AME 60611
Examination 2: **Solution**
J. M. Powers
23 November 2015

1. (20) Consider the curve in $\mathbb{R}^3$ defined parametrically by

$$
\begin{aligned}
x &= \sqrt{t},\\
y &= t,\\
z &= t.
\end{aligned}
$$

(a) Find the length of the curve from $(0,0,0)$ to $(1,1,1)$. You need not numerically evaluate the resulting integral.

(b) Find the unit tangent at the point $(1,1,1)$.

Solution

The point $(0,0,0)$ corresponds to $t=0$. The point $(1,1,1)$ corresponds to $t=1$.

By the Pythagorean theorem, we have for a differential element of arc length ds that

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

Scaling by dt, one gets

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Then making the substitutions and integrating, one gets

$$
\begin{aligned}
\frac{ds}{dt} &= \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + 1^2 + 1^2},\\
&= \sqrt{2 + \frac{1}{4t}},\\
ds &= \sqrt{2 + \frac{1}{4t}}\, dt,\\
s &= \int_0^1 \sqrt{2 + \frac{1}{4t}}\, dt,\\
&= \frac{1}{16}\left(8\sqrt{\frac{1}{t} + 8}\,t + \sqrt{2}\ln\left(4\left(\sqrt{2}\sqrt{\frac{1}{t} + 8} + 4\right)t + 1\right)\right)\Bigg|_0^1,\\
&= \frac{1}{16}\left(24 + \sqrt{2}\ln\left(17 + 12\sqrt{2}\right)\right),\\
&= 1.81161.
\end{aligned}
$$

The tangent vector is given by

$$\begin{aligned}
\mathbf{t} &= \frac{\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}, \\
&= \left.\frac{\frac{1}{2\sqrt{t}}\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}}{\sqrt{\frac{1}{4t} + 1^2 + 1^2}}\right|_{t=1}, \\
&= \frac{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3}.
\end{aligned}$$

2. (20) Consider two functions in $\mathbb{L}_2[0,1]$: $v_1 = 1$, $v_2 = t$.

   (a) Determine if v_1 and v_2 are orthonormal.

   (b) Project the Heaviside function $H(t-1/2)$ onto the space spanned by v_1 and v_2; that is, find the constants α_1, α_2 that best approximate

$$\alpha_1 v_1 + \alpha_2 v_2 \approx H(t-1/2).$$

Solution

To test for orthogonality of v_1 and v_2, one can use the inner product, which is

$$\begin{aligned}
<v_1, v_2> &= \int_0^1 v_1(t) v_2(t)\ dt, \\
&= \int_0^1 (1)t\ dt, \\
&= \left.\frac{t^2}{2}\right|_0^1, \\
&= \frac{1}{2}.
\end{aligned}$$

The inner product is not zero, so the vectors are not orthogonal, so they cannot be orthonormal.

Next, let us project $H(t-1/2)$ onto the space spanned by v_1 and v_2. So we seek α_1 and α_2 such that

$$\alpha_1 v_1 + \alpha_2 v_2 \approx H(t-1/2).$$

Take two inner products, one with v_1 and the other with v_2:

$$\begin{aligned}
<v_1, \alpha_1 v_1 + \alpha_2 v_2> &= <v_1, H(t-1/2)>, \\
<v_2, \alpha_1 v_1 + \alpha_2 v_2> &= <v_2, H(t-1/2)>
\end{aligned}$$

Using the properties of the inner product, we find then that

$$\begin{aligned}
\alpha_1 <v_1, v_1> + \alpha_2 <v_1, v_2> &= <v_1, H(t-1/2)>, \\
\alpha_1 <v_2, v_1> + \alpha_2 <v_2, v_2> &= <v_2, H(t-1/2)>.
\end{aligned}$$

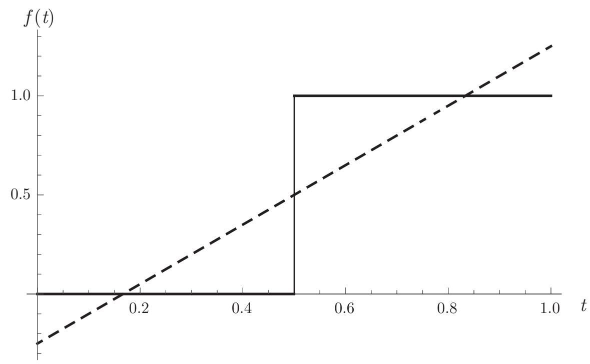

Figure 1: The function $H(t-1/2)$ and its projection onto the space spanned by the functions $v_1 = 1$ and $v_2 = t$.

In matrix form, this gives

$$\begin{pmatrix} <v_1, v_1> & <v_1, v_2> \\ <v_2, v_1> & <v_2, v_2> \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} <v_1, H(t-1/2)> \\ <v_2, H(t-1/2)> \end{pmatrix}.$$

Now replace the inner product with its integral form to get

$$\begin{pmatrix} \int_0^1 v_1 v_1 \ dt & \int_0^1 v_1 v_2 \ dt \\ \int_0^1 v_2 v_1 \ dt & \int v_2 v_2 \ dt \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \int_{1/2}^1 v_1 \ dt \\ \int_{1/2}^1 v_2 \ dt> \end{pmatrix}.$$

Now substitute for v_1 and v_2 to get

$$\begin{pmatrix} \int_0^1 (1)(1) \ dt & \int_0^1 (1)t \ dt \\ \int_0^1 t(1) \ dt & \int tt \ dt \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \int_{1/2}^1 (1) \ dt \\ \int_{1/2}^1 t \ dt> \end{pmatrix}.$$

Evaluating each of the integrals, we find

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{8} \end{pmatrix}.$$

Solving the two equations in two unknowns gives

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{2} \end{pmatrix}.$$

So the projection of $H(t-1/2)$ in the space spanned by 1 and t is

$$H(t-1/2) \approx -\frac{1}{4} + \frac{3}{2}\ t.$$

A plot of $H(t-1/2)$ and its projection is given in Figure 1.

3. (20) For $\mathbf{A} : \mathbb{C}^3 \to \mathbb{C}^2$, find the vector $\mathbf{x} \in \mathbb{C}^3$ of smallest $||\mathbf{x}||_2$ which minimizes the error norm $||\mathbf{A} \cdot \mathbf{x} - \mathbf{b}||_2$ when

$$\mathbf{A} = \begin{pmatrix} i & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$\mathbf{b} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Solution

Consider

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{x} &\approx \mathbf{b}, \\
\mathbf{A}^H \cdot \mathbf{A} \cdot \mathbf{x} &= \mathbf{A}^H \cdot \mathbf{b}, \\
\begin{pmatrix} -i & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -i & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix}, \\
\begin{pmatrix} 1 & -i & -i \\ i & 1 & 1 \\ i & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}, \\
\begin{pmatrix} 1 & -i & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Take as free variables $x_2 = s$, $x_3 = t$. Then, solving, one finds

$$x_1 = 1 + is + it.$$

So one has

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}.$$

The vectors of which s and t are coefficients span the right null space of $\mathbf{A}$. The other vector has components in both the row space and right null space of $\mathbf{A}$. Let us find the part of that vector which lies in the row space. Thus, we solve for the constants α_1, α_2 and α_3 in the linear combination of row space and right null space vectors. Note the row space vector, when cast into a column, requires use of the Hermitian transpose:

$$\alpha_1 \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

In matrix form this is

$$\begin{pmatrix} -i & i & i \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solving, we find

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} i/3 \\ -i/3 \\ -i/3 \end{pmatrix}.$$

So the solution vector is then expressed as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (i/3) \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} + (s - i/3) \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + (t - i/3) \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}.$$

The vector $\mathbf{x}$ with smallest norm is found by removing the null space components. Doing so, we find

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (i/3) \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ i/3 \\ i/3 \end{pmatrix}.$$

4. (20) Find a singular value decomposition of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solution

The singular value decomposition is

$$\mathbf{A} = \mathbf{Q}_1 \cdot \mathbf{\Sigma} \cdot \mathbf{Q}_2^T$$

The method outlined in the textbook can be used to get $\mathbf{Q}_1$, $\mathbf{\Sigma}$, and $\mathbf{Q}_2$.

With

$$\mathbf{Q}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Q}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

it is seen that

$$\mathbf{Q}_1 \cdot \mathbf{\Sigma} \cdot \mathbf{Q}_2^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A}.$$

We note that $\mathbf{Q}_1$ is an identity matrix and $\mathbf{Q}_2$ is a reflection matrix. The matrix $\mathbf{Q}_2^T$ reflects vectors about an axis inclined at an angle $\pi/4$ to the horizontal axis. The matrix $\mathbf{\Sigma}$ suppresses the second component of the reflected vector. The matrix $\mathbf{Q}_1$ maps vectors into themselves.

5. (20) Using a collocation method within the method of weighted residuals, find a one-term approximation to the solution of the following problem:

$$\frac{d^2y}{dx^2} - y = -x^3, \qquad y(0) = y(1) = 0.$$

Choose an appropriate polynomial trial function.

Solution

Let us assume that the trial function is $\phi(x) = x(1-x)$; this guarantees satisfaction of both boundary conditions.

$$y_p = cx(1-x),$$

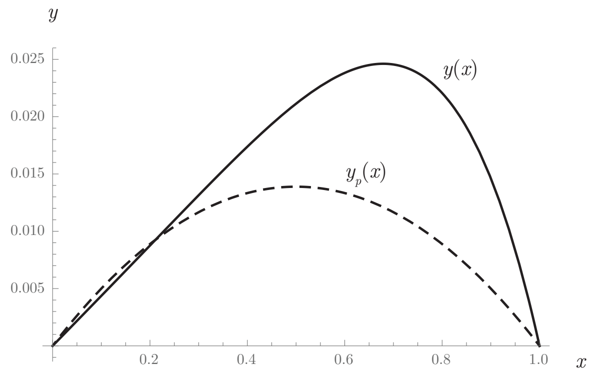

Figure 2: Exact and approximate solutions to $y'' - y = x^3$, $y(0) = y(1) = 0$.

We need to find c. Now we have the residual as

$$r(x) = \frac{d^2 y_p}{dx^2} - y_p + x^3 = -2c - c(1-x)x + x^3.$$

We need then for the weighted residual to be zero.

$$\int_0^1 \psi(x) r(x)\ dx = 0.$$

Let us take $\psi(x) = \delta(x - 1/2)$ so

$$\int_0^1 \delta(x - 1/2)\left(-2c - c(1-x)x + x^3\right)\ dx = 0.$$

Evaluating, this gives

$$-2c - c(1 - (1/2))(1/2) + (1/2)^3 = 0.$$

Simplifying, we get

$$\frac{1}{8} - \frac{9c}{4} = 0.$$

Solving gives

$$c = \frac{1}{18}.$$

Thus

$$y_p = \frac{1}{18} x(1-x).$$

The exact solution can be shown to be

$$-\frac{e^{-x}\left(e^x x^3 - e^{x+2} x^3 + 6e^x x - 6e^{x+2} x + 7e^{2x+1} - 7e\right)}{e^2 - 1}.$$

A plot is shown in Fig. 2.