AME 60611
Examination 2: SOLUTION
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23 November 2009

1. (20) For $x \in [0,3]$, find the first two terms in a Fourier-Legendre expansion of the Heaviside unit step function:

$$f(x) = H(x-1).$$

The first two Legendre polynomials are $P_o(s) = 1$, $P_1(s) = s$, for $s \in [-1,1]$.

Solution

There are a variety of ways to approach this problem. We first need to transform the variables so the domain is aligned with the domain on which the Legendre polynomials are orthogonal. We take

$$\hat{x} = a + bx.$$

We need $x \in [0,3]$ to map to $\hat{x} \in [-1,1]$. So take

$$-1 = a + b(0)$$

$$1 = a + b(3)$$

Therefore

$$a = -1, \qquad b = \frac{2}{3}.$$

and

$$\hat{x} = -1 + \frac{2}{3}x.$$

Inverting, we get

$$x = \frac{3}{2}(\hat{x}+1).$$

So expanding $H(x-1)$ corresponds to expanding

$$H\left(\frac{3}{2}(\hat{x}+1) - 1\right) = H\left(\frac{3}{2}\hat{x} + \frac{1}{2}\right).$$

So we seek the expansion

$$f(\hat{x}) = \sum_{i=0}^{1} c_i P_i(\hat{x}).$$

$$\int_{-1}^{1} f(\hat{x}) P_j(\hat{x}) d\hat{x} = \sum_{i=0}^{1} c_i \int_{-1}^{1} P_i(\hat{x}) P_j(\hat{x}) d\hat{x}.$$

$$\int_{-1}^{1} f(\hat{x}) P_j(\hat{x}) d\hat{x} = c_j \int_{-1}^{1} P_j(\hat{x}) P_j(\hat{x}) d\hat{x}.$$

$$c_j = \frac{\int_{-1}^{1} f(\hat{x}) P_j(\hat{x}) d\hat{x}}{\int_{-1}^{1} P_j(\hat{x}) P_j(\hat{x}) d\hat{x}.}$$

$$c_j = \frac{\int_{-1}^{1} H\left(\frac{3}{2}\hat{x} + \frac{1}{2}\right) P_j(\hat{x}) d\hat{x}}{\int_{-1}^{1} P_j(\hat{x}) P_j(\hat{x}) d\hat{x}.}$$

Now for $\hat{x} < -1/3$, the Heaviside function maps to zero; for $\hat{x} > -1/3$, it maps to one, so we have

$$c_j = \frac{\int_{-1/3}^{1} P_j(\hat{x}) d\hat{x}}{\int_{-1}^{1} P_j(\hat{x}) P_j(\hat{x}) d\hat{x}.}$$

So

$$c_0 = \frac{\int_{-1/3}^{1} (1) d\hat{x}}{\int_{-1}^{1} (1)(1) d\hat{x}.}$$

$$c_0 = \frac{4/3}{2} = \frac{2}{3}.$$

For the next term, we have

$$c_1 = \frac{\int_{-1/3}^{1} \hat{x} d\hat{x}}{\int_{-1}^{1} \hat{x}^2 d\hat{x}.} = \frac{\hat{x}^2/2\big|_{-1/3}^{1}}{\hat{x}^3/3\big|_{-1}^{1}} = \frac{1/2 - 1/18}{1/3 + 1/3} = \frac{2}{3}$$

So our approximation is

$$f(\hat{x}) \sim \frac{2}{3} + \frac{2}{3}\hat{x}.$$

In terms of x, we then get

$$f(x) \sim \frac{2}{3} + \frac{2}{3}\left(-1 + \frac{2}{3}x\right),$$

or

$$f(x) \sim \frac{4}{9}x.$$

The original function along with the two-term approximation is given in Figure 1. A twenty-term Fourier-Legendre approximation is given in Figure 2 for comparison.

2. (20) Given

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

where $\alpha \in \mathbb{R}^1$, $\beta \in \mathbb{R}^1$, $\gamma \in \mathbb{R}^1$ find the conditions on α, β, and γ which insure that $\mathbf{A}$ is positive definite.

Solution

For $\mathbf{A}$ to be positive definite, we must have

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} > 0$$

for all $\mathbf{x} \neq \mathbf{0}$. There are three good approaches to this:

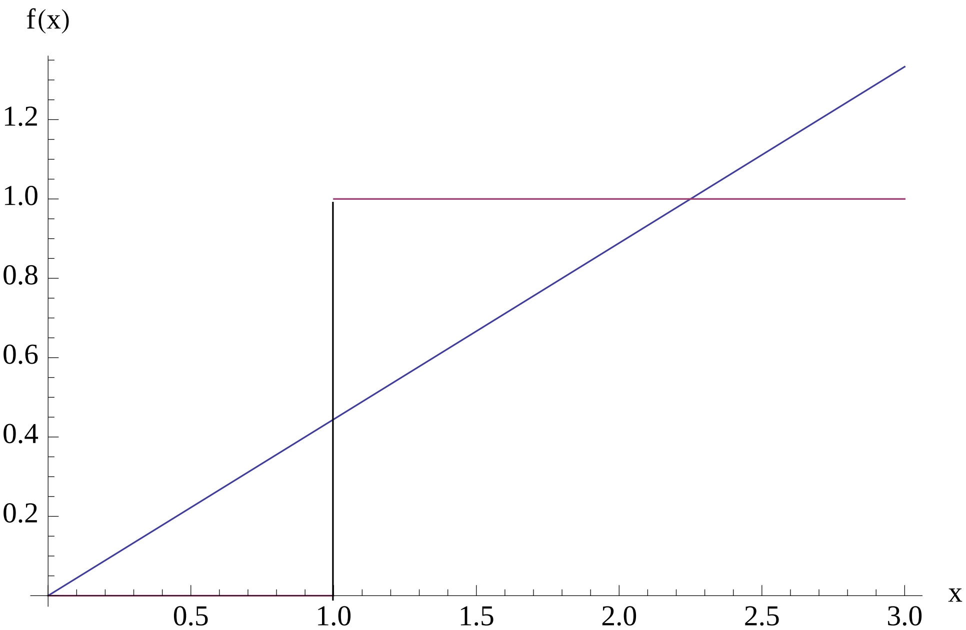

Figure 1: $H(x-1)$ and its two term Fourier-Legendre approximation for $x \in [0,3]$.

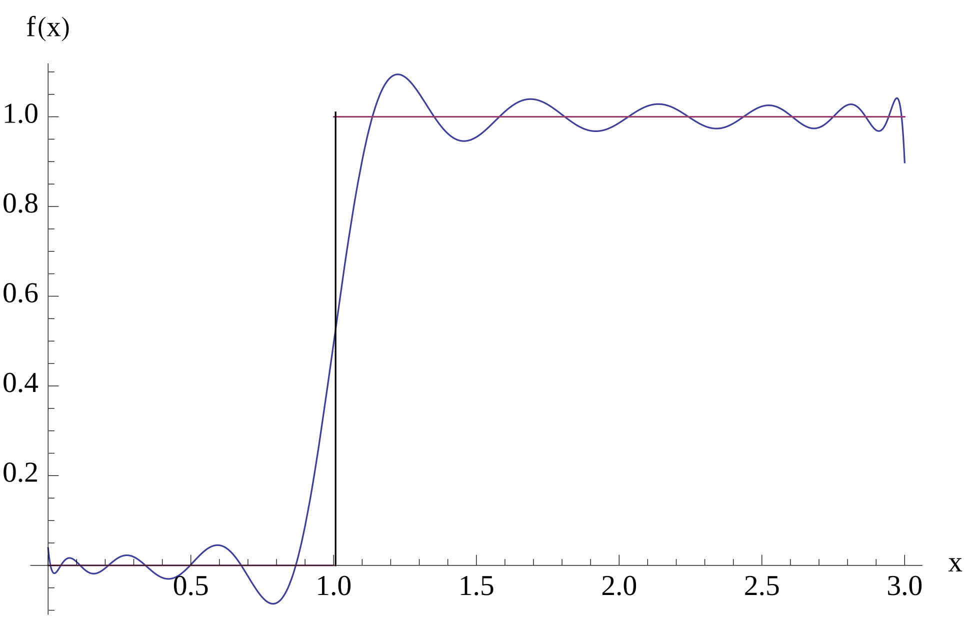

Figure 2: $H(x-1)$ and its twenty term Fourier-Legendre approximation for $x \in [0,3]$.

Approach 1

The condition for this is known to be that the eigenvalues of $\mathbf{A}$ must be positive. Note that they are already guaranteed to be real since $\mathbf{A}$ is self-adjoint. For example, if $\mathbf{x}$ happens to be an eigenvector of $\mathbf{A}$, then $\mathbf{A} \cdot \mathbf{x} = \lambda \mathbf{x}$, and our condition becomes

$$\mathbf{x}^T \cdot \lambda \mathbf{x} > 0,$$

or

$$\lambda \mathbf{x}^T \cdot \mathbf{x} > 0,$$

or

$$\lambda ||\mathbf{x}||_2^2 > 0.$$

Since $||\mathbf{x}||_2^2 > 0$, we then simply require

$$\lambda > 0.$$

Let us find the eigenvalues of $\mathbf{A}$:

$$\begin{vmatrix} \alpha - \lambda & \beta \\ \beta & \gamma - \lambda \end{vmatrix} = 0.$$

$$(\alpha - \lambda)(\gamma - \lambda) - \beta^2 = 0.$$

In the special case in which $\beta = 0$, we get the two roots $\lambda = \alpha$ and $\lambda = \gamma$. So when $\beta = 0$, we must insist that $\alpha > 0$ and $\beta > 0$ for a positive definite matrix. For general β, we expand our condition to get

$$\lambda^2 - (\alpha + \gamma)\lambda + \alpha\gamma - \beta^2 = 0.$$

We solve for λ and write the result in three equivalent forms:

$$\lambda = \frac{\alpha + \gamma \pm \sqrt{(\alpha + \gamma)^2 + 4\beta^2 - 4\alpha\gamma}}{2}, \quad (1)$$

$$\lambda = \frac{\alpha + \gamma \pm \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}, \quad (2)$$

$$\lambda = \frac{\alpha + \gamma}{2}\left(1 \pm \sqrt{1 + \frac{4(\beta^2 - \alpha\gamma)}{(\alpha + \gamma)^2}}\right). \quad (3)$$

Eq. (2) shows us that λ is guaranteed real, since the argument of the square root is never negative. Eq. (3) shows us the argument of the square root can be greater or less than unity. If $\beta^2 - \alpha\gamma > 0$, the argument of the square root will be greater than unity, and the two values of λ will be guaranteed to be one positive and one negative. If $\beta^2 < \alpha\gamma$, then the argument of the square root will be less than unity, and the term in parentheses will be guaranteed positive. We must then also insist that $\alpha + \gamma > 0$. This condition would seem to imply that we can tolerate α and γ of opposite sign, so long as their sum is positive. However, if α and γ are of opposite sign, we cannot satisfy $\beta^2 < \alpha\gamma$. So our conditions are simply

$$\alpha > 0, \qquad \gamma > 0, \qquad \beta < +\sqrt{\alpha\gamma}.$$

Roughly speaking, the matrix is positive definite if the diagonal elements are positive and large relative to the magnitude of the off-diagonal elements.

Approach 2

Consider the actual positive definite requirement explicitly:

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > 0.$$

Expand to get

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \alpha x_1 + \beta x_2 \\ \beta x_1 + \gamma x_2 \end{pmatrix} > 0.$$

Further expand to get the polynomial

$$\alpha x_1^2 + 2\beta x_1 x_2 + \gamma x_2^2 > 0.$$

This must hold for all $x_1 \in (-\infty, \infty),\ x_2 \in (-\infty, \infty)$. If $x_1 = 0$, we must have

$$\gamma x_2^2 > 0,$$

and if $x_2 = 0$, we must have

$$\alpha x_1^2 > 0$$

. Both of these conditions induce the requirements that

$$\alpha > 0, \qquad \gamma > 0.$$

Now divide by α to get

$$x_1^2 + 2\frac{\beta}{\alpha} x_1 x_2 + \frac{\gamma}{\alpha} x_2^2 > 0.$$

Now use the "complete the square" procedure to get

$$x_1^2 + 2\frac{\beta}{\alpha} x_1 x_2 + \left(\frac{\beta}{\alpha}\right)^2 x_2^2 - \left(\frac{\beta}{\alpha}\right)^2 x_2^2 + \frac{\gamma}{\alpha} x_2^2 > 0.$$

$$\left(x_1 + \frac{\beta}{\alpha} x_2\right)^2 + \left(\frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha^2}\right) x_2^2 > 0.$$

For this to hold, we need

$$\frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha^2} > 0.$$

$$\alpha\gamma - \beta^2 > 0,$$

$$\alpha\gamma > \beta^2.$$

Approach 3

This approach is an intuitive special case of that presented for three-dimensional matrices. The characteristic polynomial for the eigenvalues is, as already seen,

$$\lambda^2 - (\alpha + \gamma)\lambda + (\alpha\gamma - \beta^2) = 0.$$

This takes the form

$$\lambda^2 - I^{(1)}\lambda + I^{(2)} = 0,$$

with the so-called invariants of $\mathbf{A}$ being

$$I^{(1)} = \alpha + \beta = \mathrm{Tr}(\mathbf{A}) = \lambda^{(1)} + \lambda^{(2)},$$

$$I^{(2)} = \alpha\gamma - \beta^2 = \det(\mathbf{A}) = \lambda^{(1)}\lambda^{(2)}.$$

For positive eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$, one needs $\mathrm{Tr}(\mathbf{A}) > 0$, and $\det(\mathbf{A}) > 0$. This implies that

$$\alpha\gamma > \beta^2, \qquad \alpha > 0, \qquad \gamma > 0.$$

3. (20) For $t \in [0,1]$, find all approximate solutions available from a one-term Galerkin method applied to the differential equation and initial conditions

$$\frac{d^2y}{dt^2} + y^2 = 1, \qquad y(0) = 0, \quad \left.\frac{dy}{dt}\right|_{t=0} = 0.$$

Solution

Let us choose a trial function as

$$\phi(t) = a + bt + ct^2.$$

We need the trial function to satisfy both initial conditions. So

$$\phi(0) = 0 = a + b(0) + c(0)^2.$$

So $a = 0$. So

$$\phi(t) = bt + ct^2.$$

Now

$$\frac{d\phi}{dt} = b + 2ct.$$

And at $t = 0$, we have

$$0 = b + 2c(0).$$

So $b = 0$. Let us take $c = 1$. So our trial function is

$$\phi(t) = t^2.$$

Now seek an approximate solution of the form

$$y_a(t) = Ct^2.$$

We get the error, $e(t)$ via

$$e(t) = \frac{d^2y_a}{dt^2} + y_a^2 - 1.$$

Substituting our $y_a = Ct^2$, we get

$$e(t) = 2C + C^2t^4 - 1.$$

Now let us drive a weighted error to zero:

$$0 = \int_0^1 \psi(t)e(t)dt.$$

For the Galerkin method, we take $\psi(t) = \phi(t) = t^2$, so we solve

$$0 = \int_0^1 t^2(2C + C^2t^4 - 1)dt.$$

$$0 = \left[-\frac{t^3}{3} + \frac{2Ct^3}{3} + \frac{C^2t^7}{7}\right]_0^1.$$

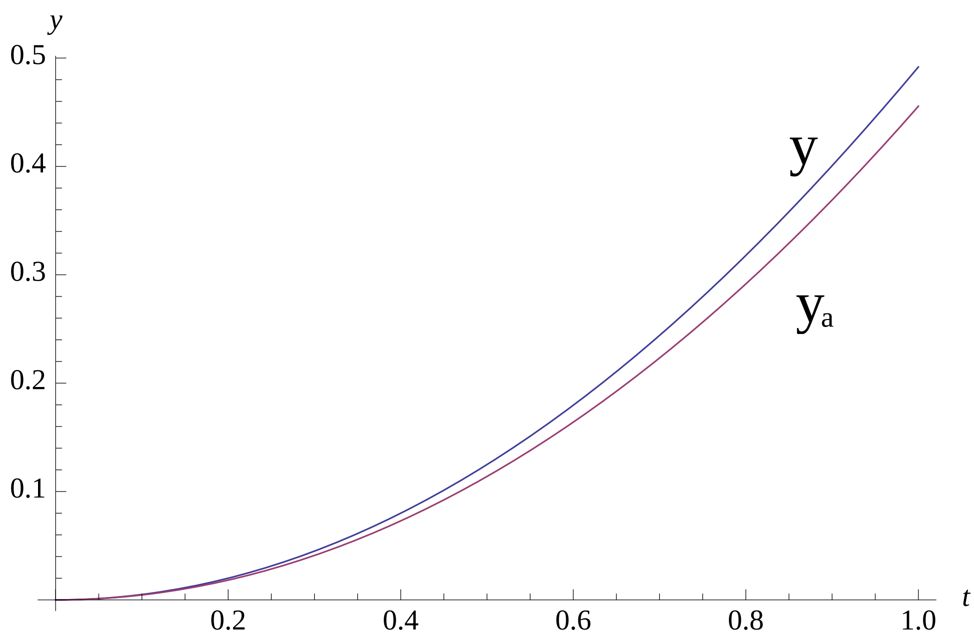

Figure 3: Plot of a high resolution numerical solution for $y(t)$ as well as $y_a(t)$ for $t \in [0, 1]$. Here the positive solution was selected.

$$0 = -\frac{1}{3} + \frac{2C}{3} + \frac{C^2}{7}.$$

Solving, we get

$$C = \frac{1}{3}\left(-7 \pm \sqrt{70}\right).$$

So our Galerkin approximation is

$$y_a(t) = \frac{1}{3}\left(-7 \pm \sqrt{70}\right) t^2.$$

Note the solution is non-unique. Taking the positive C and comparing to a high resolution numerical solution for y, we see the Galerkin approximation is good, as shown in Figure 3.

4. (20) Find the $\mathbf{x}$ of minimum $||\mathbf{x}||_2$ which minimizes the quantity $||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution

Let us apply the operator $\mathbf{A}^T$ to both sides:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Expanding, we get

$$\begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}.$$

Row echelon form gives us

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 0 \\ 0 \end{pmatrix}.$$

We take $x_2 = s$ and $x_3 = t$ to be free variables, yielding

$$x_1 = \frac{4}{5} - s - t.$$

So our solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Let us find the portion of the vector $(4/5, 0, 0)^T$ that lies in the null space. We solve for

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4/5 \\ 0 \\ 0 \end{pmatrix}.$$

Note the columns of the coefficient matrix are formed of the row space vector and the two null space vectors. Solving, we get

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4/15 \\ -4/15 \\ -4/15 \end{pmatrix}.$$

So we can write the solution vector as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{4}{15} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (s - 4/15) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + (t - 4/15) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus the vector $\mathbf{x}$ with minimum norm is found when $s = 4/15$ and $t = 4/15$ and is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{4}{15} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This value of $\mathbf{x}$ yields

$$\mathbf{A} \cdot \mathbf{x} - \mathbf{b} = \begin{pmatrix} -6/5 \\ 3/5 \end{pmatrix},$$

which gives

$$||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2 = \frac{3\sqrt{5}}{5}.$$

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5. (20) For a scalar field $\phi(x_i)$, use Cartesian index notation and prove the curl of the gradient of that scalar field is zero.

Solution

Show then that

$$\nabla \times \nabla \phi = 0.$$

Using index notation, we have

$$\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi = 0?$$

Now if ϕ is continuous and differentiable, the order of differentiation does not matter and we have

$$\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi = \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \phi$$

Thus the term $\partial^2 \phi \partial x_j \partial x_k$ is a symmetric second order tensor. And for a fixed value of i, ϵ_{ijk} is an anti-symmetric tensor. And we know the tensor inner product of an anti-symmetric tensor and a symmetric tensor is zero. QED.