AME 60611
Examination 2: SOLUTION
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1. (20) Given $x \in \mathbb{R}^1$, $f : \mathbb{R}^1 \to \mathbb{R}^1$,

$$f(x) = \sin x, \qquad x \in [-1, 1],$$

find the first two terms in a Fourier-Legendre expansion of $f(x)$. The first two Legendre polynomials are $P_o(s) = 1$, $P_1(s) = s$, $s \in [-1, 1]$.

Solution

$$f(x) = \sin x = 0 + 3x(\sin 1 - \cos 1) + \dots.$$

If one finds the first two non-zero terms, one would get

$$f(x) = \sin x = 3x(\sin 1 - \cos 1) + \frac{7}{2}(5x^3 - 3x)(14\cos 1 - 9 \sin 1) + \dots$$

2. (20) If $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$, evaluate

$$I = \oint_C \mathbf{u}^T \cdot d\mathbf{r},$$

where C is a single loop around the closed contour described by the ellipse, $x^2 + 4y^2 = 1$.

Solution

Here we happen to have

$$\mathbf{u} = \nabla \phi,$$

where the potential ϕ is

$$\phi = \frac{1}{2}(x^2 + y^2).$$

So the integral becomes

$$I = \oint_C \nabla \phi \cdot d\mathbf{r}.$$

$$I = \oint_C d\phi = 0.$$

Green's theorem can also be used to achieve the same result.

3. (20) For $x \in \overline{\mathbb{L}}_2[0,1]$, $y \in \overline{\mathbb{L}}_2[0,1]$, $t \in \mathbb{R}^1$, find the angle between the two functions

$$x(t) = it,$$

$$y(t) = t.$$

Solution

$$||x||_2 = \frac{1}{\sqrt{3}}.$$

$$||y||_2 = \frac{1}{\sqrt{3}}.$$

$$<x, y> = -\frac{i}{3}.$$

$$\cos\theta = \frac{<x, y>}{||x||_2||y||_2}.$$

$$\cos\theta = \frac{-i/3}{(1/\sqrt{3})(1/\sqrt{3})} = -i.$$

$$\theta = \cos^{-1}(-i) = \frac{\pi}{2} + 2n\pi + i\ln\left(1 + \sqrt{2}\right), \qquad n = 0, 1, 2, \ldots$$

4. (20) For $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^2$, find the vector $\mathbf{x} \in \mathbb{R}^3$ of minimum $||\mathbf{x}||_2$ which minimizes $||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Solution

$$\mathbf{x} = \frac{7}{30}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

5. (20) Find the singular value decomposition of $\mathbf{A}$ where

$$\mathbf{A} = \begin{pmatrix} 3i & 4 & 0 \end{pmatrix}.$$

Solution

One decomposition is

$$\mathbf{A} = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3i/5 & 4/5 & 0 \\ 0 & 0 & 1 \\ -4i/5 & 3/5 & 0 \end{pmatrix}.$$

The decomposition is not unique.