v \\ 1 & 6v^2 \end{vmatrix}} = \frac{-12xv^2 + 18x^2 v}{12uv^2 - 2v} = \frac{-6xv + 9x^2}{6uv - 1}.$$

2. (25) Solve

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} = x, \qquad y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$$

(You may find the error function, defined as $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\ ds$, to be useful.)

*Solution*

First note, there was a small error in the definition of the error function posted in the exam. The present version is correct.

Let $u = d^2y/dx^2$. Then the equation becomes

$$\frac{du}{dx} + xu = x, \qquad u(0) = 0.$$

This has an integrating factor of

$$e^{\int x\ dx} = e^{x^2/2}.$$

Multiply the first order ODE by the integrating factor to get

$$e^{x^2/2}\frac{du}{dx} + xe^{x^2/2}u = xe^{x^2/2}.$$

Use the product rule and obtain

$$\frac{d}{dx}\left(e^{x^2/2}u\right) = xe^{x^2/2}.$$

Integrate both sides and recover

$$e^{x^2/2}u = \int xe^{x^2/2}\ dx.$$

For the right side, take $v = x^2/2$, so $dv = x\ dx$. The integral on the right side is then

$$\int xe^{x^2/2}\ dx = \int e^v\ dv = e^v + C_1 = e^{x^2/2} + C.$$

So we get

$$e^{x^2/2}u = e^{x^2/2} + C_1,$$

or

$$u = 1 + C_1e^{-x^2/2}.$$

For $u(0) = 0$, we need $C_1 = -1$, so

$$u = 1 - e^{-x^2/2}.$$

Now return to $y$:

$$\frac{d^2y}{dx^2} = 1 - e^{-x^2/2}.$$

Integrate to get

$$\frac{dy}{dx} = x - \int_0^x e^{-s^2/2}\ ds + C_2.$$

We need $y'(0) = 0$, so $C_2 = 0$. Thus

$$\frac{dy}{dx} = x - \int_0^x e^{-s^2/2}\ ds.$$

Integrate once more to get

$$y(x) = \frac{1}{2}x^2 - \int_0^x \int_0^t e^{-s^2/2}\ ds\ dt + C_3.$$

Now $y(0) = 0$, so $C_3 = 0$. Thus,

$$y(x) = \frac{1}{2}x^2 - \int_0^x \int_0^t e^{-s^2/2}\ ds\ dt.$$

Some simplification can be done on this expression. This can be achieved by changing the order of integration. Care must be used to change the limits correctly. In $s - t$ space, the domain of integration is a triangular region bounded by $s = 0, s = t$, then by $t = 0, t = x$. This is emphasized by explicitly writing

$$y(x) = \frac{1}{2}x^2 - \int_{t=0}^{t=x} \int_{s=0}^{s=t} e^{-s^2/2}\ ds\ dt.$$

Let us now change the order of integration:

$$y(x) = \frac{1}{2}x^2 - \int_{s=0}^{s=x} \int_{t=s}^{t=x} e^{-s^2/2}\ dt\ ds.$$

We can then bring $e^{-s^2/2}$ outside the first integral to get

$$y(x) = \frac{1}{2}x^2 - \int_{s=0}^{s=x} e^{-s^2/2} \int_{t=s}^{t=x} dt\ ds.$$

The inner integral can then be taken and evaluated at the appropriate limits to get

$$y(x) = \frac{1}{2}x^2 - \int_{s=0}^{s=x} e^{-s^2/2}(x - s)\ ds.$$

Now split the integral into two parts:

$$y(x) = \frac{1}{2}x^2 - x\int_{s=0}^{s=x} e^{-s^2/2}\ ds + \int_{s=0}^{s=x} se^{-s^2/2}\ ds$$

Taking once more $v = -s^2/2$ with $dv = -sds$, we get

$$y(x) = \frac{1}{2}x^2 - x\int_{s=0}^{s=x} e^{-s^2/2}\ ds - \int_{v=0}^{v=-x^2/2} e^v\ dv$$

Integrating the final term, we find

$$y(x) = \frac{1}{2}x^2 - x\int_{s=0}^{s=x} e^{-s^2/2}\ ds - \left(e^{-x^2/2} - 1\right).$$

Rearranging, we get

$$y(x) = 1 + \frac{1}{2}x^2 - e^{-x^2/2} - x\int_0^x e^{-s^2/2}\ ds.$$

In terms of the error function, we can take $\tau = s/\sqrt{2}$. This gives $d\tau = ds/\sqrt{2}$, or $ds = \sqrt{2}d\tau$. So our solution becomes

$$y(x) = 1 + \frac{1}{2}x^2 - e^{-x^2/2} - \frac{\sqrt{2\pi}}{\sqrt{\pi}}\frac{\sqrt{2}}{\sqrt{2}}x\int_0^{x/\sqrt{2}} e^{-\tau^2}\ d\tau.$$

$$y(x) = 1 + \frac{1}{2}x^2 - e^{-x^2/2} - \frac{2\sqrt{\pi}}{\sqrt{\pi}}\frac{1}{\sqrt{2}}x\int_0^{x/\sqrt{2}} e^{-\tau^2}\ d\tau.$$

$$y(x) = 1 + \frac{1}{2}x^2 - e^{-x^2/2} - \sqrt{\frac{\pi}{2}}x\ \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right).$$

The solution is plotted in Fig. 1.

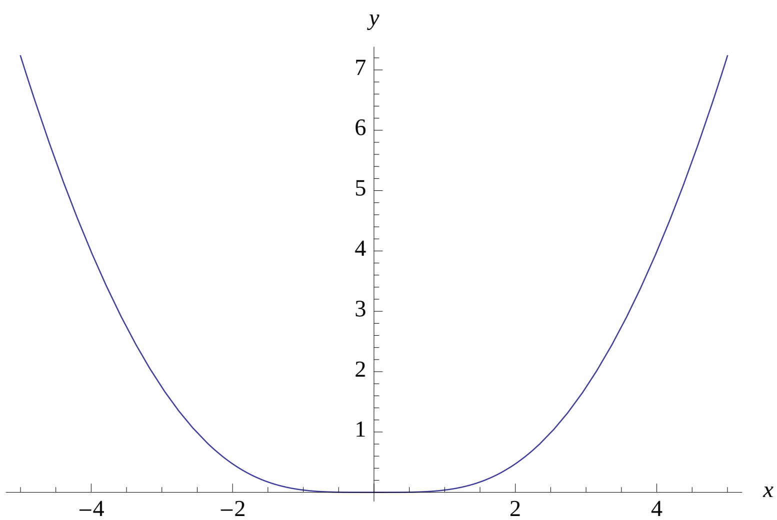


Figure 1: $y(x)$ for problem 2

3. (25) For $0 < \epsilon << 1$, use boundary layer methods to find a uniformly valid asymptotic solution to

$$\epsilon\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - y = 0, \qquad y(0) = 0,\ y(1) = 0.$$

*Solution*

The problem is linear, so if a solution exists, it will be unique. By inspection, the solution

$$y(x) = 0,$$

satisfies the differential equation and both boundary conditions, so this is the uniformly valid solution!

*Solution*

The problem is more interesting *when my typographical error is removed*, and the boundary conditions are replaced by $y(0) = y(1) = 1$. In this case, we first look for the outer solution by solving

$$-x^2 \frac{dy}{dx} - y = 0.$$

We separate variables to get

$$\frac{dy}{y} = -\frac{dx}{x^2}.$$

$$\ln y = \frac{1}{x} + C'.$$

Solving for $y$, taking $C_o = e^{C'}$, we find

$$y = C_o e^{1/x}.$$

Let us put a boundary layer of thickness $\epsilon$ at $x = 1$. So let us define a transformed distance as

$$X \equiv \frac{1-x}{\epsilon}.$$

So $dX/dx = -1/\epsilon$. By the chain rule $dy/dx = (dy/dX)(dX/dx) = -(1/\epsilon)dy/dX$. Thus $d^2y/dx^2 = (1/\epsilon^2)d^2y/dX^2$. So our ODE becomes

$$\epsilon \frac{1}{\epsilon^2}\frac{d^2y}{dX^2} + (\epsilon X - 1)^2 \frac{1}{\epsilon}\frac{dy}{dX} - y = 0.$$

$$\frac{d^2y}{dX^2} + (\epsilon X - 1)^2 \frac{dy}{dX} - \epsilon y = 0.$$

At leading order, this reduces to

$$\frac{d^2y}{dX^2} + \frac{dy}{dX} = 0.$$

This has solution

$$y = A_o + B_o e^{-X}.$$

Now at $x = 1$, we have $X = 0$ and $y = 1$. So $1 = A_o + B_o$, so we have in the layer near $x = 1$ that

$$y = A_o + (1 - A_o)e^{-X}.$$

Now as we move back towards $x = 0$, we find that $X \to \infty$, and the boundary layer solution goes to $y = A_o$. This must match to the outer solution which has $y \to C_o e$ as $x \to 1$. So we take

$$A_o = C_o e.$$

So the inner solution near $x = 1$ is

$$y = C_o e + (1 - C_o e)e^{-X} = C_o e + (1 - C_o e)e^{\frac{x-1}{\epsilon}}.$$

Now the outer solution also has a problem at $x = 0$. So let us propose a boundary layer near $x = 0$. Let us suggest a transformed variable of

$$Z = \frac{x}{\epsilon^\alpha}.$$

So $dZ/dx = \epsilon^{-\alpha}$, $dy/dx = dy/dZ dZ/dx = \epsilon^{-\alpha} dy/dZ$, and $d^2y/dx^2 = \epsilon^{-2\alpha} d^2y/dZ^2$. Our ODE become in this region

$$\epsilon^{1-2\alpha} \frac{d^2y}{dZ^2} - \epsilon^{\alpha} Z^2 \frac{dy}{dZ} - y = 0.$$

Let us balance the first and third terms. Other choices could be made, but lead to inconsistencies. So choose $\alpha$ such that $1 - 2\alpha = 0$. This gives $\alpha = 1/2$. So our scaling is $Z = x/\epsilon^{1/2}$, and our ODE becomes

$$\frac{d^2y}{dZ^2} - \epsilon^{1/2} Z^2 \frac{dy}{dZ} - y = 0.$$

At leading order, this is

$$\frac{d^2y}{dZ^2} - y = 0.$$

This has solution

$$y = D_o e^{Z} + E_o e^{-Z}.$$

At the boundary at $x = 0$, we have $Z = 0$ and $y = 1$. Thus we insist that

$$1 = D_o + E_o.$$

To prevent exponential growth away from $x = 0$, we insist next that $D_o = 0$. Thus $E_o = 1$. So we have in the inner layer that

$$y = e^{-Z}.$$

In order to match the inner layer to the outer layer we must take

$$C_o = 0.$$

So our composite solution is the two inners plus the outer (zero) minus the two common parts (both of which are zero):

$$y = e^{-x/\epsilon^{1/2}} + e^{\frac{x-1}{\epsilon}}.$$

The solution is plotted in Fig. 2.

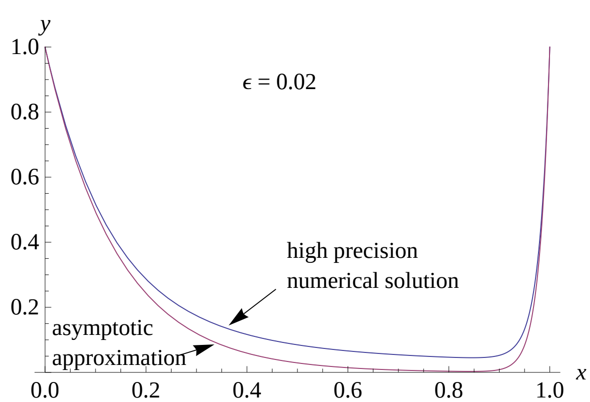


Figure 2: $y(x)$ for the modified problem 3: $\epsilon y'' - x^2 y' - y = 0, y(0) = y(1) = 1$. Here $\epsilon = 0.02$.

4. (25) Find a general solution to

$$\frac{d^2y}{dx^2} + y = x^2 + e^x.$$

*Solution*

The homogeneous part has solution

$$y = C_1 \sin x + C_2 \cos x.$$

For the particular solution, we look for solutions of the form

$$y = A_0 + A_1 x + A_2 x^2 + B_0 e^x.$$

The first and second derivatives of the particular solution are

$$\frac{dy}{dx} = A_1 + 2A_2 x + B_0 e^x.$$

$$\frac{d^2y}{dx^2} = 2A_2 + B_0 e^x.$$

Substituting into the ODE, we get

$$2A_2 + B_0 e^x + A_0 + A_1 x + A_2 x^2 + B_0 e^x = x^2 + e^x.$$

Regrouping, we find

$$(2A_2 + A_0)x^0 + (A_1)x^1 + (A_2 - 1)x^2 + (2B_0 - 1)e^x = 0.$$

Since all the functions of $x$ are linearly independent, we insist that their coefficients be zero, leading to a solution of

$$A_0 = -2,$$
$$A_1 = 0,$$
$$A_2 = 1,$$
$$B_0 = \frac{1}{2}.$$

Thus the general solution is

$$y = C_1 \sin x + C_2 \cos x - 2 + x^2 + \frac{1}{2}e^x.$$