AME 60611
Examination 2: Solution
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1. (20) Consider the lines in $\mathbb{E}^3$ given by

$$x = y = z$$

and

$$x + y = x - y + 1 = z - 1.$$

It is straightforward to find the distance from a point on one line to a point on the other. Find the coordinates of the point on each line which minimizes this distance, and find the value of the distance.

Solution

Let us first give a parametric description of each line. The first is simply $x = y = z = t$, which yields

$$x = t$$
$$y = t$$
$$z = t.$$

The second is $x + y = x - y + 1 = z - 1 = s$, which yields

$$x + y = s$$
$$x - y + 1 = s$$
$$z = s + 1$$

Solve for x and y in the second set. In parametric form, the second line is given by

$$\begin{aligned} x &= -\frac{1}{2} + s, \\ y &= \frac{1}{2}, \\ z &= s + 1. \end{aligned}$$

Now the square of the Euclidean distance from generic point on the first line to a generic point on the second is

$$\ell^2 = (t + 1/2 - s)^2 + (t - 1/2)^2 + (t - s - 1)^2.$$

Now if ℓ is minimized, ℓ^2 is as well, so we will seek values of s and t which drive ℓ^2 to a minimum. At such minima, we must have $\partial \ell^2/\partial t = \partial \ell^2/\partial s = 0$. Forming the partial derivatives, we find

$$\frac{\partial \ell^2}{\partial t} = 2(t + 1/2 - s) + 2(t - 1/2) + 2(t - s - 1) = 0.$$

$$\frac{\partial \ell^2}{\partial s} = -2(t + 1/2 - s) - 2(t - s - 1) = 0.$$

Expanding, we get

$$6t - 4s = 2$$
$$4t - 4s = 1$$

which has solution

$$s = \frac{1}{4}; \qquad t = \frac{1}{2}$$

So the square of the distance between these two points is

$$\ell^2 = (1/2 + 1/2 - 1/4)^2 + (1/2 - 1/2)^2 + (1/2 - 1/4 - 1)^2 = (3/4)^2 + 0^2 + (-3/4)^2 = \frac{9}{8}.$$

And the distance is thus

$$\boxed{\ell = \frac{3}{2\sqrt{2}}}.$$

The coordinates of the points on the two lines are thus

$$\begin{pmatrix} t \\ t \\ t \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}}, \qquad \text{and} \qquad \begin{pmatrix} -\frac{1}{2} + s \\ \frac{1}{2} \\ s + 1 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ \frac{5}{4} \end{pmatrix}}$$

2. (25) Find $\mathbf{x}$ of minimum $||\mathbf{x}||_2$ which minimizes $||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2$ when

$$\mathbf{A} = \begin{pmatrix} 1 & i \\ 2 & 2i \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution

All 2×2 sub-determinants of $\mathbf{A}$ are singular. However, many 1×1 sub-determinants are non-singular. Thus, the rank of $\mathbf{A}$ is one; that is, it is not a full rank matrix. Therefore, this is likely a simultaneously over- and under- constrained system. Let us operate on both sides of the "equation" by the conjugate transpose of $\mathbf{A}$:

$$\mathbf{A}^H \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{A}^H \cdot \mathbf{b}.$$

$$\begin{pmatrix} 1 & 2 & 0 \\ -i & -2i & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 2 & 2i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ -i & -2i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 5i \\ -5i & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3i \end{pmatrix}.$$

Use Gaussian elimination to find

$$\begin{pmatrix} 5 & 5i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Thus x_2 is a free variable, which we will take to be $x_2 = s$. The first equation then becomes

$$5x_1 = 3 - 5is.$$

Solving, we get

$$x_1 = \frac{3}{5} - is.$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ 0 \end{pmatrix} + s \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 & i \end{pmatrix}^H = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is in the row space of $\mathbf{A}$. The vector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is in the right null space of $\mathbf{A}$. The vector $\begin{pmatrix} \frac{3}{5} \\ 0 \end{pmatrix}$ lies in a linear combination of the row space and right null space of $\mathbf{A}$. Let us decompose that vector into

$$\begin{pmatrix} \frac{3}{5} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

We invert and find

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{3i}{10} \end{pmatrix},$$

so

$$\begin{pmatrix} \frac{3}{5} \\ 0 \end{pmatrix} = \frac{3}{10} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{3i}{10} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{3}{10} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \left(s + \frac{3i}{10} \right) \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

The $\mathbf{x}$ with the minimum $||\mathbf{x}||_2$ occurs when $s = -3i/10$; this gives

$$\boxed{\mathbf{x} = \begin{pmatrix} \frac{3}{10} \\ \frac{-3i}{10} \end{pmatrix}.}$$

The residual itself in satisfying the original equation is

$$\mathbf{r} = \mathbf{A} \cdot \mathbf{x} - \mathbf{b} = \begin{pmatrix} 1 & i \\ 2 & 2i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{10} \\ -\frac{3i}{10} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{6}{5} \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ -1 \end{pmatrix}.$$

The magnitude of the residual is

$$||\mathbf{r}||_2 = \sqrt{\left(-\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + (-1)^2} = \sqrt{\frac{6}{5}}.$$

The norm of the vector $\mathbf{x}$ is

$$||\mathbf{x}||_2 = \sqrt{\begin{pmatrix} \frac{3}{10} & -\frac{3i}{10} \end{pmatrix} \begin{pmatrix} \frac{3}{10} \\ \frac{3i}{10} \end{pmatrix}} = \frac{3}{5\sqrt{2}}.$$

As an alternative approach to finding the $\mathbf{x}$ with the smallest norm, we can consider

$$\mathbf{x} = \begin{pmatrix} \frac{3}{5} - is \\ s \end{pmatrix}.$$

Now s could be complex, so we allow

$$s = s_R + is_I,$$

where s_R and s_I are both real. So

$$\mathbf{x} = \begin{pmatrix} \frac{3}{5} - is_R + s_I \\ s_R + is_I \end{pmatrix}.$$

Taking the norm, we get

$$\begin{aligned} ||\mathbf{x}||_2^2 &= \begin{pmatrix} \frac{3}{5} + is_R + s_I & s_R - is_I \end{pmatrix} \begin{pmatrix} \frac{3}{5} - is_R + s_I \\ s_R + is_I \end{pmatrix}, \\ &= \frac{9}{25} + \frac{6}{5}s_I + s_R^2 + s_I^2 + s_R^2 + s_I^2, \\ &= \frac{9}{25} + \frac{6}{5}s_I + 2s_R^2 + 2s_I^2. \end{aligned}$$

Now, we need to choose s so as to minimize $||\mathbf{x}||_2$, which is equivalent to minimizing $||\mathbf{x}||_2^2$.

$$\frac{\partial}{\partial s_R}||\mathbf{x}||_2^2 = 4s_R = 0,$$
$$\frac{\partial}{\partial s_I}||\mathbf{x}||_2^2 = \frac{6}{5} + 4s_I = 0,$$

Solving, we get

$$s_R = 0, \qquad s_I = -\frac{3}{10}.$$

So we have

$$s = -\frac{3}{10}i.$$

So we get

$$\mathbf{x} = \begin{pmatrix} \frac{3}{5} - is \\ s \end{pmatrix} = \begin{pmatrix} \frac{3}{5} - i\left(-\frac{3}{10}i\right) \\ -\frac{3}{10}i \end{pmatrix} = \boxed{\begin{pmatrix} \frac{3}{10} \\ -\frac{3}{10}i \end{pmatrix}.}$$

3. (25) In $\mathbb{L}_2[0,1]$, we have the linearly independent functions $u_1 = t$, $u_2 = t^2$. Project the function $f(t) = t^3$ onto the space spanned by u_1 and u_2; thus, find the best α_1 and α_2 to approximate $f(t) \simeq \alpha_1 u_1 + \alpha_2 u_2$.

Solution

Start with

$$\alpha_1 u_1 + \alpha_2 = f(t)$$

Take the inner product with u_1 and u_2 to get

$$<u_1, \alpha_1 u_1> + <u_1, \alpha_2 u_2> = <u_1, f(t)>.$$

$$<u_2, \alpha_1 u_1> + <u_2, \alpha_2 u_2> = <u_2, f(t)>.$$

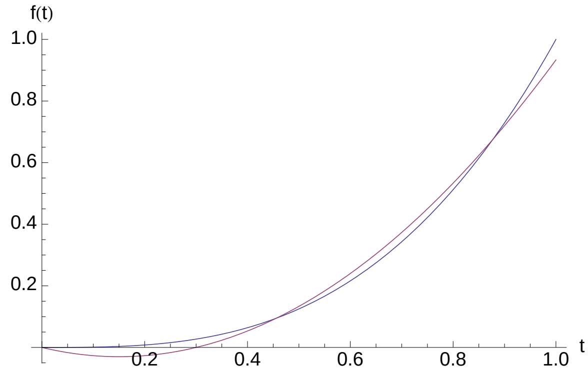

Figure 1: Plot of $f(t) = t^3$ (blue curve) and its approximation, $-(2/5)t + (4/3)t^2$ (red curve).

Regroup to get

$$\alpha_1 \langle u_1, u_1 \rangle + \alpha_2 \langle u_1, u_2 \rangle = \langle u_1, f(t) \rangle.$$

$$\alpha_1 \langle u_2, u_1 \rangle + \alpha_2 \langle u_2, u_2 \rangle = \langle u_2, f(t) \rangle.$$

In matrix form, we have

$$\begin{pmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \langle u_1, f(t) \rangle \\ \langle u_2, f(t) \rangle \end{pmatrix}$$

In terms of our functions

$$\begin{pmatrix} \int_0^1 t^2 \, dt & \int_0^1 t^3 \, dt \\ \int_0^1 t^3 \, dt & \int_0^1 t^4 \, dt \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \int_0^1 t^4 \, dt \\ \int_0^1 t^5 \, dt \end{pmatrix}$$

Evaluating, we get

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{6} \end{pmatrix}$$

Inverting, we find

$$\alpha_1 = -\frac{2}{5}, \qquad \alpha_2 = \frac{4}{3}.$$

So

$$\boxed{f(t) = t^3 \simeq -\frac{2}{5}t + \frac{4}{3}t^2.}$$

See Figure 1 for a comparison of $f(t)$ with its approximation.

4. (25) Use a one-term Galerkin method with a polynomial basis function to estimate the solution to the differential equation

$$\frac{d^3y}{dx^3} + y = x, \qquad y(0) = 0, \ y(1) = 0, \ y'(0) = 0.$$

Solution

Assume, for the one term expansion, that the Galerkin-projected approximate solution is

$$y_p = c\phi(x).$$

where $\phi(x)$ is a polynomial which satisfies the boundary conditions. Let as assume a third order polynomial.

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Now $\phi(x)$ must satisfy the same boundary conditions as $y(x)$. So at $x = 0$, we have

$$\phi(0) = 0 = a_0 + a_1(0) + a_2 0^2 + a_3 0^3.$$

Thus $a_0 = 0$. The first derivative is then

$$\phi'(x) = a_1 + 2a_2 x + 3a_3 x^2.$$

At $x = 0$ one must satisfy the derivative boundary condition

$$\phi'(0) = 0 = a_1 + 2a_2(0) + 3a_3 0^2.$$

Thus $a_1 = 0$. So we have

$$\phi(x) = a_2 x^2 + a_3 x^3.$$

At $x = 1$, we must have

$$\phi(1) = 0 = a_2 + a_3.$$

Take $a_2 = 1$, thus $a_3 = -1$, and

$$\phi(x) = x^2(1 - x).$$

Now, we have

$$y_p = cx^2(1 - x).$$

The residual with this approximation is

$$r = \frac{d^3 y_p}{dx^3} + y_p - x,$$

which evaluates to

$$r = -6c + cx^2(1 - x) - x$$

Now for the Galerkin method we need

$$<\phi, r> = 0.$$

Thus, we c such that

$$\int_0^1 \phi(x) r(x)\ dx = 0.$$

$$\int_0^1 x^2(1 - x)(-6c + cx^2(1 - x) - x)\ dx = 0.$$

$$\int_0^1 (-6cx^2 + (6c - 1)x^3 + (1 + c)x^4 - 2cx^5 + cx^6)\ dx = 0.$$

$$-2cx^3 + \left(\frac{3c}{2} - \frac{1}{4}\right) x^4 + \frac{1 + c}{5} x^5 - \frac{cx^6}{3} + \frac{cx^6}{7}\Big|_0^1 = 0.$$

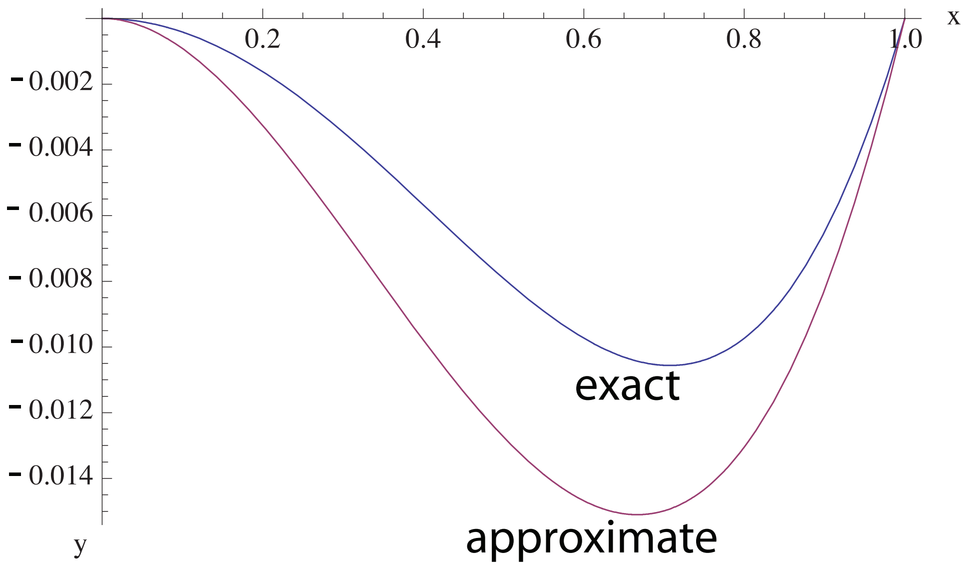

Figure 2: Plot of exact and Galerkin approximation solutions.

$$-2c + \frac{3c}{2} - \frac{1}{4} + \frac{1+c}{5} - \frac{c}{3} + \frac{c}{7} = 0.$$

Solve for c and get

$$c = -\frac{21}{206}.$$

So

$$\boxed{y_p = -\frac{21}{206}x^2(1-x).}$$

The exact solution can be obtained from computer algebra, but is lengthy. It can, however, easily be plotted and compared with the Galerkin approximation. See Figure 2.

5. (5) Use Cartesian index notation to prove the identity

$$\nabla^T \cdot (\nabla \times \mathbf{u}) = 0.$$

Solution

Consider the equation in Cartesian index notation:

$$\frac{\partial}{\partial x_i}\epsilon_{ijk}\frac{\partial}{\partial x_j}u_k = 0?,$$

$$\epsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_k = 0?,$$

For a given k, we have that ϵ_{ijk} is an anti-symmetric tensor in i and j. And for a given k, the term $\partial^2 u_k / \partial x_i \partial x_j$ is a symmetric tensor in i and j. Since the tensor inner product of an anti-symmetric tensor and a symmetric tensor is zero, the identity is proved.