Problem 1

Kaplan, p. 436: 2c,i

Find the first three nonzero terms of the following Taylor series:

c) $\ln(1+x)^2$ about $x = 0$

i) $\operatorname{arctanh} x$ about $x = 0$

The definition of a Taylor Series is:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{\partial^n f}{\partial x^n}\right|_{x=x_0} (x - x_0)^n \tag{1}$$

Part c

Using the definition from Equation 1 for (2c), the summary in Table 1 below is produced.

n	$\frac{\partial^n f}{\partial x^n}$	$\left.\frac{\partial^n f}{\partial x^n}\right\|_{x=x_0}$	$\frac{1}{n!}\left.\frac{\partial^n f}{\partial x^n}\right\|_{x=x_0}$	$\frac{1}{n!}\left.\frac{\partial^n f}{\partial x^n}\right\|_{x=x_0}(x-x_o)^n$
0	$\log(1+x)^2$	0	0	0
1	$\frac{2\log(1+x)}{1+x}$	0	0	0
2	$\frac{2}{(1+x)^2} - \frac{2\log(1+x)}{(1+x)^2}$	2	1	x^2
3	$-\frac{6}{(1+x)^3} + \frac{4\log(1+x)}{(1+x)^3}$	-6	-1	$-x^3$
4	$\frac{22}{(1+x)^4} - \frac{12\log(1+x)}{(1+x)^4}$	22	$\frac{11}{12}$	$\frac{11}{12}x^4$

Table 1: Development of Taylor Series for $\ln(x+1)^2$

Summing the elements of the final column, the three term approximation is:

$$f \approx x^2 - x^3 + \frac{11}{12}x^4$$

Part i

Using the definition from Equation 1 for (2c), the summary in Table 2 below is produced.

n	$\frac{\partial^n f}{\partial x^n}$	$\left.\frac{\partial^n f}{\partial x^n}\right\|_{x=x_0}$	$\frac{1}{n!}\left.\frac{\partial^n f}{\partial x^n}\right\|_{x=x_0}$	$\frac{1}{n!}\left.\frac{\partial^n f}{\partial x^n}\right\|_{x=x_0}(x-x_o)^n$
0	$\operatorname{arctanh} x$	0	0	0
1	$\frac{1}{1-x^2}$	1	1	x
2	$\frac{2x}{(1-x^2)^2}$	0	0	0
3	$\frac{8x^2}{(1-x^2)^3} + \frac{2}{(1-x^2)^2}$	2	$\frac{1}{3}$	$\frac{x^3}{3}$
4	$\frac{48x^3}{(1-x^2)^4} + \frac{24x}{(1-x^2)^3}$	0	0	0
5	$\frac{384x^4}{(1-x^2)^5} + \frac{288x^2}{(1-x^2)^4} + \frac{24}{(1-x^2)^3}$	24	$\frac{1}{5}$	$\frac{x^5}{5}$

Table 2: Development of Taylor Series for $\operatorname{arctanh} x$

Summing the elements of the final column, the three term approximation is:

$$f \approx x + \frac{x^3}{3} + \frac{x^5}{5}$$

Problem 2

Kaplan, p. 649: 5

Show that

$$J_0(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{x^{2n}}{(n!)^2} \tag{2}$$

Satisfies Bessel's equation of order 0 in the form

$$xy'' + y' + xy = 0 \tag{3}$$

Computing the fist and second derivatives of Equation 2 as

$$J_0'(x) = \sum_{n=1}^{\infty} 2n \left(-\frac{1}{4}\right)^n x^{2n-1}/(n!)^2$$
$$J_0''(x) = \sum_{n=1}^{\infty} 2n(2n-1) \left(-\frac{1}{4}\right)^n x^{2n-2}/(n!)^2$$

Remapping these indices of the summation to $n = 0$

$$J_0'(x) = \sum_{n=0}^{\infty} 2(n+1) \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^n x^{2n+1}/((n+1)!)^2$$
$$J_0''(x) = \sum_{n=0}^{\infty} 2(n+2)(2n+1) \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^n x^{2n}/((n+1)!)^2 \tag{4}$$

Placing the definitions from Equations 2 and 4, into Equation 3 and combining like terms,

$$\sum_{n=0}^{\infty} \left[\left\{(2n+2)(2n+1)\left(-\frac{1}{4}\right) + 2(n+1)\left(-\frac{1}{4}\right) + (n+1)^2\right\}\left(-\frac{1}{4}\right)^n x^{2n+1}/(n+1)!\right] = 0 \tag{5}$$

For this to be true for the summation it must hold for all values of x and n, therefore

$$(2n+2)(2n+1)\left(-\frac{1}{4}\right) + 2(n+1)\left(-\frac{1}{4}\right) + (n+1)^2 = 0 \tag{6}$$

And upon expanding Equation 6, we see that this is true and therefore the series from Equation 2 satisfies Equation 3.

Problem 3

Course notes, 4.1a

Solve as a series in x for $x > 0$ about the point $x = 0$:

$$\begin{aligned} x^2y'' - 2xy' + (x+1)y &= 0 \\ y(1) &= 1 \\ y(4) &= 0 \end{aligned} \tag{7}$$

Examining Equation 7, and recognizing that is of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Since $P(0) = 0$, is not an ordinary point. Examining $xQ(x)/P(x) = -2$ and $x^2Q(x)/P(x) = x + 1$ are both analytic at $x = 0$ therefore, this is a regular singular point problem. Therefore there exist a solution for y is of the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{8}$$

Plugging this assumed form into the differential equation and simplifying yields,

$$\left[\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r}\right] + \left[\sum_{n=0}^{\infty} -2a_n(n+r)x^{n+r}\right] + \left[\sum_{n=0}^{\infty} a_n x^{n+r+1}\right] + \left[\sum_{n=0}^{\infty} a_n x^{n+r}\right] = 0$$

Combining like powers of x into a single summation and simplifying the result

$$\left[\sum_{n=0}^{\infty} a_n[(n+r)^2 - 3(n+r) + 1]x^{n+r}\right] + \left[\sum_{n=0}^{\infty} a_n x^{n+r+1}\right] = 0$$

Removing the $n = 0$ term from the first summation and remapping the index back to zero,

$$a_0(r^2 - 3r + 1)x^r + \left[\sum_{n=0}^{\infty} a_{n+1}[(n+r+1)^2 - 3(n+r+1) + 1]x^{n+r+1}\right] + \left[\sum_{n=0}^{\infty} a_n x^{n+r+1}\right] = 0$$

Combining the two summations

$$a_0(r^2 - 3r + 1)x^r + \sum_{n=0}^{\infty} [a_n + a_{n+1}\{(n+r+1)^2 - 3(n+r+1) + 1\}]x^{n+r+1} = 0$$

The term outside the summation gives us the indicial equation

$$r^2 - 3r + 1 = 0$$

Therefore $r = \frac{3\pm\sqrt{5}}{2}$. Using one of these values, the term outside of the summation is zero and therefore the coefficient of x^{n+r+1} inside the summation must be zero for all values of x and n. Therefore,

$$\begin{aligned} a_n + a_{n+1}((n+r+1)^2 - 3(n+r+1) + 1) &= 0 \\ a_{n+1} &= -\frac{a_n}{(n+r+1)^2 - 3(n+r+1) + 1} \end{aligned}$$

For $r = \frac{3+\sqrt{5}}{2}$

$$a_{n+1} = -\frac{a_n}{(n+1+\sqrt{5})(n+1)} = -\frac{a_0}{(n+1)!\prod_{i=1}^{n+1}(i+\sqrt{5})}$$

Therefore, the first solution to Equation 7 is,

$$y_1 = a_0 x^{\frac{1}{2}(3+\sqrt{5})}\left(1 - \frac{1}{1+\sqrt{5}}x + \frac{1}{14+6\sqrt{5}}x^2 + \frac{1}{24}\left(4\sqrt{5} - 9\right)x^3 + \ldots\right)$$

For $r = \frac{3-\sqrt{5}}{2}$

$$b_{n+1} = -\frac{b_n}{(n+1-\sqrt{5})(n+1)} = -\frac{b_0}{(n+1)!\prod_{i=1}^{n+1}(i-\sqrt{5})}$$

Therefore, the second solution to Equation 7 is,

$$y_2 = b_0 x^{\frac{1}{2}(3-\sqrt{5})}\left(1 + \frac{1}{\sqrt{5}-1}x + \frac{1}{8}\left(7+3\sqrt{5}\right)x^2 + \frac{(7+3\sqrt{5})}{24\left(\sqrt{5}-3\right)}x^3 + \dots\right)$$

Combining the two solutions, $y = y_1 + y_2$, then considering the boundary conditions $a_0 = 0.675038$ and $b_0 = 0.183975$. Therefore, we can plot the solution below.

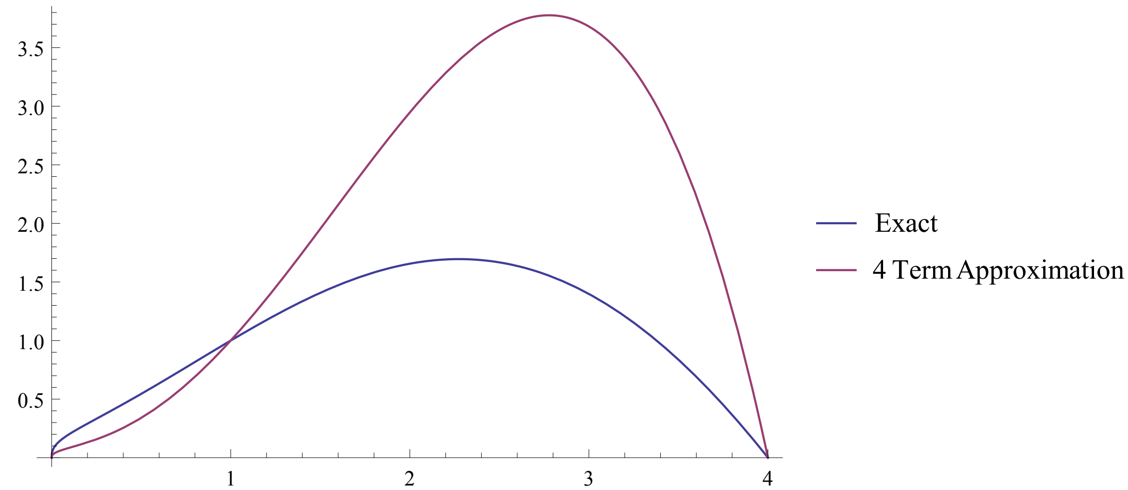

Figure 1: Exact and approximate solution to $x^2y'' - 2xy' + (x+1)y = 0$

Problem 4

Course notes, 4.2

Find two-term expansions for each of the roots of:

$$(x-1)(x+3)(x-3\lambda)+1=0$$

where λ is large.
Multiplying out the terms:

$$x^3+(2-3\lambda)x^2-(6\lambda+3)x+(1+9\lambda)=0$$

Dividing through by λ, and substituting in a small number ϵ for $\frac{1}{\lambda}$,

$$\epsilon x^3+(2\epsilon-3)x^2-(6+3\epsilon)x+(\epsilon+9)=0 \tag{9}$$

Examining this problem, we see that as $\epsilon\to 0$, a solution is lost. Therefore, we must change variables, we can perform the transformation,

$$X=\frac{x}{\epsilon^\alpha} \tag{10}$$

Using the transformation from Equation 10 on Equation 9,

$$X^3\epsilon^{3\alpha+1}-3X^2\epsilon^{2\alpha}+2X^2\epsilon^{2\alpha+1}-3X(\epsilon+2)\epsilon^\alpha+\epsilon+9=0 \tag{11}$$

The two highest order terms of X are in the same order of ϵ if $1+3\alpha=2\alpha$, Therefore, we demand that $\alpha=-1$. With this, Equation 11 becomes,

$$X^3+X^2(2\epsilon-3)-3X\epsilon(\epsilon+2)+\epsilon^2(\epsilon+9)=0 \tag{12}$$

Let x be expressed as a sum in ϵ,

$$X=\sum_{n=0}^{\infty}X_n\epsilon^n\approx X_0+X_1\epsilon+X_2\epsilon^2+X_3\epsilon^3 \tag{13}$$

Therefore, separating on orders of ϵ and recognizing that since the expansion is 0 each term must also equal 0, the equations become

$$\begin{aligned}
X_0^3-3X_0^2&=0\\
3X_1X_0^2+2X_0^2-6X_1X_0-6X_0&=0\\
3X_2X_0^2+3X_1^2X_0+4X_1X_0-6X_2X_0-3X_0-3X_1^2-6X_1+9&=0\\
X_1^3+2X_1^2+6X_0X_2X_1-6X_2X_1-3X_1+4X_0X_2-6X_2+3X_0^2X_3-6X_0X_3+1&=0
\end{aligned}$$

Sequentially solving these equations until we have two terms for each solution, we find

$$\begin{aligned}
(X_0,X_1,X_2)&=\left(0,-3,-\frac{1}{12}\right)\\
(X_0,X_1,X_2)&=\left(0,1,\frac{1}{12}\right)\\
(X_0,X_1,X_2,X_3)&=\left(3,0,0,-\frac{1}{9}\right)
\end{aligned} \tag{14}$$

Then combining the values from Equation 14 and the initial approximation from Equation 13, we find,

$$X\approx-\frac{\epsilon^2}{12}-3\epsilon,\frac{\epsilon^2}{12}+\epsilon \text{ or } 3-\frac{\epsilon^3}{9}$$

Therefore

$$\begin{aligned}
x&\approx-\frac{\epsilon}{12}-3,\frac{\epsilon}{12}+1,\ \text{or}\ \frac{3}{\epsilon}-\frac{\epsilon^2}{9}\\
x&\approx-\frac{1}{12\lambda}-3,\frac{1}{12\lambda}+1,\ \text{or}\ 3\lambda-\frac{1}{9\lambda^2}
\end{aligned}$$

Problem 5

Course notes, 4.11b

Find all solutions through $O(\epsilon^2)$, where ϵ is a small parameter, and compare with the exact result for $\epsilon = 0.01$.

$$2\epsilon x^4 + 2(2\epsilon+1)x^3 + (7-2\epsilon)x^2 - 5x - 4 = 0 \tag{15}$$

Starting this problem, we see that as $\epsilon \to 0$, a solution is lost. Therefore, we must change variables, we can perform the transformation,

$$X = \frac{x}{\epsilon^\alpha} \tag{16}$$

Using the transformation from Equation 16 on Equation 15,

$$2\epsilon^{4\alpha+1}X^4 + 2(2\epsilon^{3\alpha+1} + \epsilon^{3\alpha})X^3 + (7\epsilon^{2\alpha} - 2\epsilon^{2\alpha+1})X^2 - 5\epsilon^\alpha X - 4 = 0 \tag{17}$$

The two highest order terms of X are in the same order of ϵ if $4\alpha+1 = 3\alpha$, Therefore, we demand that $\alpha = -1$. With this, Equation 17 becomes,

$$2\epsilon^{-3}X^4 + 2(2\epsilon^{-2} + \epsilon^{-3})X^3 + (7\epsilon^{-2} - 2\epsilon^{-1})X^2 - 5\epsilon^{-1}X - 4 = 0 \tag{18}$$

In order to solve this problem, we start by assuming that X can be written in the form,

$$X = \sum_{n=0}^{\infty} X_n\epsilon^n = X_0 + X_1\epsilon + X_2\epsilon^2 + X_3\epsilon^3 + \dots \tag{19}$$

Then placing the expansion from Equation 19 into Equation 18, we find,

$$\begin{aligned} &2\epsilon^{-3}(X_0 + X_1\epsilon + X_2\epsilon^2 + X_3\epsilon^3 + \dots)^4 + 2(2\epsilon^{-2} + \epsilon^{-3})(X_0 + X_1\epsilon + X_2\epsilon^2 + X_3\epsilon^3 + \dots)^3 + \\ &(7\epsilon^{-2} - 2\epsilon^{-1})(X_0 + X_1\epsilon + X_2\epsilon^2 + X_3\epsilon^3 + \dots)^2 - 5\epsilon^{-1}(X_0 + X_1\epsilon + X_2\epsilon^2 + X_3\epsilon^3 + \dots) - 4 = 0 \end{aligned} \tag{20}$$

Expanding this and separating by powers of ϵ and recognizing that if the sum of all terms is 0 then each order of ϵ must be 0 as well. Examining these equations,

$$\begin{aligned} &\epsilon^0 & -2X_0^4 - 2X_0^3 &= 0 \\ &\epsilon^1 & -8X_1X_0^3 - 4X_0^3 - 6X_1X_0^2 - 7X_0^2 &= 0 \\ &\epsilon^2 & -8X_2X_0^3 - 12X_1^2X_0^2 - 12X_1X_0^2 - 6X_2X_0^2 + 2X_0^2 - 6X_1^2X_0 - 14X_1X_0 + 5X_0 &= 0 \\ &\epsilon^3 & -8X_3X_0^3 - 24X_1X_2X_0^2 - 12X_2X_0^2 - 6X_3X_0^2 - 8X_1^3X_0 - 12X_1^2X_0 + 4X_1X_0 & \\ & & -12X_1X_2X_0 - 14X_2X_0 - 2X_1^3 - 7X_1^2 + 5X_1 + 4 &= 0 \\ &\vdots & &\vdots \end{aligned} \tag{21}$$

Solving Equations 21 we see,

$$(X_0, X_1, X_2, X_3) = \left(0, -4, -\frac{32}{5}, -\frac{31232}{875}\right), \left(0, -\frac{1}{2}, -\frac{1}{12}, -\frac{11}{378}\right), \left(0, 1, -\frac{4}{15}, \frac{752}{3375}\right), \left(-1, \frac{3}{2}, \frac{27}{4}, \frac{71}{2}\right) \tag{22}$$

Therefore using the definition from Equation 19, the data from Equation 22 and the transformation defined in Equation 16, the $O(\epsilon^2)$ solutions are,

$$\begin{aligned} X &= -4\epsilon - \frac{32}{5}\epsilon^2 - \frac{31232}{875}\epsilon^3 &\to\quad x &= -\frac{31232}{875}\epsilon^2 - \frac{32}{5}\epsilon - 4 \\ X &= -\frac{1}{2}\epsilon - \frac{1}{12}\epsilon^2 - \frac{11}{378}\epsilon^3 &\to\quad x &= -\frac{11}{378}\epsilon^2 - \frac{1}{12}\epsilon - \frac{1}{2} \\ X &= \epsilon - \frac{4}{15}\epsilon^2 + \frac{752}{3375}\epsilon^3 &\to\quad x &= \frac{752}{3375}\epsilon^2 - \frac{4}{15}\epsilon + 1 \\ X &= -1 + \frac{3}{2}\epsilon + \frac{27}{4}\epsilon^2 + \frac{71}{2}\epsilon^3 &\to\quad x &= \frac{71}{2}\epsilon^2 + \frac{27}{4}\epsilon + \frac{3}{2} - \frac{1}{\epsilon} \end{aligned}$$

Now, in the case of $\epsilon = 0.01$, our estimate of the roots of Equation 15 would be

$$x = -4.06757, -0.500836, 0.997356, -98.429$$

While the exact solution is,

$$x = -4.06783, -0.500836, 0.997355, -98.4287$$

The $O(\epsilon^2)$ approximation provides an excellent approximation of the roots, the maximum relative error is 6.5×10^{-5}.

Problem 6

Course notes, 4.12

Find three terms of a solution of

$$x + \epsilon \cos(x + 2\epsilon) = \frac{\pi}{2} \tag{23}$$

where ϵ is a small parameter. For $\epsilon = 0.2$, compare the best asymptotic solution with the exact solution.
In order to solve this problem, we start by assuming that x can be written in the form,

$$x = \sum_{n=0}^{\infty} x_n \epsilon^n = x_0 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \dots \tag{24}$$

Then placing the approximation from 24 into Equation 23, we find,

$$(x_0 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \dots) + \epsilon \cos((x_0 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \dots) + 2\epsilon) = \frac{\pi}{2} \tag{25}$$

Performing a Taylor series expansion in ϵ about $\epsilon = 0$, splitting this up by powers of ϵ,

$$\begin{aligned} \epsilon^0 &: & x_0 - \frac{\pi}{2} &= 0 \\ \epsilon^1 &: & x_1 + \cos(x_0) &= 0 \\ \epsilon^2 &: & x_2 - (x_1 + 2)\sin(x_0) &= 0 \\ \epsilon^3 &: & x_3 - x_2\sin(x_0) - \frac{1}{2}(x_1+2)^2\cos(x_0) &= 0 \\ &\vdots & &\vdots \end{aligned} \tag{26}$$

Solving for the terms in Equation 26, we find $x_0 = \frac{\pi}{2}, x_1 = 0, x_2 = 2, x_3 = 2$. Therefore the three term asymptotic solution of Equation 23 is,

$$x = \frac{\pi}{2} + 2\epsilon^2 + 2\epsilon^3$$

In the case of $\epsilon = 0.2$, this yields $x_{Appx} = 1.6668$, while the exact solution is $x_{Exact} = 1.6658$.

Problem 7

Course notes, 4.16

The solution of the matrix equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{y}$ can be written as $\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{y}$. Find the perturbation solution of

$$(\mathbf{A} + \epsilon \mathbf{B}) \cdot \mathbf{x} = \mathbf{y}, \tag{27}$$

where ϵ is a small parameter.
Assuming x is of the form

$$\mathbf{x} = \sum_{n=0}^{\infty} \epsilon^n \mathbf{x}_n = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \epsilon^2 \mathbf{x}_2 + \dots \tag{28}$$

This means that Equation 27 can be written as,

$$(\mathbf{A} + \epsilon \mathbf{B}) \cdot (\mathbf{x}_0 + \epsilon \mathbf{x}_1 + \epsilon^2 \mathbf{x}_2 + \dots) = \mathbf{y} \tag{29}$$

Distributing through the dot product and grouping by powers of ϵ, Equation 29 becomes,

$$(\mathbf{A} \cdot \mathbf{x}_0 - \mathbf{y}) + \epsilon(\mathbf{B} \cdot \mathbf{x}_0 + \mathbf{A} \cdot \mathbf{x}_1) + \epsilon^2(\mathbf{B} \cdot \mathbf{x}_1 + \mathbf{A} \cdot \mathbf{x}_2) + \cdots = 0 \tag{30}$$

Since the summation of the terms is zero, each power of ϵ must be as well therefore,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{x}_0 &= \mathbf{y} \\ \mathbf{B} \cdot \mathbf{x}_0 + \mathbf{A} \cdot \mathbf{x}_1 &= \mathbf{0} \\ \mathbf{B} \cdot \mathbf{x}_1 + \mathbf{A} \cdot \mathbf{x}_2 &= \mathbf{0} \\ &\vdots \end{aligned}$$

Solving each line for the unknown $\mathbf{x}$,

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{A}^{-1} \cdot \mathbf{y} \\ \mathbf{x}_1 &= -\mathbf{A}^{-1} \cdot \mathbf{B} \cdot \mathbf{x}_0 \\ \mathbf{x}_2 &= -\mathbf{A}^{-1} \cdot \mathbf{B} \cdot \mathbf{x}_1 \\ &\vdots \end{aligned}$$

Back substituting, we find that in general,

$$\mathbf{x}_n = \mathbf{A}^{-1} \cdot \left(-\mathbf{B} \cdot \mathbf{A}^{-1}\right)^n \cdot \mathbf{y} \tag{31}$$

Therefore using the definition of $\mathbf{x}$ from Equation 28 and the values of $\mathbf{x}_n$ from Equation 31, the perturbation solution of Equation 27 is,

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \left[\sum_{n=0}^{\infty} \left(-\epsilon \mathbf{B} \cdot \mathbf{A}^{-1}\right)^n \right] \cdot \mathbf{y}$$

Problem 8

Course notes, 4.17

Find all solutions of

$$\epsilon x^4 + x - 2 = 0 \tag{32}$$

approximately, if ϵ is small and positive. If $\epsilon = 0.001$, compare the exact solution obtained numerically with the asymptotic solution.

Note as $\epsilon \to 0$, the equation becomes singular. Let

$$X = \frac{x}{\epsilon^\alpha} \tag{33}$$

Using the transformation from Equation 33 on Equation 32,

$$\epsilon^{4\alpha+1} X^4 + \epsilon^\alpha X - 2 = 0 \tag{34}$$

The two highest order terms of X are in the same order of ϵ if $4\alpha + 1 = \alpha$, Therefore, we demand that $\alpha = -\frac{1}{3}$. With this, Equation 34 becomes,

$$\epsilon^{-\frac{1}{3}} X^4 + \epsilon^{-\frac{1}{3}} X - 2 = 0 \tag{35}$$

In order to solve this problem, we start by assuming that X can be written in the form,

$$X = \sum_{n=0}^{\infty} X_n \epsilon^{\frac{n}{3}} = X_0 + X_1 \epsilon^{\frac{1}{3}} + X_2 \epsilon^{\frac{2}{3}} + X_3 \epsilon + \dots \tag{36}$$

Then placing the expansion from Equation 36 into Equation 35, we find,

$$(X_0 + X_1 \epsilon^{\frac{1}{3}} + X_2 \epsilon^{\frac{2}{3}} + X_3 \epsilon + \dots)^4 + (X_0 + X_1 \epsilon^{\frac{1}{3}} + X_2 \epsilon^{\frac{2}{3}} + X_3 \epsilon + \dots) - 2\epsilon^{\frac{1}{3}} = 0 \tag{37}$$

Expanding this and separating by powers of ϵ and recognizing that if the sum of all terms is 0 then each order of ϵ must be 0 as well. Examining these equations,

$$\begin{aligned}
&\epsilon^0 && X_0^4 + X_0 = 0 \\
&\epsilon^{\frac{1}{3}} && 4X_1 X_0^3 + X_1 - 2 = 0 \\
&\epsilon^{\frac{2}{3}} && 4X_2 X_0^3 + 6X_1^2 X_0^2 + X_2 = 0 \\
&\epsilon^1 && 4X_3 X_0^3 + 12 X_1 X_2 X_0^2 + 4X_1^3 X_0 + X_3 = 0 \\
&\vdots && \vdots
\end{aligned} \tag{38}$$

Solving Equations 38 we see,

$$\begin{aligned}
(X_0, X_1, X_2, X_3) = &\left(-1, -\frac{2}{3}, \frac{8}{9}, -\frac{160}{81}\right), \\
&(0, 2, 0, 0), \\
&\left(\sqrt[3]{-1}, -\frac{2}{3}, \frac{8}{9}\left(\sqrt[3]{-1} - 1\right), \frac{160\sqrt[3]{-1}}{81}\right), \\
&\left(-(-1)^{2/3}, -\frac{2}{3}, \frac{8}{9}\left(-1 - (-1)^{2/3}\right), \frac{1}{81}(-160)(-1)^{2/3}\right)
\end{aligned} \tag{39}$$

Therefore using the definition from Equation 36, the data from Equation 39 and the transformation defined in Equation 33, the $O(\epsilon)$ solutions are,

$$\begin{aligned}
X &= \frac{8\epsilon^{2/3}}{9} - \frac{1}{3} 2\sqrt[3]{\epsilon} - \frac{160\epsilon}{81} - 1 \\
X &= 2\sqrt[3]{\epsilon} \\
X &= \frac{8}{9}\left(\sqrt[3]{-1} - 1\right)\epsilon^{2/3} - \frac{1}{3} 2\sqrt[3]{\epsilon} + \frac{1}{81}\sqrt[3]{-1} 160\epsilon + \sqrt[3]{-1} \\
X &= \frac{8}{9}\left(-1 - (-1)^{2/3}\right)\epsilon^{2/3} - \frac{1}{3} 2\sqrt[3]{\epsilon} - \frac{160}{81}(-1)^{2/3}\epsilon - (-1)^{2/3}
\end{aligned}$$

Performing the inverse transformation to convert X to x,

$$
\begin{aligned}
x &= -\frac{160\epsilon^{2/3}}{81} + \frac{8\sqrt[3]{\epsilon}}{9} - \frac{1}{\sqrt[3]{\epsilon}} - \frac{2}{3} \\
x &= 2 \\
x &= \frac{\frac{8}{9}\left(\sqrt[3]{-1} - 1\right)\epsilon^{2/3} - \frac{1}{3}2\sqrt[3]{\epsilon} + \frac{1}{81}\sqrt[3]{-1}160\epsilon + \sqrt[3]{-1}}{\sqrt[3]{\epsilon}} \\
x &= \frac{-72\left(1 + (-1)^{2/3}\right)\epsilon^{2/3} - 54\sqrt[3]{\epsilon} - 160(-1)^{2/3}\epsilon - 81(-1)^{2/3}}{81\sqrt[3]{\epsilon}}
\end{aligned}
$$

Using this approximation when $\epsilon = 0.001$, we find the approximate roots are

$$
\begin{aligned}
x_1 &= -10.5975 \\
x_2 &= 2 \\
x_3 &= 4.29877 + 8.75434i \\
x_4 &= 4.29877 - 8.75434i
\end{aligned}
$$

and the exact roots are

$$
\begin{aligned}
x_1 &= -10.5934 \\
x_2 &= 1.98449 \\
x_3 &= 4.30446 - 8.75258i \\
x_4 &= 4.30446 + 8.75258i
\end{aligned}
$$

Problem 9

Course notes, 4.18

Obtain the first two terms of an approximate solution to

$$\begin{aligned} \ddot{x} + 3(1+\epsilon)\dot{x} + 2x = 0, \text{ with} \\ x(0) = 2(1+\epsilon), \\ \dot{x}(0) = -3(1+2\epsilon), \end{aligned} \tag{40}$$

for small ϵ. Compare with the exact solution graphically in the range $0 \le t \le 1$ for (a) $\epsilon = 0.1$, (b) $\epsilon = 0.25$ and (c) $\epsilon = 0.5$.

Letting x be of the form $x = x_0 + \epsilon x_1 + \dots$. Therefore the second derivative takes the form, $\ddot{x} = \dot{x}_0 + \epsilon \dot{x}_1 + \dots$ and the second derivative takes the form, $\ddot{x} = \ddot{x}_0 + \epsilon \ddot{x}_1 + \dots$. Equation 40 then becomes,

$$(\ddot{x}_0 + \epsilon\ddot{x}_1 + \dots) + 3(1+\epsilon)(\dot{x}_0 + \epsilon\dot{x}_1 + \dots) + 2(x_0 + \epsilon x_1 + \dots) = 0, \text{ with } x(0) = 2(1+\epsilon),\ \dot{x}(0) = -3(1+2\epsilon)$$

Combining the sums with the same power of ϵ, and recognizing that all orders of ϵ are linearly independent, since the of all powers of ϵ must be zero, each order of ϵ sums to zero, ie,

$$\begin{array}{llll} \epsilon^0: & x_0''(t) + 3x_0'(t) + 2x_0(t) = 0, & x_0(0) = 2, & x_0'(1) = -3 \\ \epsilon^1: & x_1''(t) + 3x_1'(t) + 2x_1(t) + 3x_0' = 0, & x_1(0) = 2, & x_1'(t) = -6 \end{array}$$

Solving these sequentially (note that since the initial conditions are of mixed order of ϵ they are separated as well),

$$\begin{aligned} x_0 &= e^{-2t}\left(e^t + 1\right) \\ x_1 &= e^{-2t}\left(-6t + e^t(3t+1) + 1\right) \end{aligned}$$

We find the exact and the two term approximate solution to Equation 40 to be,

$$x_{Appx} = e^{-2t}\left(-6t\epsilon + e^t(3t\epsilon + \epsilon + 1) + \epsilon + 1\right)$$

Plotting the cases of $\epsilon = 0.1$, $\epsilon = 0.25$ and $\epsilon = 0.5$.

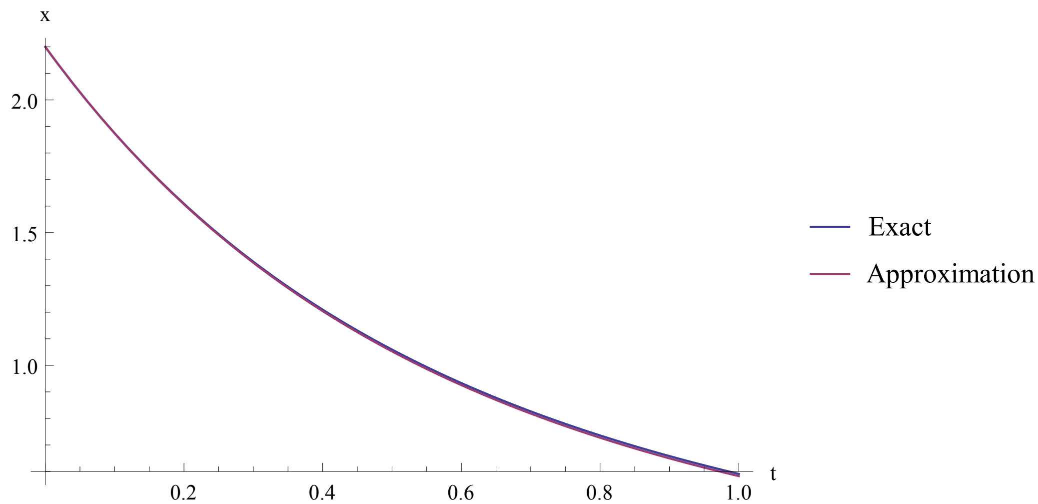

Figure 2: $\epsilon = 0.1$

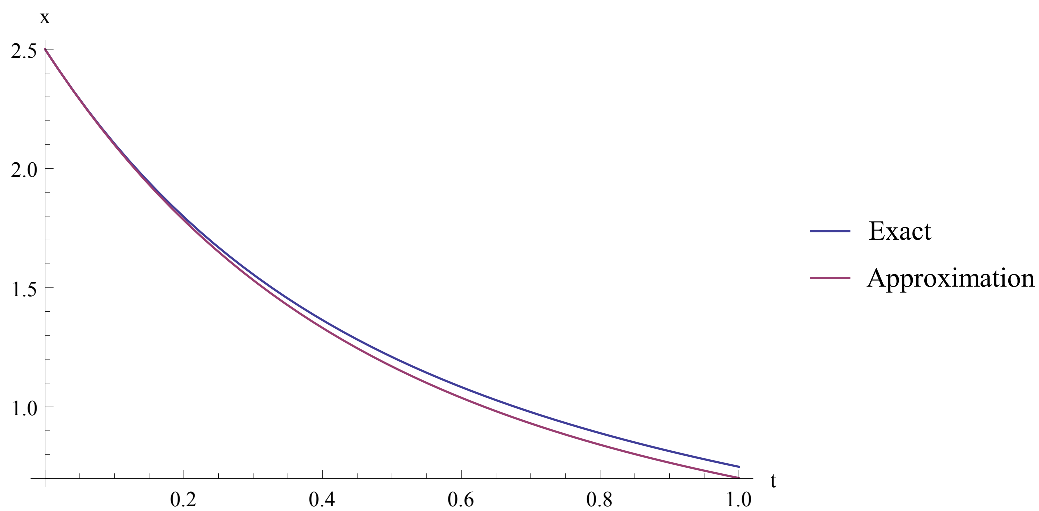

Figure 3: $\epsilon = 0.25$

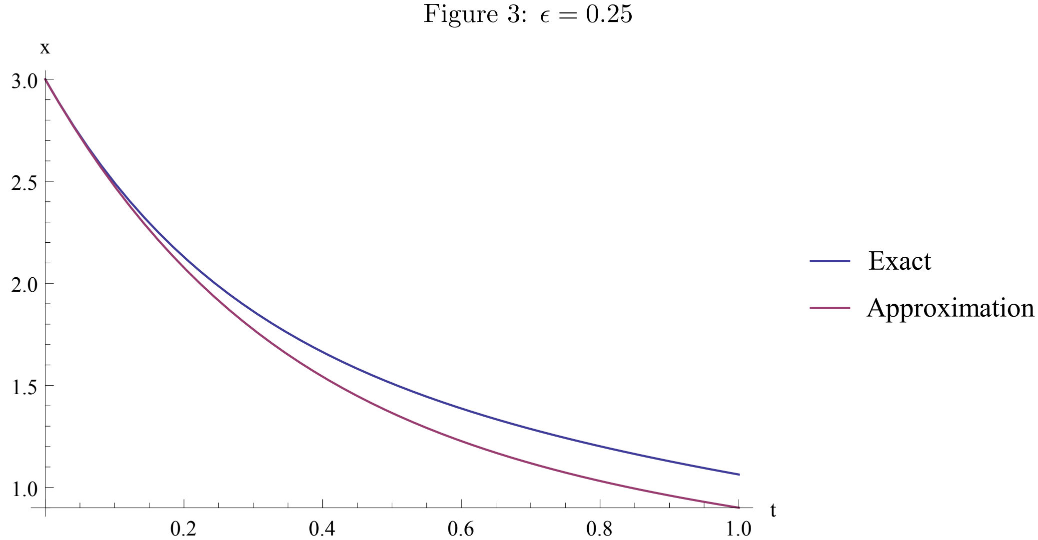

Figure 4: $\epsilon = 0.5$

Problem 10

Course notes, 4.58

Find the solution of the transcendental equation

$$\sin x = \epsilon \cos 2x, \tag{41}$$

near $x = \pi$ for small positive ϵ.
If we substitute $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$ into Equation 41, we find

$$\sin\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots\right) = \epsilon \cos 2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) \tag{42}$$

Performing a Taylor Series on Equation 42 about $\epsilon = 0$ and collecting by powers of ϵ,

$$\begin{aligned}
\epsilon^0 :& \quad \sin(x_0) = 0 \\
\epsilon^1 :& \quad \cos(x_0)x_1 - \cos(2x_0) = 0 \\
\epsilon^2 :& \quad -\frac{1}{2}\sin(x_0)x_1^2 + 2\sin(2x_0)x_1 + \cos(x_0)x_2 = 0 \\
\epsilon^3 :& \quad -\frac{1}{6}\cos(x_0)x_1^3 + 2\cos(2x_0)x_1^2 - \sin(x_0)x_2x_1 + 2\sin(2x_0)x_2 + \cos(x_0)x_3 = 0 \\
\vdots \; & \quad \vdots
\end{aligned} \tag{43}$$

Solving Equation 43, we see $x_0 = \pi$, $x_1 = -1$, $x_2 = 0$, $x_3 = \frac{11}{6}$. Therefore

$$x = \pi - \epsilon + \frac{11}{6}\epsilon^3 + \dots$$