AME 60611 Homework #2 Solution

1. Kaplan p. 105: 2b Find $\frac{\partial(x,y)}{\partial(s,t)}$ for $s=0$, $t=0$ if
$x=(z^2+w^2)^{1/2}$, $y=w(z^2+w^2)^{-1/2}$, $z=(s+t+1)^{-1}$, $w=(2s-t+1)^{-1}$

When $s=0$ and $t=0 \Rightarrow z=(0+0+1)^{-1}=1$ $w=(2(0)-0+1)^{-1}$
$w=1$.

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

To compute these partials, we need to find
$\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}$

$$\frac{\partial z}{\partial s} = (-1)(s+t+1)^{-2}\Big|_{\substack{s=0\\t=0}} = (-1)(0+0+1)^{-2} = -1$$

$$\frac{\partial z}{\partial t} = (-1)(s+t+1)^{-2}\Big|_{\substack{s=0\\t=0}} = (-1)(0+0+1)^{-2} = -1$$

$$\frac{\partial w}{\partial s} = (-1)(2)(2s-t+1)^{-2}\Big|_{\substack{s=0\\t=0}} = (-1)(2)(0-0+1)^{-2} = -2$$

$$\frac{\partial w}{\partial t} = (-1)(-1)(2s-t+1)^{-2}\Big|_{\substack{s=0\\t=0}} = (-1)(-1)(0-0+1)^{-2} = 1$$

$$\frac{\partial x}{\partial z} = \left(\tfrac{1}{2}\right)(2z)(z^2+w^2)^{-1/2}\Big|_{\substack{z=1\\w=1}} = \left(\tfrac{1}{2}\right)(2)(1+1)^{-1/2} = (1)(2)^{-1/2} = \frac{1}{\sqrt{2}}$$

$$\frac{\partial x}{\partial w} = \left(\tfrac{1}{2}\right)(2w)(z^2+w^2)^{-1/2}\Big|_{\substack{z=1\\w=1}} = \left(\tfrac{1}{2}\right)(2)(1+1)^{-1/2} = \frac{1}{\sqrt{2}}$$

$$\frac{\partial y}{\partial z} = \left(-\tfrac{1}{2}\right)(2z)w(z^2+w^2)^{-3/2}\Big|_{\substack{z=1\\w=1}} \left(-\tfrac{1}{2}\right)(2)(1)(1+1)^{-3/2} = \frac{-1}{2\sqrt{2}}$$

$$\frac{\partial y}{\partial w} = (1)(z^2+w^2)^{-1/2} + w(2w)\left(\tfrac{-1}{2}\right)(z^2+w^2)^{-3/2}\Big|_{\substack{z=1\\w=1}} = (1+1)^{-1/2} + (1)(2)\left(\tfrac{-1}{2}\right)(1+1)^{-3/2}$$
$$= \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial s} = \left(\tfrac{1}{\sqrt{2}}\right)(-1) + \left(\tfrac{1}{\sqrt{2}}\right)(-2) = \frac{-3}{\sqrt{2}}$$

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial t} = \left(\tfrac{1}{\sqrt{2}}\right)(-1) + \left(\tfrac{1}{\sqrt{2}}\right)(1) = 0$$

$$\frac{\partial y}{\partial s} = \frac{\partial y}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial s} = \left(\tfrac{-1}{2\sqrt{2}}\right)(-1) + \left(\tfrac{1}{2\sqrt{2}}\right)(-2) = \frac{-1}{2\sqrt{2}}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial t} = \left(\tfrac{-1}{2\sqrt{2}}\right)(-1) + \left(\tfrac{1}{2\sqrt{2}}\right)(1) = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

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1. cont.

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

$$= \left(\frac{-3}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - (0)\left(\frac{-1}{2\sqrt{2}}\right) = \boxed{\frac{-3}{2}}$$

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2. Kaplan p. 117: 8 b,f in thermodynamics the variables p (pressure), T (temperature), U (internal energy), and V (volume) occur. For each substance these are related by two equations, so that any two of the four variables can be chosen as independent, the other two then being dependent. In addition, the second law of thermodynamics implies the relation

$$\frac{\partial U}{\partial V} - T\frac{\partial p}{\partial T} + p = 0, \quad \text{when } V \text{ and } T \text{ are independent.}$$

Show that this relation can be written in each of the following forms:

b) $\frac{\partial T}{\partial V} + T\frac{\partial p}{\partial U} - p\frac{\partial T}{\partial U} = 0$ (U, V independent)

f) $T\frac{\partial(p,V)}{\partial(T,U)} - p\frac{\partial V}{\partial U} - 1 = 0$ (T, U independent)

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b) Assume $dT = \alpha\, dV + \beta\, dU$ $\quad dp = \gamma\, dV + \delta\, dU$

$\alpha = \frac{\partial T}{\partial V}$ $\quad \beta = \frac{\partial T}{\partial U}$ $\quad \gamma = \frac{\partial p}{\partial V}$ $\quad \delta = \frac{\partial p}{\partial U}$

$\Rightarrow \beta\, dU = dT - \alpha\, dV$ $\quad dp = \gamma\, dV + \delta\left(\frac{1}{\beta}\right)(dT - \alpha\, dV)$

$dp = \left(\frac{\delta}{\beta}\right)dT + \left(\gamma - \frac{\alpha\delta}{\beta}\right)dV$

From part (a) $dU = a\,dV + b\,dT$ $\quad dp = c\,dV + e\,dT$

We have $\quad dU = -\frac{\alpha}{\beta}dV + \frac{1}{\beta}dT$ $\quad dp = \left(\gamma - \frac{\alpha\delta}{\beta}\right)dV + \left(\frac{\delta}{\beta}\right)dT$

Therefore $a = \frac{-\alpha}{\beta}$ $\quad b = \frac{1}{\beta}$ $\quad c = \gamma - \frac{\alpha\delta}{\beta}$ $\quad e = \frac{\delta}{\beta}$

Plugging this into $a - Te + p = 0$

$\Rightarrow \frac{-\alpha}{\beta} - T\left(\frac{\delta}{\beta}\right) + p = 0$ multiply by $-\beta$ gives

$\alpha + T\delta - p\beta = 0$ plug in the values for α, δ, β

$\frac{\partial T}{\partial V} + T\frac{\partial p}{\partial U} - p\frac{\partial T}{\partial U} = 0$ ✓

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2. cont.) Assume $dV = \alpha dT + \beta du$ $\quad dp = \gamma dT + \delta du$

$$\alpha = \frac{\partial V}{\partial T} \quad \beta = \frac{\partial V}{\partial u} \quad \gamma = \frac{\partial p}{\partial T} \quad \delta = \frac{\partial p}{\partial u}$$

$$\Rightarrow du = \frac{1}{\beta} dV - \frac{\alpha}{\beta} dT \qquad dp = \gamma dT + \delta\left(\frac{1}{\beta} dV - \frac{\alpha}{\beta} dT\right)$$

$$dp = \frac{\delta}{\beta} dV + \left(\gamma + \frac{-\alpha\delta}{\beta}\right) dT$$

From part (a) $du = a\,dV + b\,dT$ $\quad dp = c\,dV + e\,dT$

we have $du = \frac{1}{\beta} dV - \frac{\alpha}{\beta} dT$ $\quad dp = \frac{\delta}{\beta} dV + \left(\gamma - \frac{\alpha\delta}{\beta}\right) dT$

$$a = \frac{1}{\beta} \quad b = -\frac{\alpha}{\beta} \quad c = \frac{\delta}{\beta} \quad e = \gamma - \frac{\alpha\delta}{\beta}$$

Plugging this into $a - Te + p = 0$

$\Rightarrow \frac{1}{\beta} - T\left(\gamma - \frac{\alpha\delta}{\beta}\right) + p = 0$ multiplying by $-\beta$ gives

$$-1 + T(\beta\gamma - \alpha\delta) - \beta p = 0 \qquad T(\beta\gamma - \alpha\delta) - p\beta - 1 = 0$$

Note that $\frac{\partial(p,V)}{\partial(T,u)} = \begin{vmatrix} \frac{\partial p}{\partial T} & \frac{\partial p}{\partial u} \\ \frac{\partial V}{\partial T} & \frac{\partial V}{\partial u} \end{vmatrix} = \begin{vmatrix} \gamma & \delta \\ \alpha & \beta \end{vmatrix} = \gamma\beta - \alpha\delta$

$$\Rightarrow T \frac{\partial(p,V)}{\partial(T,u)} - p \frac{\partial V}{\partial u} - 1 = 0 \checkmark$$

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3. Kaplan p. 121: 3a, b: also plot lines of constant u and v in the x, y plane. plot lines of constant x and y in the u, v plane. Find the metric tensor G.

Given the mapping $x = u^2 - v^2$ $y = 2uv$,

a) Compute its Jacobian,

b) evaluate $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial v}{\partial x}\right)_y$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = \boxed{4(u^2+v^2)}$$

Let $f = u^2 - v^2 - x$ $g = 2uv - y$ and using cramer's rule

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} -(-1) & -2v \\ 0 & 2u \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}} = \frac{2u}{4(u^2+v^2)} = \boxed{\frac{u}{2(u^2+v^2)}}$$

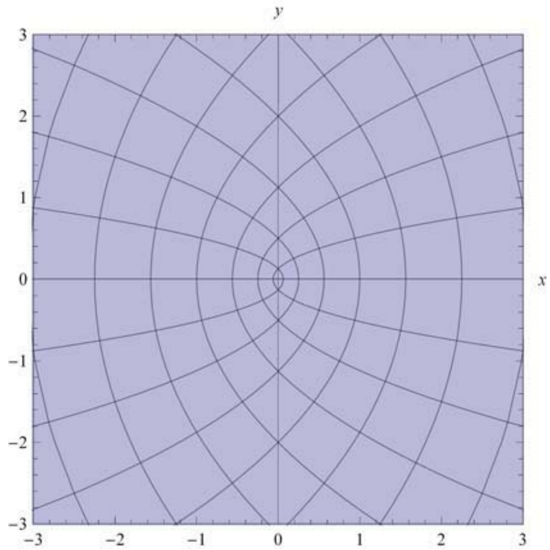
$$\left(\frac{\partial v}{\partial x}\right)_y = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} 2u & -(-1) \\ 2v & 0 \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}} = \frac{-2v}{4(u^2+v^2)} = \boxed{\frac{-v}{2(u^2+v^2}}$$

$$G = J^T J = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}^T \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = \begin{bmatrix} 2u & 2v \\ -2v & 2u \end{bmatrix} \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$$

$$= \begin{bmatrix} (4u^2 + 4v^2) & ((-2uv)+2uv) \\ (-4uv+4uv) & (4v^2+4u^2) \end{bmatrix} = \boxed{\begin{bmatrix} 4(u^2+v^2) & 0 \\ 0 & 4(u^2+v^2) \end{bmatrix}}$$

In[1]:=
```
ParametricPlot[{u^2 - v^2, 2 u v}, {u, -2, 2},
 {v, -2, 2}, PlotRange → 3, AxesLabel → {x, y}]
```

Out[1]=



In[2]:=
```
myuv = Solve[{y == 2 u v, x == u^2 - v^2}, {u, v}] // Simplify
```

Out[2]=

$$\left\{\left\{u \to -\frac{\sqrt{x - \sqrt{x^2 + y^2}}}{\sqrt{2}},\ v \to \frac{\sqrt{x - \sqrt{x^2 + y^2}}\ \left(x + \sqrt{x^2 + y^2}\right)}{\sqrt{2}\ y}\right\},\right.$$

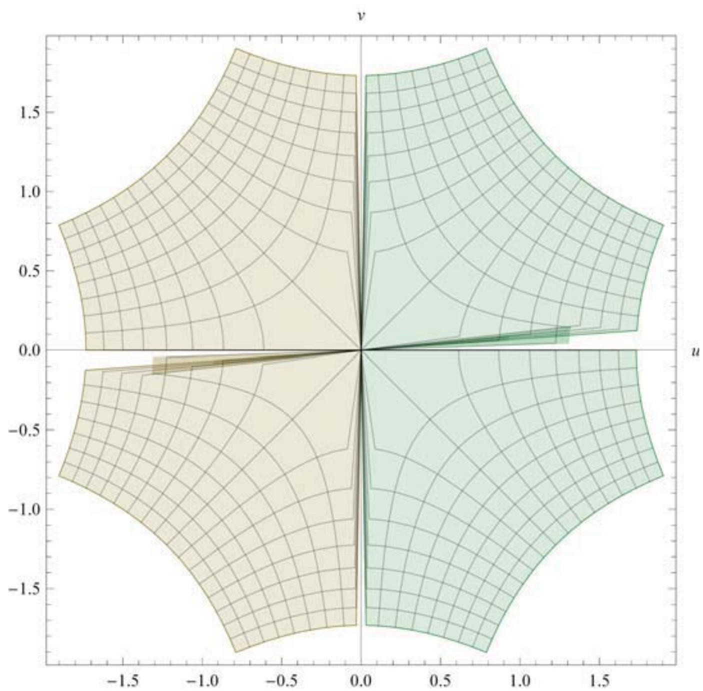
$$\left\{u \to \frac{\sqrt{x - \sqrt{x^2 + y^2}}}{\sqrt{2}},\ v \to -\frac{\sqrt{x - \sqrt{x^2 + y^2}}\ \left(x + \sqrt{x^2 + y^2}\right)}{\sqrt{2}\ y}\right\},$$

$$\left\{u \to -\frac{\sqrt{x + \sqrt{x^2 + y^2}}}{\sqrt{2}},\ v \to \frac{\left(x - \sqrt{x^2 + y^2}\right)\sqrt{x + \sqrt{x^2 + y^2}}}{\sqrt{2}\ y}\right\},$$

$$\left.\left\{u \to \frac{\sqrt{x + \sqrt{x^2 + y^2}}}{\sqrt{2}},\ v \to \frac{\left(-x + \sqrt{x^2 + y^2}\right)\sqrt{x + \sqrt{x^2 + y^2}}}{\sqrt{2}\ y}\right\}\right\}$$

In[3]:=
```
ParametricPlot[{{u, v} /. myuv[[1]], {u, v} /. myuv[[2]], {u, v} /. myuv[[3]],
  {u, v} /. myuv[[4]]}, {x, -3, 3}, {y, -3, 3} , AxesLabel → {u, v}]
```

Out[3]=



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4. Kaplan p. 121: 5 Given the mapping

$x = f(u,v,w) \qquad y = g(u,v,w) \qquad z = h(u,v,w)$,

with Jacobian $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$, show that for the inverse functions one has

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(v,w)} \qquad \frac{\partial u}{\partial y} = \frac{1}{J}\frac{\partial(z,x)}{\partial(v,w)} \qquad \frac{\partial u}{\partial z} = \frac{1}{J}\frac{\partial(x,y)}{\partial(v,w)}$$

$$\frac{\partial v}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(w,u)} \qquad \frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial(z,x)}{\partial(w,u)} \qquad \frac{\partial v}{\partial z} = \frac{1}{J}\frac{\partial(x,y)}{\partial(w,u)}$$

$$\frac{\partial w}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(u,v)} \qquad \frac{\partial w}{\partial y} = \frac{1}{J}\frac{\partial(z,x)}{\partial(u,v)} \qquad \frac{\partial w}{\partial z} = \frac{1}{J}\frac{\partial(x,y)}{\partial(u,v)}.$$

$f^* = f(u,v,w) - x \qquad g^* = g(u,v,w) - y \qquad h^* = h(u,v,w) - z$

$$\frac{\partial(f^*,g^*,h^*)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f^*}{\partial u} & \frac{\partial f^*}{\partial v} & \frac{\partial f^*}{\partial w} \\ \frac{\partial g^*}{\partial u} & \frac{\partial g^*}{\partial v} & \frac{\partial g^*}{\partial w} \\ \frac{\partial h^*}{\partial u} & \frac{\partial h^*}{\partial v} & \frac{\partial h^*}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix} = \frac{\partial(f,g,h)}{\partial(u,v,w)} = J$$

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f^*,g^*,h^*)}{\partial(x,v,w)}}{\frac{\partial(f^*,g^*,h^*)}{\partial(u,v,w)}} = \frac{\begin{vmatrix} -\frac{\partial f^*}{\partial x} & \frac{\partial f^*}{\partial v} & \frac{\partial f^*}{\partial w} \\ -\frac{\partial g^*}{\partial x} & \frac{\partial g^*}{\partial v} & \frac{\partial g^*}{\partial w} \\ -\frac{\partial h^*}{\partial x} & \frac{\partial h^*}{\partial v} & \frac{\partial h^*}{\partial w} \end{vmatrix}}{J} = \frac{\begin{vmatrix} -(-1) & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ 0 & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ 0 & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}}{J}$$

$$= \frac{(1)\begin{vmatrix} \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{\begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}}{J} = \boxed{\frac{1}{J}\frac{\partial(y,z)}{\partial(v,w)} = \frac{\partial u}{\partial x}}$$

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial(f^*,g^*,h^*)}{\partial(y,v,w)}}{\frac{\partial(f^*,g^*,h^*)}{\partial(u,v,w)}} = \frac{\begin{vmatrix} 0 & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ -(-1) & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ 0 & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{(-1)\begin{vmatrix} \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{\begin{vmatrix} \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{vmatrix}}{J}$$

$$= \frac{1}{J}\begin{vmatrix} \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \\ \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(z,x)}{\partial(v,w)} = \frac{\partial u}{\partial y}}$$

4. cont.

$$\frac{\partial u}{\partial z} = \frac{\dfrac{\partial(f^*, g^*, h^*)}{\partial(z, v, w)}}{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u, v, w)}} = \frac{\begin{vmatrix} 0 & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ 0 & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ -(-1) & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{(1)\begin{vmatrix} \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \end{vmatrix}}{J}$$

$$= \frac{1}{J}\begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(x, y)}{\partial(v, w)} = \frac{\partial u}{\partial z}}$$

$$\frac{\partial v}{\partial x} = \frac{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u, x, w)}}{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u, v, w)}} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & -(-1) & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & 0 & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & 0 & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{-(1)\begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{(1)\begin{vmatrix} \frac{\partial g}{\partial w} & \frac{\partial g}{\partial u} \\ \frac{\partial h}{\partial w} & \frac{\partial h}{\partial u} \end{vmatrix}}{J}$$

$$= \frac{1}{J}\begin{vmatrix} \frac{\partial y}{\partial w} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial w} & \frac{\partial z}{\partial u} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(y, z)}{\partial(w, u)} = \frac{\partial v}{\partial x}}$$

$$\frac{\partial v}{\partial y} = \frac{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u, y, w)}}{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u, v, w)}} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & 0 & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & -(-1) & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & 0 & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{(1)\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{(-1)\begin{vmatrix} \frac{\partial h}{\partial u} & \frac{\partial h}{\partial w} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial w} \end{vmatrix}}{J}$$

$$= \frac{-(-1)}{J}\begin{vmatrix} \frac{\partial h}{\partial w} & \frac{\partial h}{\partial u} \\ \frac{\partial f}{\partial w} & \frac{\partial f}{\partial u} \end{vmatrix} = \frac{1}{J}\begin{vmatrix} \frac{\partial z}{\partial w} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial w} & \frac{\partial x}{\partial u} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(z, x)}{\partial(w, u)} = \frac{\partial v}{\partial y}}$$

$$\frac{\partial v}{\partial z} = \frac{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u, z, w)}}{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u, v, w)}} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & 0 & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & 0 & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & -(-1) & \frac{\partial h}{\partial w} \end{vmatrix}}{J} = \frac{-(1)\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial w} \end{vmatrix}}{J} = \frac{\begin{vmatrix} \frac{\partial f}{\partial w} & \frac{\partial f}{\partial u} \\ \frac{\partial g}{\partial w} & \frac{\partial g}{\partial u} \end{vmatrix}}{J}$$

$$= \frac{1}{J}\begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial u} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(x, y)}{\partial(w, u)} = \frac{\partial v}{\partial z}}$$

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4. cont.

$$\frac{\partial w}{\partial x} = \frac{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u,v,x)}}{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u,v,w)}} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & -(-1) \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & 0 \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & 0 \end{vmatrix}}{J} = \frac{(1)\begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}}{J}$$

$$= \frac{1}{J}\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial w}{\partial x}}$$

$$\frac{\partial w}{\partial y} = \frac{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u,v,y)}}{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u,v,w)}} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & 0 \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & -(-1) \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & 0 \end{vmatrix}}{J} = \frac{-(1)\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}}{J}$$

$$= \frac{1}{J}\begin{vmatrix} \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{vmatrix} = \frac{1}{J}\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(z,x)}{\partial(u,v)} = \frac{\partial w}{\partial y}}$$

$$\frac{\partial w}{\partial z} = \frac{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u,v,z)}}{\dfrac{\partial(f^*, g^*, h^*)}{\partial(u,v,w)}} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & 0 \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & 0 \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & -(-1) \end{vmatrix}}{J} = \frac{(1)\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}}{J}$$

$$= \frac{1}{J}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \boxed{\frac{1}{J}\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial w}{\partial z}}$$

AME 60611 Homework #2 Solution

5. Kaplan p. 159: 5d Find the critical points of the following functions, classify, and graph the level curves of the functions:

d) $z = \dfrac{x}{x^2+y^2}$

---

critical points occur when $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

$$\frac{\partial z}{\partial x} = (1)(x^2+y^2)^{-1} + x(-1)(2x)(x^2+y^2)^{-2}$$

$$= \frac{(x^2+y^2)}{(x^2+y^2)^2} + \frac{-2x^2}{(x^2+y^2)^2} = \frac{x^2-2x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\Rightarrow x^2 = y^2$ for $\frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial y} = (-1)x(2y)(x^2+y^2)^{-2} = \frac{-2xy}{(x^2+y^2)^2} = 0$ when x and/or $y = 0$.

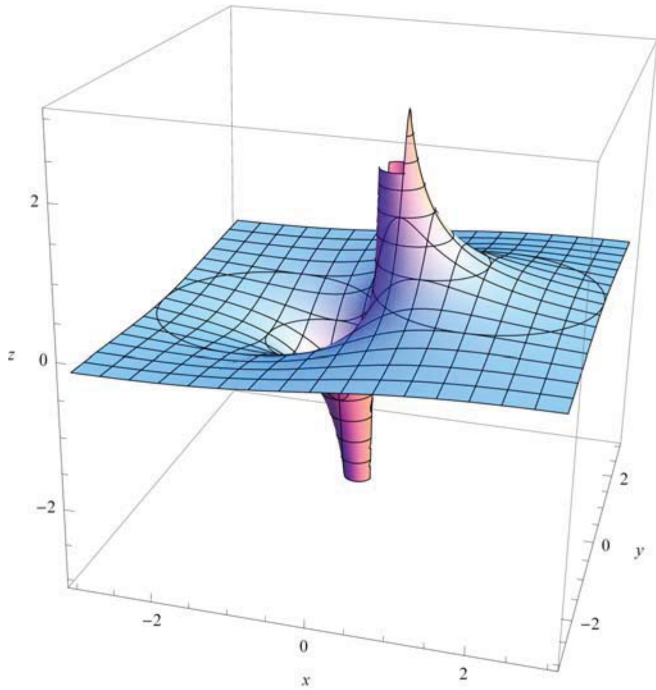
Therefore, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ when $x = y = 0$.

However $z = \frac{0}{0^2+0^2} = \frac{0}{0} \Rightarrow z$ is undefined at $x=y=0$

Therefore since $(x,y) = (0,0)$ is a discontinuity and the only possible critical point, the function $z = \frac{x}{x^2+y^2}$ has no critical points.

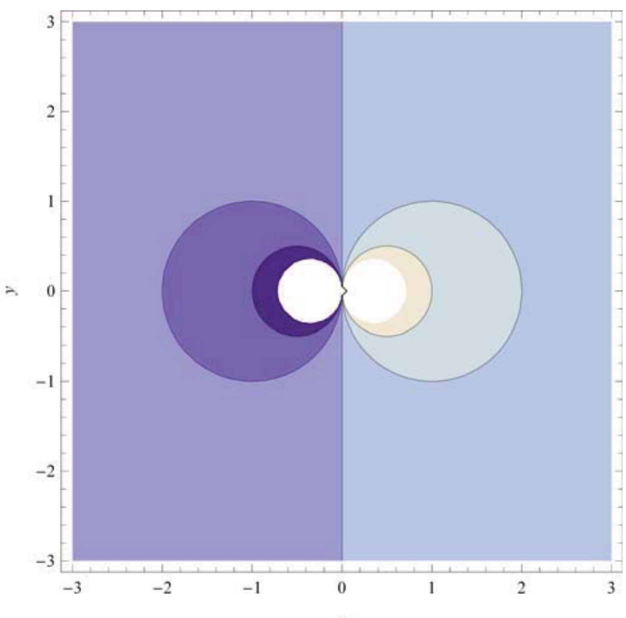
In[5]:=
```
ContourPlot3D[x / (x^2 + y^2) == z, {x, -3, 3},
  {y, -3, 3}, {z, -3, 3}, AxesLabel → {x, y, z}]
```

Out[5]=



In[6]:=
```
ContourPlot[x / (x^2 + y^2), {x, -3, 3}, {y, -3, 3}, FrameLabel → {x, y}]
```

Out[6]=



AME 60611 Homework #2 Solution

6: Kaplan p. 159: 7

Find the point of the curve

$$x^2 - xy + y^2 - z^2 = 1, \quad x^2 + y^2 = 1$$

nearest to the origin (0,0,0).

The distance is defined as $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$, and this is what we want to minimize.

For this problem $(x_2, y_2, z_2) = (0,0,0)$

$\Rightarrow f = \sqrt{x_1^2 + y_1^2 + z_1^2}$ Note that the minimum for this function is the same as $f^2 = x_1^2 + y_1^2 + z_1^2$

The constraints can be rewritten as

$g_1 = x^2 - xy + y^2 - z^2 - 1 = 0 \qquad g_2 = x^2 + y^2 - 1 = 0$

$f^* = f^2 + \lambda_1 g_1 + \lambda_2 g_2$ ← new function to minimize

Critical points occur when $\frac{\partial f^*}{\partial x} = \frac{\partial f^*}{\partial y} = \frac{\partial f^*}{\partial z} = 0$

$$f^* = x^2 + y^2 + z^2 + \lambda_1(x^2 - xy + y^2 - z^2 - 1) + \lambda_2(x^2 + y^2 - 1)$$

$$\frac{\partial f^*}{\partial x} = 2x + \lambda_1(2x - y) + \lambda_2(2x) = 0$$

$$\frac{\partial f^*}{\partial y} = 2y + \lambda_1(2y - x) + \lambda_2(2y) = 0$$

$$\frac{\partial f^*}{\partial z} = 2z + \lambda_1(-2z) = 0 \Rightarrow 2z(1 - \lambda_1) = 0$$

$\Rightarrow$ Either $z = 0$ or $\lambda_1 = 1$

Also have $g_1 = 0$ and $g_2 = 0$ to give 5 equations to find 5 unknowns $(x, y, z, \lambda_1, \lambda_2)$

AME 60611 Homework #2 Solution

6. cont.) Plugging the 5 equations into Mathematica gives

x	y	z	λ_1	λ_2	f^2	
-1	0	0	0	-1	1	
0	-1	0	0	-1	1	min points
0	1	0	0	-1	1	
1	0	0	0	-1	1	
$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{-i}{\sqrt{2}}$	1	$-\frac{3}{2}$	imaginary z	
$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{i}{\sqrt{2}}$	1	$-\frac{5}{2}$	imaginary z	
$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	1	$-\frac{5}{2}$	$\frac{3}{2}$	
$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$-\frac{5}{2}$	$\frac{3}{2}$	max points
$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	1	$-\frac{5}{2}$	$\frac{3}{2}$	
$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$-\frac{5}{2}$	$\frac{3}{2}$	
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{-i}{\sqrt{2}}$	1	$-\frac{3}{2}$	imaginary z	
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{i}{\sqrt{2}}$	1	$-\frac{3}{2}$	imaginary z	

Therefore the points closest to the origin are

$$\boxed{(-1,0,0) \quad (0,-1,0) \quad (0,1,0) \quad (1,0,0)}$$

AME 60611 Homework #2 Solution

7. Kaplan p. 166: 6c Show that the following sets of functions are functionally dependent:

c) $f = x^2y - xy^2 + xyz \qquad g = xy + x - y + z$

$h = x^2 + y^2 + z^2 - 2yz + 2xz$

A set of functions are functionally dependent when the determinant of the Jacobian equals zero.

So when
$$\frac{\partial(f,g,h)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{vmatrix} = \frac{\partial f}{\partial x}\left(\frac{\partial g}{\partial y}\frac{\partial h}{\partial z} - \frac{\partial g}{\partial z}\frac{\partial h}{\partial y}\right) - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\frac{\partial h}{\partial z} - \frac{\partial g}{\partial z}\frac{\partial h}{\partial x}\right) + \frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x}\right)$$

$$= (2xy - y^2 + yz)((x-1)(2z-2y+2x) - (1)(2y-2z))$$
$$-(x^2 - 2xy + xz)((y+1)(2z-2y+2x) - (1)(2x+2z))$$
$$+(xy)((y+1)(2y-2z) - (x-1)(2x+2z))$$

$$= (y)(2x-y+z)(2xz - 2xy + 2x^2 - \cancel{2z} + \cancel{2y} - 2x - \cancel{2y} + \cancel{2z})$$
$$-(x)(x-2y+z)(2yz - 2y^2 + 2xy + \cancel{2z} - 2y + \cancel{2x} - \cancel{2x} - \cancel{2z})$$
$$+(xy)(2y^2 - 2yz + 2y - \cancel{2z} - 2x^2 - 2xz + 2x + \cancel{2z})$$

$$= 2(xy)(2x-y+z)(z-y+x-1)$$
$$-2(xy)(x-2y+z)(z-y+x-1)$$
$$+2(xy)(y(y-z+1) + x(-x-z+1))$$

$$= 2(xy)((z-y+x-1)(2x-y+\cancel{z} - x + 2y - \cancel{z}) + y(y-z+1) + x(-x-z+1))$$

$$= 2(xy)((z-y+x-1)(x+y) + y^2 - yz + y + -x^2 - xz + x)$$

$$= 2(xy)(\cancel{xz} + \cancel{yz} - \cancel{xy} - \cancel{y^2} + \cancel{x^2} + \cancel{xy} - \cancel{x} - \cancel{y} + \cancel{y^2} - \cancel{yz} + \cancel{y} - \cancel{x^2} - \cancel{xz} + \cancel{x})$$

$= 2(xy)(0) = 0$ Since the determinant of the Jacobian equals zero, the functions are functionally dependent.

AME 60611 Homework #2 Solution

8 Kaplan p 210: 16 In E^2 let (ξ^1, ξ^2) be standard coordinates and let (x^1, x^2) be new coordinates given by $x^1 = 3\xi^1 + 2\xi^2$ $x^2 = 4\xi^1 + 3\xi^2$. Find the (x^i) components of the following tensors, for which the components in standard coordinates are given

b) v^i, where $V^1 = \xi^1 \cos\xi^2$ $V^2 = \xi^1 \sin\xi^2$

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \Rightarrow \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = \frac{1}{3(3)-2(4)}\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

$$v^i = \sum_{j=1}^{n} \frac{\partial x^i}{\partial \xi^j} V^j \Rightarrow \begin{aligned} v^1 &= \frac{\partial x^1}{\partial \xi^1} V^1 + \frac{\partial x^1}{\partial \xi^2} V^2 \\ v^2 &= \frac{\partial x^2}{\partial \xi^1} V^1 + \frac{\partial x^2}{\partial \xi^2} V^2 \end{aligned} \Rightarrow \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} \end{bmatrix}\begin{bmatrix} V^1 \\ V^2 \end{bmatrix}$$

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} \xi^1 \cos\xi^2 \\ \xi^1 \sin\xi^2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} (3x^1-2x^2)\cos(-4x^1+3x^2) \\ (3x^1-2x^2)\sin(-4x^1+3x^2) \end{bmatrix}$$

$$v^1 = 3(3x^1-2x^2)\cos(-4x^1+3x^2) + 2(3x^1-2x^2)\sin(-4x^1+3x^2)$$
$$v^2 = 4(3x^1-2x^2)\cos(-4x^1+3x^2) + 3(3x^1-2x^2)\sin(-4x^1+3x^2)$$

$$\boxed{\begin{aligned} v^1 &= (3x^1-2x^2)[3\cos(-4x^1+3x^2) + 2\sin(-4x^1+3x^2)] \\ v^2 &= (3x^1-2x^2)[4\cos(-4x^1+3x^2) + 3\sin(-4x^1+3x^2)] \end{aligned}}$$

AME 60611 Homework #2 Solution

9. Course notes: 1.18 For the parabolic coordinates

$$\xi^1 = x^1 x^2 \cos x^3$$
$$\xi^2 = x^1 x^2 \sin x^3$$
$$\xi^3 = \frac{1}{2}\left((x^2)^2 - (x^1)^2\right)$$

Find the Jacobian matrix $\underline{J}$ and the metric tensor $\underline{G}$. Find the transformation $x^i = x^i(\xi^j)$. Plot lines of constant x^1 and x^2 in the ξ^1 and ξ^2 plane.

Jacobian $\underline{J} = \begin{bmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^1}{\partial x^3} \\ \frac{\partial \xi^2}{\partial x^1} & \frac{\partial \xi^2}{\partial x^2} & \frac{\partial \xi^2}{\partial x^3} \\ \frac{\partial \xi^3}{\partial x^1} & \frac{\partial \xi^3}{\partial x^2} & \frac{\partial \xi^3}{\partial x^3} \end{bmatrix} = \boxed{\begin{bmatrix} x^2\cos x^3 & x^1\cos x^3 & -x^1x^2\sin x^3 \\ x^2\sin x^3 & x^1\sin x^3 & x^1x^2\cos x^3 \\ -x^1 & x^2 & 0 \end{bmatrix} = \underline{J}}$

Metric Tensor $\underline{G} = \underline{J}^T \underline{J} = \begin{bmatrix} x^2\cos x^3 & x^2\sin x^3 & -x^1 \\ x^1\cos x^3 & x^1\sin x^3 & x^2 \\ -x^1x^2\sin x^3 & x^1x^2\cos x^3 & 0 \end{bmatrix}\begin{bmatrix} x^2\cos x^3 & x^1\cos x^3 & -x^1x^2\sin x^3 \\ x^2\sin x^3 & x^1\sin x^3 & x^1x^2\cos x^3 \\ -x^1 & x^2 & 0 \end{bmatrix}$

$$\underline{G} = \begin{bmatrix} (x^2)^2+(x^1)^2 & x^1x^2 - x^1x^2 & 0 \\ x^1x^2 - x^1x^2 & (x^1)^2+(x^2)^2 & 0 \\ 0 & 0 & (x^1x^2)^2 \end{bmatrix} = \boxed{\begin{bmatrix} (x^1)^2+(x^2)^2 & 0 & 0 \\ 0 & (x^1)^2+(x^2)^2 & 0 \\ 0 & 0 & (x^1x^2)^2 \end{bmatrix} = \underline{G}}$$

Note that $\frac{\xi^2}{\xi^1} = \frac{\cancel{x^1x^2}\sin x^3}{\cancel{x^1x^2}\cos x^3} = \tan x^3 \Rightarrow x^3 = \arctan\left(\frac{\xi^2}{\xi^1}\right)$

and $2\xi^3 = (x^2)^2 - (x^1)^2$ $\qquad (\xi^1)^2 + (\xi^2)^2 = (x^1x^2)^2\cos^2 x^3 + (x^1x^2)^2\sin^2 x^3$

$\Rightarrow (\xi^1)^2 + (\xi^2)^2 = (x^1)^2(x^2)^2 \Rightarrow (x^1)^2 = \frac{1}{(x^2)^2}\left((\xi^1)^2+(\xi^2)^2\right)$

$\qquad (x^2)^2 = \frac{1}{(x^1)^2}\left((\xi^1)^2+(\xi^2)^2\right)$

$\Rightarrow 2\xi^3 = \frac{1}{(x^1)^2}\left((\xi^1)^2+(\xi^2)^2\right) - (x^1)^2 = (x^2)^2 - \frac{1}{(x^2)^2}\left((\xi^1)^2+(\xi^2)^2\right)$

This then gives

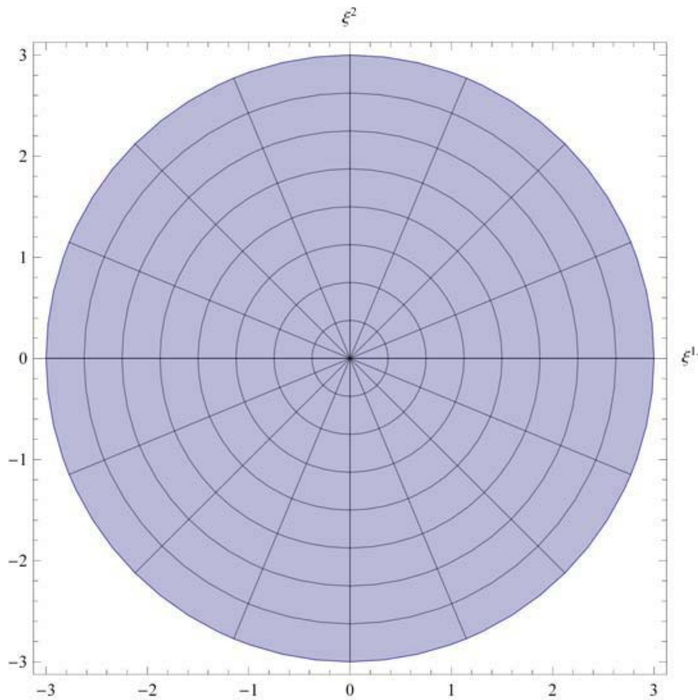
$$\boxed{\begin{aligned} x^1 &= \pm\sqrt{-\xi^3 \pm \sqrt{(\xi^1)^2+(\xi^2)^2+(\xi^3)^2}} \qquad x^2 = \pm\sqrt{\xi^3 \pm \sqrt{(\xi^1)^2+(\xi^2)^2+(\xi^3)^2}} \\ x^3 &= \arctan\left(\frac{\xi^2}{\xi^1}\right) \end{aligned}}$$

```
ξ1 == x1 x2 Cos[x3];
ξ2 == x1 x2 Sin[x3];
ξ3 == (1 / 2) (x2^2 - x1^2);
```

Note that from these equations we have that tan(θ) = ξ2/ξ1, which gives that cos(θ) = $\xi 1 \Big/ \sqrt{\xi 1\,^\wedge 2 + \xi 2\,^\wedge 2}$. This gives that ξ1 = x1 x2 $\xi 1 \Big/ \sqrt{\xi 1\,^\wedge 2 + \xi 2\,^\wedge 2}$, which can be simplified to ξ1^2+ξ2^2 = (x1 x2)^2. So for constant values of x1 and x2, this is the equation for a circle.

In[4]:=
```
ParametricPlot[{x1x2 Cos[x3], x1x2 Sin[x3] },
 {x1x2, -3, 3}, {x3, 0, 2 Pi}, AxesLabel → {ξ^1., ξ^2}]
```

Out[4]=



AME 60611 Homework #2 Solution

10. Course notes: 1.19 For the parabolic coordinate system of the previous problem, find $\nabla^T \cdot \vec{u}$ where $\vec{u}$ is an arbitrary vector

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Recall $G = \begin{bmatrix} (x^1)^2+(x^2)^2 & 0 & 0 \\ 0 & (x^1)^2+(x^2)^2 & 0 \\ 0 & 0 & (x^1x^2)^2 \end{bmatrix} \Rightarrow g = ((x^1)^2+(x^2)^2)((x^1)^2+(x^2)^2)(x^1x^2)^2$

$$g = ((x^1)^2+(x^2)^2)^2(x^1x^2)^2 = (x^1x^2((x^1)^2+(x^2)^2))^2$$

$$\Rightarrow \sqrt{g} = x^1x^2((x^1)^2+(x^2)^2) = (x^1)^3x^2 + x^1(x^2)^3$$

$$\nabla^T \cdot \vec{u} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g}\, u^i)$$

$$= \frac{1}{x^1x^2((x^1)^2+(x^2)^2)} \frac{\partial}{\partial x^i}\left(((x^1)^3x^2 + x^1(x^2)^3)u^i\right)$$

$$= \frac{1}{x^1x^2((x^1)^2+(x^2)^2)}\left[\frac{\partial}{\partial x^1}\left(((x^1)^3x^2 + x^1(x^2)^3)u^1\right) + \frac{\partial}{\partial x^2}\left(((x^1)^3x^2 + x^1(x^2)^3)u^2\right) + \frac{\partial}{\partial x^3}\left(((x^1)^3x^2 + x^1(x^2)^3)u^3\right)\right]$$

$$= \frac{1}{x^1x^2((x^1)^2+(x^2)^2)}\left[3(x^1)^2x^2u^1 + (x^2)^3u^1 + ((x^1)^3x^2 + x^1(x^2)^3)\frac{\partial u^1}{\partial x^1} + (x^1)^3u^2 + 3x^1(x^2)^2u^2 + ((x^1)^3x^2 + x^1(x^2)^3)\frac{\partial u^2}{\partial x^2} + ((x^1)^3x^2 + x^1(x^2)^3)\frac{\partial u^3}{\partial x^3}\right]$$

$$= \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} + \frac{(3(x^1)^2x^2 + (x^2)^3)u^1}{(x^1x^2)((x^1)^2+(x^2)^2)} + \frac{((x^1)^3 + 3x^1(x^2)^2)}{(x^1x^2)((x^1)^2+(x^2)^2)}u^2$$

$$\boxed{\nabla^T \cdot \vec{u} = \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} + \frac{u^1}{x^1} + \frac{u^2}{x^2} + \frac{2(x^1u^1 + x^2u^2)}{((x^1)^2+(x^2)^2)}}$$