AME 60611
Examination 1: Solution
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1. (25) Consider the curve $\mathcal{C}$ defined by the intersection of two surfaces: 1) the unit sphere

$$x^2 + y^2 + z^2 = 1,$$

and 2) the plane

$$x + y + z = 1.$$

Find the minimum value of y on $\mathcal{C}$ and the values of x and z on $\mathcal{C}$ where y takes on its minimum value.

Solution

This can be done via Lagrange multipliers or a more direct approach, and simply eliminate either z or x, get a single equation, and minimize y using the approach of freshman calculus. That is the approach I take here.

The two surfaces are shown in Figure 1. in Figure 2. Eliminating z, one gets

$$x^2 + y^2 + (1 - x - y)^2 = 1.$$

Expanding, one finds

$$x^2 + y^2 - x + xy - y = 0.$$

Solving for $y(x)$ via the quadratic equation, one gets

$$y = \frac{1 - x \pm \sqrt{1 + 2x - 3x^2}}{2}.$$

The critical points are found at values of x for which $dy/dx = 0$. Taking the derivative, one finds

$$\frac{dy}{dx} = \frac{1}{2}\left(-1 \pm \frac{1 - 3x}{\sqrt{1 + 2x - 3x^2}}\right).$$

Setting the derivative to 0 for the critical points, one finds

$$\begin{aligned}
0 &= \frac{1}{2}\left(-1 \pm \frac{1 - 3x}{\sqrt{1 + 2x - 3x^2}}\right), \\
1 &= \pm\frac{1 - 3x}{\sqrt{1 + 2x - 3x^2}}, \\
\sqrt{1 + 2x - 3x^2} &= \pm(1 - 3x), \\
1 + 2x - 3x^2 &= 1 - 6x + 9x^2, \\
8x - 12x^2 &= 0, \\
4x(2 - 3x) &= 0.
\end{aligned}$$

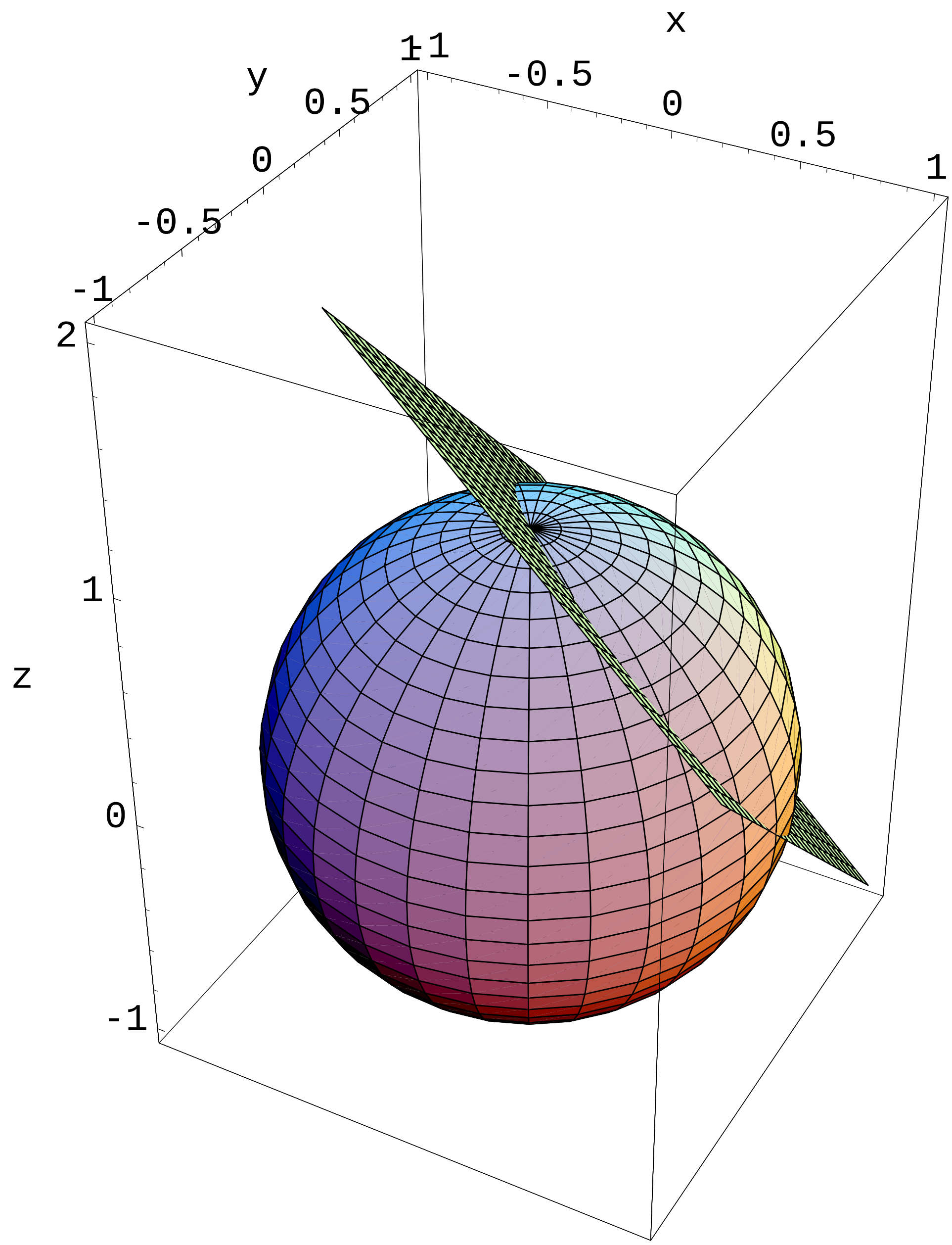

Figure 1: Plot of two intersecting surfaces given by the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 1$.

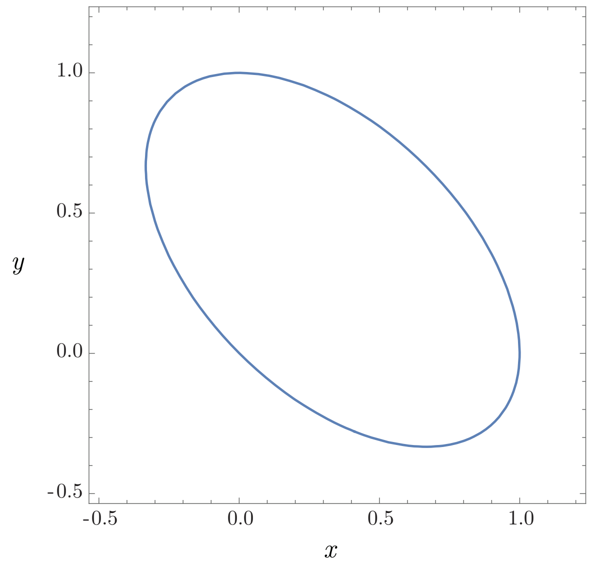

Figure 2: Plot of $\mathcal{C} : x^2 + y^2 - x + xy - y = 0$, which is the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 1$.

This yields

$$x = 0, \qquad x = \frac{2}{3}.$$

When $x = 0$, $y = 0$ or $y = 1$. When $x = 2/3$, $y = -1/3$ or $y = 2/3$. One could do a second derivative test, which would reveal what seems obvious, that y takes on a mininum value at $x = 2/3$. For this point, we find

$$z = 1 - x - y = 1 - \frac{2}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}.$$

So y is minimized on $\mathcal{C}$ at the point $(x, y, z) = (2/3, -1/3, 2/3)$. This is obvious when one plots the curve $\mathcal{C}$, as done in Figure 2.

The same solution can be achieved with the alternative *Lagrange multipliers approach.* With this approach, one must remember that y is being minimized. One could then take several approaches. Let us pose the function to be minimized as

$$y = 1 - x - z.$$

The constraint must not contain y. So take the constraint as

$$0 = 1 - x^2 - z^2 - (1 - x - z)^2.$$

The constraint can be rewritten as

$$0 = x - x^2 + z - xz - z^2.$$

So, we can take the Lagrange multiplier formulation as

$$y = 1 - x - z + \lambda(x - x^2 + z - xz - z^2).$$

Now take the appropriate partial derivatives

$$\begin{aligned} \frac{\partial y}{\partial x} &= -1 + \lambda(1 - 2x - z) = 0, \\ \frac{\partial y}{\partial z} &= -1 + \lambda(1 - x - 2z) = 0. \end{aligned}$$

Combine these with the constraint to form three equations in three unknowns, x, z, and λ.

$$\begin{aligned} -1+\lambda(1-2x-z) &= 0, \\ -1+\lambda(1-x-2z) &= 0, \\ x-x^2+z-xz-z^2 &= 0. \end{aligned}$$

Leaving out the solution details, there are two roots for these equations:

$$(x,z,\lambda) = \left(\frac{2}{3},\frac{2}{3},-1\right), \quad (0,0,1).$$

For the first root, we get $y=-1/3$. For the second root, we get $y=1$. Obviously, the first root gives the minimum y. So, the solution is the point

$$(x,y,z) = \left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right).$$

The other root corresponds to a maximum value of y on $\mathcal{C}$.

2. (25) Consider

$$x\frac{dy}{dx}-y^2+y=0, \qquad y(0)=-1.$$

Determine a solution if a solution exists. If it exists, determine whether it is unique.

Solution

This equation is singular at $x=0$, so we expect some potential troubles, especially since the initial condition is specified at $x=0$. There are two straightforward ways to deal with this problem: 1) as a Bernoulli equation, whose approach I outline in detail in the following paragraphs, or 2) separation of variables, in which one gets

$$\frac{dy}{y^2-y}=\frac{dx}{x},$$

followed by a partial fraction expansion of $1/(y^2-y)$ and integration of what remains.

Here is the approach treating the equation as a Bernoulli equation. Rearranging the equation, we get

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{x}y^2.$$

This is a Bernoulli equation. Define then,

$$\begin{aligned} u &\equiv \frac{1}{y}, \\ y &= \frac{1}{u}, \\ \frac{dy}{dx} &= -\frac{1}{u^2}\frac{du}{dx}. \end{aligned}$$

Replacing y with u in the ODE, we get

$$\begin{aligned} -\frac{1}{u^2}\frac{du}{dx} + \frac{1}{x}\frac{1}{u} = \frac{1}{x}\frac{1}{u^2}, \\ \frac{du}{dx} - \frac{1}{x}u &= -\frac{1}{x}. \end{aligned}$$

The integrating factor is

$$\exp\left(\int\left(\frac{-1}{x}\right)dx\right) = \exp\left(-\ln x\right) = \frac{1}{x}.$$

Multiplying the ODE by the integrating factor, we get

$$\begin{aligned} \frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u &= -\frac{1}{x^2}, \\ \frac{d}{dx}\left(\frac{u}{x}\right) &= -\frac{1}{x^2}, \\ \frac{u}{x} &= \frac{1}{x} + C, \\ u &= 1 + Cx, \\ \frac{1}{y} &= 1 + Cx, \\ y &= \frac{1}{1+Cx} \end{aligned}$$

Now, when $x = 0$, $y = 1$, so the initial condition is not satisfied. Thus, a solution does not exist.

In fact a solution exists *only* for the initial condition $y(0) = 1$. However, in that case, the solution is *not unique*, since for all $C \in \mathbb{R}^1$, the differential equation and initial condition are satisfied.

3. (25) Use the Green's function method to find the general solution on the domain $x \in [0, \infty)$ to

$$\frac{dy}{dx} + y = f(x), \qquad y(0) = 1.$$

It can help to transform y to a new dependent variable to render the boundary condition to be homogeneous.

Solution

We need homogeneous boundary conditions, so take

$$z = y - 1.$$

Despite the hint, most students did not understand what was suggested here, and missed this essential step for the Green's function method.

The transformed problem is

$$\frac{dz}{dx} + (z+1) = f(x), \qquad z(0) = 0.$$

Take now

$$h(x) = f(x) - 1,$$

so that

$$\frac{dz}{dx} + z = h(x), \qquad z(0) = 0.$$

Here the operator $\mathbf{L}$ is

$$\mathbf{L} = \frac{d}{dx} + 1.$$

Consider first $x < s$. Solving for $\mathbf{L}g = 0$, we get

$$\begin{aligned} \frac{dg}{dx} + g &= 0, \\ g &= Ae^{-x} \end{aligned}$$

Many students chose the *incorrect* path (or some permutation) of ignoring terms and solving $dg/dx = 0$; $g = C$. This does not work.

Now g must satisfy the boundary conditions on x so

$$g(0) = 0 = Ae^0.$$

Therefore $A = 0$, and

$$g = 0, \qquad x < s.$$

Now for $x > s$, we have

$$\begin{aligned} \frac{dg}{dx} + g &= 0, \\ g &= Be^{-x}. \end{aligned}$$

The highest order derivative is unity, so g itself must suffer a jump at $x = s$. Most students missed this important point. The jump for an equation of order n occurs at the $n-1$ level. So for this case there is no jump in the first derivative, as there is for second order ODEs; the jump is on g itself. Since the leading coefficient on dy/dx is unity, the jump on g is also unity:

$$\begin{aligned} g(s+\epsilon) - g(s-\epsilon) &= 1, \\ Be^{-s+\epsilon} - 0 &= 1, \\ B &= e^{s-\epsilon}, \\ \lim_{\epsilon \to 0} B &= e^s. \end{aligned}$$

Therefore one gets

$$g(x,s) = e^{s-x}, \qquad x > s.$$

So the general solution is

$$\begin{aligned} z &= \int_0^x g(x,s)h(s)\ ds + \int_x^\infty g(x,s)h(s)\ ds, \\ &= \int_0^x e^{s-x}\ h(s)\ ds + \int_x^\infty 0\ h(s)\ ds, \\ &= e^{-x} \int_0^x e^s h(s)\ ds, \\ y - 1 &= e^{-x} \int_0^x e^s (f(s) - 1)\ ds, \\ y(x) &= 1 + e^{-x} \int_0^x e^s (f(s) - 1)\ ds. \end{aligned}$$

This can be simplified to form

$$y(x) = e^{-x}\left(1 + \int_0^x e^s f(s)\ ds\right).$$

Let us test our solution in the case where $f(x) = 2$. Then

$$\begin{aligned} y(x) &= e^{-x}\left(1 + \int_0^x 2e^s\ ds\right), \\ &= e^{-x}\left(1 + 2\ e^s\ |_0^x\right), \\ &= e^{-x}\left(1 + 2\left(e^x - 1\right)\right), \\ &= 2 - e^{-x}. \end{aligned}$$

The boundary condition is satisfied:

$$y(0) = 2 - e^0 = 1.$$

The differential equation, $dy/dx + y = 2$ is satisfied as it reduces to

$$e^{-x} + 2 - e^{-x} = 2.$$

4. (25) If $0 < \epsilon << 1$, $x \in [0, 1]$, find an appropriate $O(1)$ and $O(\epsilon)$ solution for

$$x\frac{dy}{dx} - \epsilon y = 0, \qquad y(1) = 1.$$

Compare to the exact solution.

Solution

Many students got this right. One surprise was that many students chose to *mistakenly* expand $x = x_0 + \epsilon x_1 + \ldots$. That is a misunderstanding of the approach of the method.

First try a regular expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots.$$

Substituting into the ODE and IC, we get

$$x\frac{d}{dx}\left(y_0 + \epsilon y_1 + \ldots\right) - \epsilon\left(y_0 + \ldots\right) = 0, \qquad y_0(1) + \epsilon y_1(1) + \ldots = 1.$$

At leading order we get then

$$x\frac{dy_0}{dx} = 0, \qquad y_0(1) = 1.$$

For $x \neq 0$, the unique solution is

$$y_0 = 1.$$

A disturbingly large number of students decided to take some permutation of the *incorrect* step of $dy_0 = (0/x)dx$; $dy_0 = dx$, $y_0 = x + C$. This leads one far afield.

At $O(\epsilon)$, one gets

$$\begin{aligned}
x\frac{dy_1}{dx} &= y_0, \qquad y_1(1) = 0,\\
x\frac{dy_1}{dx} &= 1,\\
\frac{dy_1}{dx} &= \frac{1}{x},\\
y_1 &= \ln x + C,\\
0 &= \ln(1) + C,\\
0 &= C,\\
y_1 &= \ln x,\\
y &\sim 1 + \epsilon \ln x + \ldots.
\end{aligned}$$

Obviously, this solution encounters problems as $x \to 0$. In fact the first term is as large as the second when

$$\epsilon \ln x \sim 1,$$

$$x \sim e^{-1/\epsilon}.$$

Try then the stretching

$$X = \frac{x}{e^{-1/\epsilon}}.$$

This gives

$$x = e^{-1/\epsilon}X, \qquad dx = e^{-1/\epsilon}dX.$$

The ODE becomes then

$$\begin{aligned}
e^{-1/\epsilon}Xe^{1/\epsilon}\frac{dy}{dX} - \epsilon y &= 0, \qquad y(1) = 1,\\
X\frac{dy}{dX} - \epsilon y &= 0.
\end{aligned}$$

This is unchanged from the orginal, so the stretching does no good!

Let's try to get an exact solution. The equation is first order linear, with an integrating factor of $\exp(\int -\epsilon/x\ dx) = \exp(-\epsilon \ln x) = 1/x^\epsilon$, and can be solved with standard methods:

$$\begin{aligned}
\frac{dy}{dx} - \frac{\epsilon}{x}y &= 0,\\
\frac{1}{x^\epsilon}\frac{dy}{dx} - \frac{\epsilon}{x^{1+\epsilon}}y &= 0,\\
\frac{d}{dx}\left(\frac{y}{x^\epsilon}\right) &= 0,\\
\frac{y}{x^\epsilon} &= C,\\
y &= Cx^\epsilon,\\
1 &= C1^\epsilon,\\
1 &= C,\\
y &= x^\epsilon.
\end{aligned}$$

To get a Taylor series of the *exact* solution, it is helpful to re-express it as

$$y = e^{\epsilon \ln x}.$$

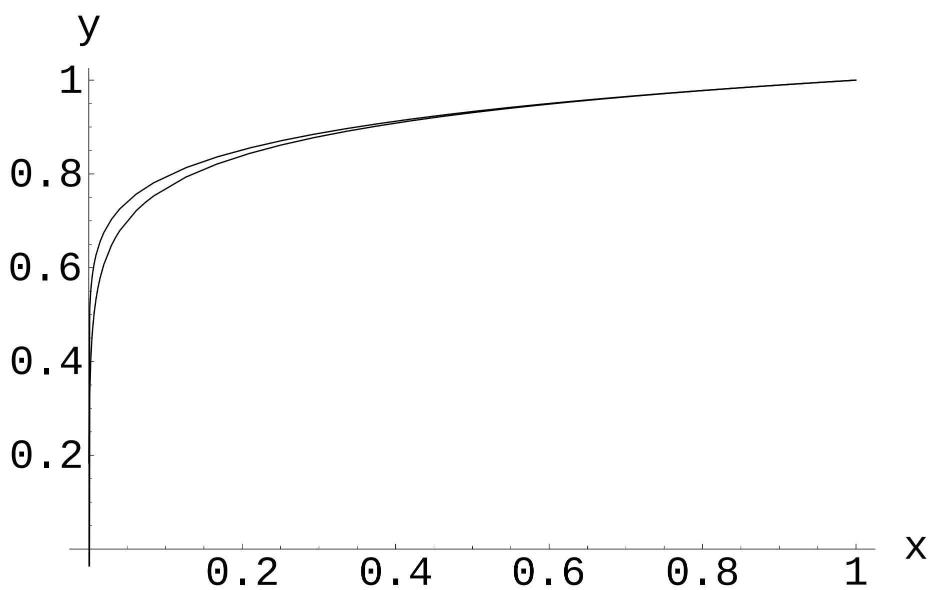

Figure 3: Exact solution, $y = x^\epsilon$, and two-term asymptotic solution $y \sim 1 + \epsilon \ln x$ for $\epsilon = 0.1$.

Now the Taylor series expansion of the exact solution about $\epsilon = 0$ yields

$$y = 1 + \epsilon \ln x + \frac{1}{2}\left(\epsilon \ln x\right)^2 + \ldots + \frac{1}{n!}\left(\epsilon \ln x\right)^n .$$

The ratio test tells us about the convergence of the series and gives the ratio of the n-term to the $n-1$-term, r, as

$$r = \frac{\frac{1}{n!}\left(\epsilon \ln x\right)^n}{\frac{1}{(n-1)!}\left(\epsilon \ln x\right)^{n-1}} = \frac{\epsilon}{n} \ln x.$$

For any fixed values of ϵ and x, other than zero, the ratio of terms goes to zero as $n \to \infty$, thus the series is convergent for $x \neq 0$. So in fact, there is nothing wrong with the outer solution that was found earlier, except for the singularity at $x = 0$. Note that the exact solution gives $y(0) = 0$, and is well behaved for $x \in [0, 1]$.

The exact solution and the two term asymptotic solution are shown in Figure 3.