AME 60611
Examination 1: SOLUTION
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1. (25) Consider the transformation relating Cartesian (ξ^1, ξ^2) to a new coordinate system, (x^1, x^2):

$$\begin{aligned} \xi^1 &= x^1, \\ \xi^2 &= -x^1 + x^2. \end{aligned}$$

   (a) Find the Jacobian matrix, the metric tensor, determine if the mapping is orthogonal, area- and orientation-preserving.

   (b) Sketch lines of constant x^1 and x^2 in the (ξ^1, ξ^2) plane.

   (c) For a known function, $\phi(\xi^1, \xi^2)$, find a representation for $\partial\phi/\partial x^1$ and $\partial\phi/\partial x^2$ using appropriate transformation rules.

Solution

There were not too many major issues with this problem. A few too many people insisted the transformation was orthogonal, despite correctly plotting lines of constant x^1 and x^2 which obviously showed non-orthogonality.

The Jacobian is

$$\mathbf{J} = \frac{\partial \xi^i}{\partial x^j} = \begin{pmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}.$$

We see that

$$\boxed{\det \mathbf{J} = 1 - 0 = 1.}$$

Thus, the mapping is area- and orientation-preserving.

The metric tensor is

$$\mathbf{G} = \mathbf{J}^T \cdot \mathbf{J} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}}.$$

Because the off-diagonal elements are non-zero, the mapping is non-orthogonal.

A sketch of the mapping is given in Fig. 1.

From the chain rule, we have

$$\underbrace{\begin{pmatrix} \frac{\partial \phi}{\partial x^1} \\ \frac{\partial \phi}{\partial x^2} \end{pmatrix}}_{\nabla_{\mathbf{x}}\phi} = \underbrace{\begin{pmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^2}{\partial x^1} \\ \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^2}{\partial x^2} \end{pmatrix}}_{\mathbf{J}^T} \cdot \underbrace{\begin{pmatrix} \frac{\partial \phi}{\partial \xi^1} \\ \frac{\partial \phi}{\partial \xi^2} \end{pmatrix}}_{\nabla_{\boldsymbol{\xi}}\phi} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \phi}{\partial \xi^1} \\ \frac{\partial \phi}{\partial \xi^2} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{\partial \phi}{\partial \xi^1} - \frac{\partial \phi}{\partial \xi^2} \\ \frac{\partial \phi}{\partial \xi^1} \end{pmatrix}}.$$

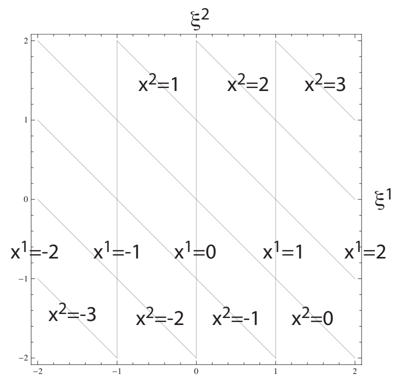

Figure 1: Coordinate transformation

2. (25) Consider on the domain $x \in [0, \infty)$ the differential equation and initial condtion

$$\epsilon \frac{dy}{dx} + y = f(x), \qquad y(0) = 0.$$

(a) For any ϵ, large or small, use the Green's function method to find a solution of the form

$$y(x) = \int_0^\infty g(x,s) f(s) ds.$$

(b) Find $y(x)$ via the Green's function for $f(x) = 1$ and show from direct expansion of the Green's function solution that $y(x) \to f(x) = 1$ as $\epsilon \to 0$.

(c) Discuss the solution for $y(x)$ when $f(x) = 1$ and $0 < \epsilon \ll 1$ in the context of boundary layer theory.

Solution

This problem caused a lot of difficulty and those who became confused should read this solution carefully. Many people did get it essentially correct.

We break the domain $x \in [0, \infty]$ into two parts $x \in [0, s)$, $x \in (s, \infty)$. We then seek a Green's function, $g(x, s)$ which satisfies $\epsilon dg/dx + g = \delta(x - s)$.

For $x < s$, this reduces to solving

$$\epsilon \frac{dg}{dx} + g = 0,$$

$$\frac{dg}{dx} + \frac{1}{\epsilon} g = 0,$$

$$g = C_1 e^{-x/\epsilon},$$

$$g(0) = 0 = C_1 e^{-0/\epsilon} = C_1.$$

So for $x < s$,

$$g = 0.$$

For $x > s$, we have

$$\frac{dg}{dx} + \frac{1}{\epsilon} g = 0,$$

$$g = C_2 e^{-x/\epsilon}.$$

Now at $x = s$, we have the jump condition

$$\underbrace{g|_{s+\delta}}_{C_2 e^{-(s+\delta)/\epsilon}} - \underbrace{g|_{s-\delta}}_{C_1 e^{-(s-\delta)/\epsilon}=0} = \frac{1}{\epsilon}.$$

Thus

$$C_2 e^{-(s+\delta)/\epsilon} = \frac{1}{\epsilon}.$$

This gives

$$C_2 = \frac{1}{\epsilon} e^{(s+\delta)/\epsilon}.$$

Letting $\delta \to 0$, we get

$$C_2 = \frac{1}{\epsilon} e^{s/\epsilon},$$

thus

$$g = \frac{1}{\epsilon} e^{(s-x)/\epsilon}, \qquad x > s.$$

Thus, the Green's function solution is

$$y(x) = \int_0^x \underbrace{g(x,s)}_{(1/\epsilon)e^{(s-x)/\epsilon}} f(s) ds + \int_x^\infty \underbrace{g(x,s)}_{0} f(s) ds.$$

So the solution reduces to

$$\boxed{y(x) = \frac{1}{\epsilon} \int_0^x f(s) e^{(s-x)/\epsilon} ds.}$$

Now when $f(x) = 1$, the Green's function solution is

$$y(x) = \frac{1}{\epsilon} \int_0^x (1) e^{(s-x)/\epsilon} ds.$$

$$y(x) = \frac{1}{\epsilon} \left(\epsilon e^{(s-x)/\epsilon} \right) \Big|_0^x .$$

$$y(x) = \frac{1}{\epsilon} \left(\epsilon e^{(x-x)/\epsilon} - \left(\epsilon e^{(0-x)/\epsilon} \right) \right)$$

$$y(x) = \frac{1}{\epsilon} \left(\epsilon - \epsilon e^{-x/\epsilon} \right)$$

$$\boxed{y(x) = 1 - e^{-x/\epsilon}.}$$

As $\epsilon \to 0$ for $x > 0$, we see

$$\boxed{\lim_{\epsilon \to 0} y(x) \to 1.}$$

There is a boundary layer near $x = 0$. Its thickness, by inspection of the exact solution is ϵ. One could easily use boundary layer theory to show the outer solution is 1 and find the corresponding inner solution and matching.

3. (25) Find the most general solution to

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = x.$$

Solution

Most people got full credit on this problem. A few did not recognize that the characteristic polynomial was easily factorizable.

Assuming solutions of the form $y = Ce^{rx}$ leads to the characteristic polynomial

$$r^3 + 3r^2 + 3r + 1 = 0.$$

The factors as

$$(r+1)^3 = 0.$$

This gives three repeated roots of $r = -1$. Thus, we have complementary functions of e^{-x}, xe^{-x} and x^2e^{-x}.

We assume a particular solution of the form

$$y_p = a + bx.$$

Substituting into the differential equation, we find

$$3b + (a + bx) = x.$$

Regrouping, we get

$$x^0(3b + a) + x^1(b - 1) = 0.$$

Because x^0 and x^1 are linearly independent, we must have

$$3b + a = 0, \qquad b - 1 = 0.$$

Thus,

$$b = 1, \qquad a = -3.$$

So $y_p = -3 + x$.

The total solution is

$$\boxed{y(x) = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} - 3 + x.}$$

4. (25) Find the general solution to

$$\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 0.$$

Solution

This problem was a little tricky and very few saw it through to the end. The errors were many and disparate, so most students should read through the solution carefully.

If we take $v = dy/dx$, we recognize this as a first order equation:

$$v^2 - \frac{dv}{dx} - 2xv = 0.$$

Rearranging, we get

$$\frac{dv}{dx} + 2xv = v^2.$$

This is recognized to be a Bernoulli equation with $P(x) = 2x$, $Q(x) = 1$ and $n = 2$. As such, we take

$$u = v^{1-n} = v^{-1}.$$

Thus

$$v = u^{-1}$$

and

$$\frac{dv}{dx} = -\frac{1}{u^2}\frac{du}{dx}.$$

So our Bernoulli equation becomes

$$-\frac{1}{u^2}\frac{du}{dx} + \frac{2x}{u} = \frac{1}{u^2}.$$

Multiplying by $-u^2$, we get

$$\frac{du}{dx} - 2xu = -1.$$

This is a first order linear equation. The integrating factor is

$$e^{-\int 2x} = e^{-x^2},$$

so multiplying by it, we get

$$e^{-x^2}\frac{du}{dx} - 2xe^{-x^2}u = -e^{-x^2},$$

$$\frac{d}{dx}\left(e^{-x^2}u\right) = -e^{-x^2}.$$

Integrating, we get

$$e^{-x^2}u = C_1 - \int_0^x e^{-s^2}\,ds.$$

$$u = e^{x^2}\left(C_1 - \int_0^x e^{-s^2}\,ds\right).$$

And since $v = 1/u$, we find

$$v = \frac{e^{-x^2}}{C_1 - \int_0^x e^{-s^2}\,ds}.$$

And since $v = dy/dx$, we get

$$\frac{dy}{dx} = \frac{e^{-x^2}}{C_1 - \int_0^x e^{-s^2}\,ds}.$$

Now let $w(x) = \int_0^x e^{-s^2}\,ds$. So $dw/dx = e^{-x^2}$. Thus, our equation becomes

$$\frac{dy}{dx} = \frac{\frac{dw}{dx}}{C_1 - w}.$$

$$dy = \frac{dw}{C_1 - w}$$

$$y = C_2 + \ln\left(\frac{-1}{C_1 - w}\right).$$

$$y = C_2 + \ln\left(\frac{1}{-C_1 + w}\right).$$

$$y = C_2 - \ln\left(w - C_1\right).$$

$$y = C_2 - \ln\left(\int_0^x e^{-s^2} ds - C_1\right).$$

In terms of the error function, $\mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-s^2} ds$, we could say

$$\boxed{y(x) = C_2 - \ln\left(\frac{\sqrt{\pi}}{2}\mathrm{erf}(x) - C_1\right).}$$

For $C_1 = -1$, $C_2 = 0$, the solution is plotted in Fig. 2.

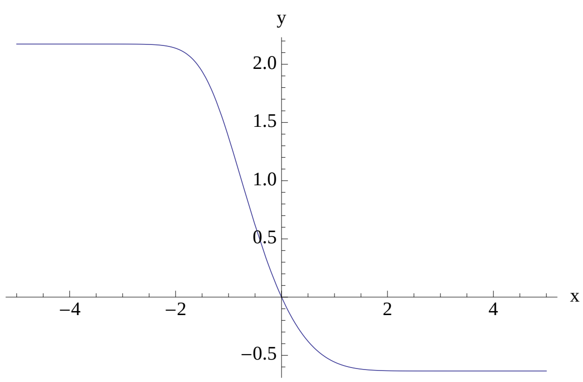

Figure 2: Solution $y(x)$ when $C_1 = -1$, $C_2 = 0$