LECTURE NOTES ON MATHEMATICAL METHODS II

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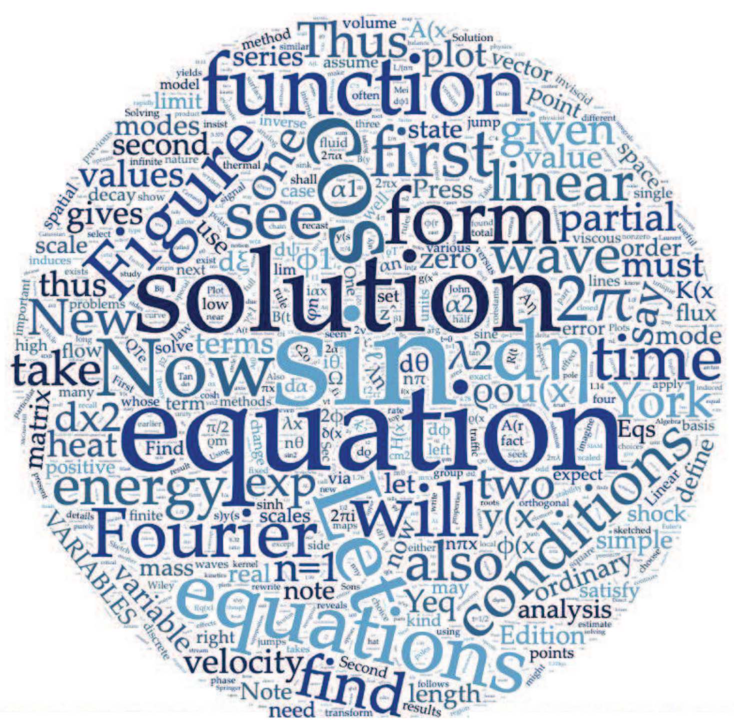

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Preface

These are lecture notes for AME 60612 Mathematical Methods II, the second of a pair of courses on applied mathematics taught in the Department of Aerospace and Mechanical Engineering of the University of Notre Dame. Most of the students in this course are beginning graduate students in engineering coming from a variety of backgrounds. The course objective is to survey topics in applied mathematics, with the focus being on partial differential equations. Specific topics include physical motivations, classification, separation of variables, one-dimensional waves, similarity, complex variables, integral transform methods, and integral equations.

These notes emphasize method and technique over rigor and completeness; the student should call on textbooks and other reference materials. It should also be remembered that practice is essential to learning; the student would do well to apply the techniques presented by working as many problems as possible. The notes, along with much information on the course, can be found at `https://www3.nd.edu/~powers/ame.60612`. At this stage, members of the class have permission to download the notes. I ask that you not distribute them.

These notes may have typographical errors. Do not hesitate to identify those to me. I would be happy to hear further suggestions as well.

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Chapter 1

Physical problem formulation

see Mei, Chapter 1,

Here we consider mathematical formulation of physical problems in terms of partial differential equations.

1.1 Simple wave propagation

Consider the scenario of Fig. 1.1. Here a material whose density of mass ρ varies with position x and time t, i.e. $\rho = \rho(x,t)$, flows with constant velocity a in a tube of constant cross-sectional area A. One can consider the SI units of mass to be kg, those of ρ to be kg/m^3, x to be m, t to be s, and a to be m/s. At the entrance, we are at position x_1. At the exit, we are at position $x_1 + \Delta x$. Standard geometry tells us the volume bounded within the tube is $V = A\Delta x$. Also indicated is the distance a material particle will have propagated in a small increment of time Δt, that distance being $a\Delta t$.

Certainly it is possible to define an average density within the volume, denoted with an over-bar:

$$\overline{\rho}(t) = \frac{1}{\Delta x}\int_{x_1}^{x_1+\Delta x} \rho(x,t)\ dx. \tag{1.1}$$

Let us invoke a common physical principle known as mass conservation and see how this principle can be cast as a partial differential equation. One way to consider the mass conservation principle is to insist for a fixed volume, such as ours, that the change in mass within the volume can only be ascribed to mass entering and exiting the surface bounding the volume. We might say

$$\underbrace{\text{total mass @ } (t+\Delta t) \text{ - total mass @ } t}_{\text{unsteady}} = \underbrace{\text{mass flux in} - \text{mass flux out}}_{\text{advection and } \underbrace{\text{diffusion}}_{=0}}. \tag{1.2}$$

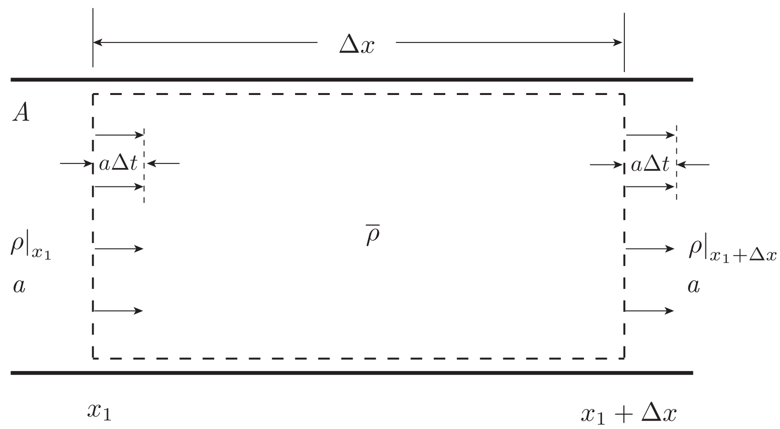


Figure 1.1: Schematic of mass advection.

The term on the left side of Eq. (1.2) is known as the "unsteady" term as it accounts for the change in mass. The terms on the right side are those physical processes which can induce change, namely mass entering and exiting the volume. In general, one can expect the physical processes of advection and diffusion to allow mass changes. Here for simplicity, we will ignore diffusion.

With mass m within the volume given in terms of density as

$$m = \int_{x_1}^{x_1+\Delta x} \rho(x,t) A \, dx = \overline{\rho} \underbrace{A\Delta x}_{V} = \overline{\rho} V, \tag{1.3}$$

our mass conservation statement has the mathematical expression

$$\begin{aligned} m|_{t+\Delta t} - m|_t &= m_{in} - m_{out}, &(1.4)\\ &= -\left(m_{out} - m_{in}\right). &(1.5)\end{aligned}$$

By inspection of Fig. 1.1, we see that

$$\begin{aligned} m_{in} &= \rho|_{x_1} A a \Delta t, &(1.6)\\ m_{out} &= \rho|_{x_1+\Delta x} A a \Delta t. &(1.7)\end{aligned}$$

Therefore Eq. (1.5) can be recast as

$$\begin{aligned} m|_{t+\Delta t} - m|_t &= -\left(\rho|_{x_1+\Delta x} A a \Delta t - \rho|_{x_1} A a \Delta t\right), &(1.8)\\ \frac{m|_{t+\Delta t} - m|_t}{\Delta t} &= -\left(\rho|_{x_1+\Delta x} A a - \rho|_{x_1} A a\right). &(1.9)\end{aligned}$$

As $\Delta t \to 0$, Eq. (1.9) reduces to

$$\frac{dm}{dt} = -\left(\rho|_{x_1+\Delta x} A a - \rho|_{x_1} A a\right). \tag{1.10}$$

It is the form of Eq. (1.10) which is often considered to be the fundamental form expressing mass conservation. We have not insisted here on any continuity properties for ρ. Here however, for simplicity, we shall assume continuity of ρ, and return to operate on Eq. (1.9) as follows:

$$A\Delta x \frac{\overline{\rho}|_{t+\Delta t} - \overline{\rho}|_t}{\Delta t} = -\left(\rho|_{x_1+\Delta x} Aa - \rho|_{x_1} Aa \right), \tag{1.11}$$

$$\frac{\overline{\rho}|_{t+\Delta t} - \overline{\rho}|_t}{\Delta t} = -a \frac{\rho|_{x_1+\Delta x} - \rho|_{x_1}}{\Delta x}. \tag{1.12}$$

Now as we let $\Delta x \to 0$, $\overline{\rho} \to \rho$ by the mean value theorem, assuming continuity of ρ. In important cases to be studied in Sec. 4.1 in which the volume contains internal discontinuities, we will not be able to make such an assumption. Then employing the definition of the partial derivative, we arrive at

$$\frac{\partial \rho}{\partial t} = -a \frac{\partial \rho}{\partial x}. \tag{1.13}$$

Rearranging, we get the classical form of what is known as a *linear advection equation*, which is a type of partial differential equation.

$$\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} = 0. \tag{1.14}$$

We can formally integrate Eq. (1.14) to recover our original integral form.

$$\int_{x_1}^{x_1+\Delta x} \left(\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} \right) dx = \underbrace{\int_{x_1}^{x_1+\Delta x} 0 \, dx}_{0}, \tag{1.15}$$

$$\int_{x_1}^{x_1+\Delta x} \frac{\partial \rho}{\partial t} \, dx + a \int_{x_1}^{x_1+\Delta x} \frac{\partial \rho}{\partial x} \, dx = 0. \tag{1.16}$$

We use Leibniz's[1] rule and the fundamental theorem of calculus to then get

$$\frac{d}{dt} \int_{x_1}^{x_1+\Delta x} \rho \, dx + a \left(\rho|_{x_1+\Delta x} - \rho|_{x_1} \right) = 0, \tag{1.17}$$

$$A\Delta x \frac{d\overline{\rho}}{dt} + aA \left(\rho|_{x_1+\Delta x} - \rho|_{x_1} \right) = 0, \tag{1.18}$$

$$\frac{dm}{dt} = \rho|_{x_1} aA - \rho|_{x_1+\Delta x} aA. \tag{1.19}$$

We might then say that the time rate of change of mass enclosed is equal to the difference of the mass flux in and the mass flux out.

[1]Gottfried Wilhelm Leibniz, 1646-1716, German polymath.

It is obvious why Eq. (1.14) is an advection equation. Let us examine why it is linear. If we take the differential operator $\mathbf{L}$ to be

$$\mathbf{L} = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}, \tag{1.20}$$

Eq. (1.14) is stated as

$$\mathbf{L}\rho = 0. \tag{1.21}$$

The operator $\mathbf{L}$ is linear because it can be shown to satisfy the properties of a linear operator:

$$\mathbf{L}(\rho + \phi) = \mathbf{L}\rho + \mathbf{L}\phi, \tag{1.22}$$
$$\mathbf{L}(\alpha\rho) = \alpha\mathbf{L}\rho, \tag{1.23}$$

where $\rho = \rho(x,t)$, $\phi = \phi(x,t)$, and α is a constant.

Let us imagine that we are given an initial distribution of ρ:

$$\rho(x,0) = f(x). \tag{1.24}$$

Then it is easy to show that a solution which satisfies the linear advection equation, Eq. (1.14) and the initial condition is

$$\rho(x,t) = f(x - at). \tag{1.25}$$

Let us consider how this can be understood through the use of the general language of coordinate transformations. We may imagine that our original coordinate system (x,t) maps to a more convenient coordinate system which we will call (ξ,τ):

$$x = x(\xi,\tau), \tag{1.26}$$
$$t = t(\xi,\tau). \tag{1.27}$$

We will find the following to be useful. We get expressions for the differentials to be

$$dx = \frac{\partial x}{\partial \xi}\,d\xi + \frac{\partial x}{\partial \tau}\,d\tau, \tag{1.28}$$
$$dt = \frac{\partial t}{\partial \xi}\,d\xi + \frac{\partial t}{\partial \tau}\,d\tau. \tag{1.29}$$

In matrix form this is

$$\begin{pmatrix} dx \\ dt \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \tau} \\ \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \tau} \end{pmatrix}}_{=\mathbf{J}} \begin{pmatrix} d\xi \\ d\tau \end{pmatrix}. \tag{1.30}$$

Here the *Jacobian*[2] of the transformation is defined as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \tau} \\ \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \tau} \end{pmatrix}. \tag{1.31}$$

Our goal is to select a coordinate transformation which renders the solution of Eq. (1.14) to be obvious. How to make such a choice is in general difficult. Leaving aside the important question of how to make such a choice, we select our new set of coordinates to be given by the linear transformation

$$x(\xi, \tau) = \xi + a\tau, \tag{1.32}$$

$$t(\xi, \tau) = \tau. \tag{1.33}$$

In matrix form, we have

$$\begin{pmatrix} x \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} \xi \\ \tau \end{pmatrix}. \tag{1.34}$$

Here the Jacobian $\mathbf{J}$ of the transformation is

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \tau} \\ \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \tag{1.35}$$

We have

$$J = |\mathbf{J}| = \det \mathbf{J} = 1. \tag{1.36}$$

The transformation is nonsingular. Expanding our notion of area to think of area in (x, t) space, the transformation is area- and orientation-preserving We have the unique inverse transformation

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}}_{\mathbf{J}^{-1}} \begin{pmatrix} x \\ t \end{pmatrix}, \tag{1.37}$$

or simply,

$$\xi = x - at, \tag{1.38}$$

$$\tau = t. \tag{1.39}$$

To transform Eq. (1.14) into the new coordinate system, we need rules for how the partial derivatives transform. The chain rule tells us

$$\frac{\partial \rho}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial \rho}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial \rho}{\partial t}, \tag{1.40}$$

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial x}{\partial \tau}\frac{\partial \rho}{\partial x} + \frac{\partial t}{\partial \tau}\frac{\partial \rho}{\partial t}. \tag{1.41}$$

[2]Carl Gustav Jacob Jacobi, 1804-1851, German mathematician.

In matrix form, this is

$$\begin{pmatrix} \frac{\partial \rho}{\partial \xi} \\ \frac{\partial \rho}{\partial \tau} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial t}{\partial \xi} \\ \frac{\partial x}{\partial \tau} & \frac{\partial t}{\partial \tau} \end{pmatrix}}_{\mathbf{J}^T} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial t} \end{pmatrix}, \tag{1.42}$$

$$= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial t} \end{pmatrix}. \tag{1.43}$$

Inverting, we find

$$\begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}}_{\mathbf{J}^{T^{-1}}} \begin{pmatrix} \frac{\partial \rho}{\partial \xi} \\ \frac{\partial \rho}{\partial \tau} \end{pmatrix}. \tag{1.44}$$

This is in short

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial \xi}, \tag{1.45}$$

$$\frac{\partial \rho}{\partial t} = -a\frac{\partial \rho}{\partial \xi} + \frac{\partial \rho}{\partial \tau}. \tag{1.46}$$

Now we apply these transformation rules to our physical equation, Eq. (1.14), to recast it as

$$\underbrace{\cancel{-a\frac{\partial \rho}{\partial \xi}} + \frac{\partial \rho}{\partial \tau}}_{\partial\rho/\partial t} + a\underbrace{\cancel{\frac{\partial \rho}{\partial \xi}}}_{\partial\rho/\partial x} = 0, \tag{1.47}$$

$$\frac{\partial \rho}{\partial \tau} = 0. \tag{1.48}$$

Integrating, we get

$$\rho = \rho(\xi), \tag{1.49}$$

$$= \rho(x - at). \tag{1.50}$$

To satisfy the initial condition, Eq. (1.24), we must then insist that

$$\rho(x,t) = \rho(x - at) = f(x - at). \tag{1.51}$$

Physically, this indicates that the initial signal $f(x)$ maintains its structure but is advected in the direction of increasing x with velocity a. Note remarkably, that f may contain discontinuous jumps. If we focus on a point with $\xi = \xi_0$, a constant, we can see how this describes signal propagation. At $\xi = \xi_0$, we have $\rho = \rho_0$. And we can say

$$\xi_0 = x - at. \tag{1.52}$$

Taking the time derivative, we get

$$\frac{d\xi_0}{dt} = \frac{dx}{dt} - a, \tag{1.53}$$

$$0 = \frac{dx}{dt} - a, \tag{1.54}$$

$$\frac{dx}{dt} = a. \tag{1.55}$$

That is, a point where ξ and thus ρ, remain constant propagates at constant velocity a.

One can use the rules for differentiation to check if the differential equation is satisfied. With $\xi = x - at$, we have

$$\rho(x,t) = f(\xi), \tag{1.56}$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial \xi}{\partial t}\frac{df}{d\xi}, \tag{1.57}$$

$$= -a\frac{df}{d\xi}, \tag{1.58}$$

$$\frac{\partial \rho}{\partial x} = \frac{\partial \xi}{\partial x}\frac{df}{d\xi}, \tag{1.59}$$

$$= \frac{df}{d\xi}. \tag{1.60}$$

Thus,

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = -a\frac{df}{d\xi} + a\frac{df}{d\xi} = 0. \tag{1.61}$$

Let us use a common but less rigorous method to solve Eq. (1.14). This method will clearly expose some important notions for more complicated systems and also identify the nature of the signal propagation. For this discussion, we will imagine the $\rho(x,t)$ is continuous and everywhere differentiable, though it is possible to relax these assumptions. If so, the rules of calculus of many variables tell us the total differential $d\rho$ is given by

$$d\rho = \frac{\partial \rho}{\partial t}\,dt + \frac{\partial \rho}{\partial x}\,dx. \tag{1.62}$$

Let us scale both sides by dt to get

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt}. \tag{1.63}$$

Consider now curves within (x,t) space on which

$$\frac{dx}{dt} = a, \qquad x(0) = x_0. \tag{1.64}$$

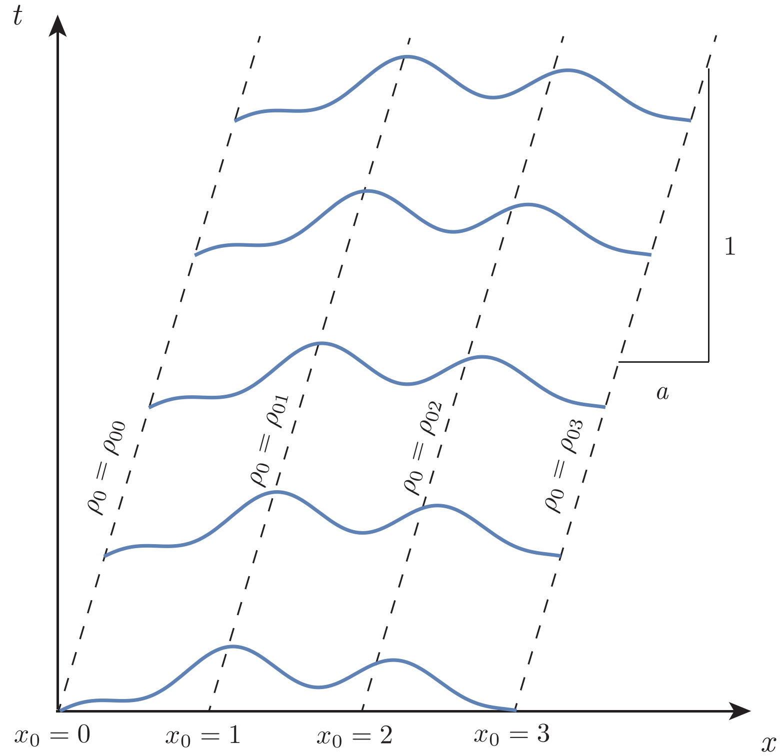


Figure 1.2: Sketch of propagation of ρ via linear advection with velocity a.

Such curves form a family of parallel lines given by

$$x = at + x_0, \tag{1.65}$$

where x_0 can take on many different values. On these curves, which are known as the *characteristics* of the system, Eq. (1.63), a purely mathematical construct, reduces to

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x}; \qquad x = at + x_0. \tag{1.66}$$

Employing the mathematical construct of Eq. (1.66) within our physical principle of Eq. (1.14), we obtain

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0, \qquad x = at + x_0, \tag{1.67}$$

$$\rho = \rho_0, \qquad x = at + x_0. \tag{1.68}$$

That is to say, on a given characteristic curve, ρ maintains that value that it had at $t = 0$, ρ_0. The value of ρ_0 can vary from characteristic to characteristic! This is sketched in Fig. 1.2.

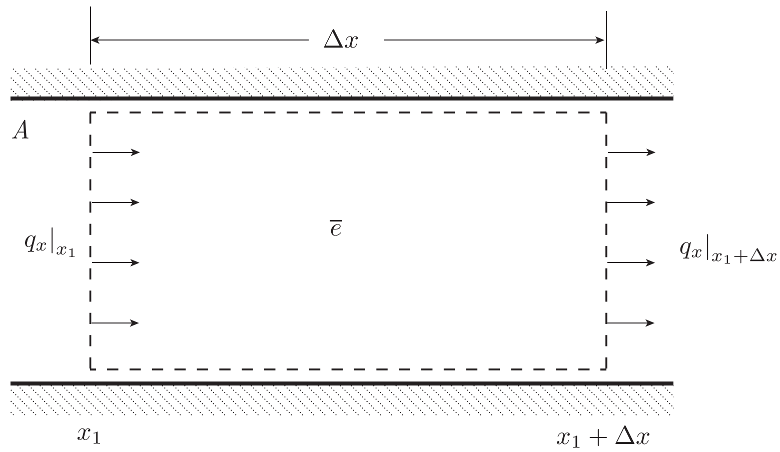


Figure 1.3: Sketch of one-dimensional energy diffusion.

1.2 One-dimensional unsteady energy diffusion

Let us perform a similar physical derivation of the so-called heat equation, a partial differential equation which is a manifestation of the first law of thermodynamics in combined with an experimentally known relationship for the heat flux. The equation will describe the process of energy diffusion. As depicted in Fig. 1.3 consider a volume V of dimension A by Δx. We describe the heat flux in the x direction as q_x. At the left boundary x_1, we have diffusive heat flux in which we notate as $q_x|_{x_1}$. At the right boundary at $x_1 + \Delta x$, we have diffusive heat flux out of $q_x|_{x_1+\Delta x}$. Recall the units of heat flux are J/m^2/s. We assume the walls are thermally insulated; thus, there is no heat flux through the walls. The total energy within the volume is E with units of J. We also have the specific energy $e = E/m$ with units of J/kg, where m is the mass enclosed within V. We suppress any addition of mass into V, so that m can be considered a constant.

We allow for the specific energy e to vary with space and time: $e = e(x, t)$. Certainly it is possible to define an average specific energy within the volume, denoted with an over-bar:

$$\overline{e}(t) = \frac{1}{\Delta x} \int_{x_1}^{x_1+\Delta x} e(x,t)\, dx. \tag{1.69}$$

Our physical principle is the first law of thermodynamics, that being

$$\text{change in total energy} = \text{heat in} - \underbrace{\text{work out}}_{=0}. \tag{1.70}$$

There is no work for our system. But there is heat flux through the system boundaries. In a combination of symbols and words, we can say

$$\underbrace{\text{total energy @ } t + \Delta t - \text{total energy @ } t}_{\text{unsteady}} = \underbrace{\text{energy flux in} - \text{energy flux out}}_{\underbrace{\text{advection}}_{=0} \text{ and diffusion}}. \tag{1.71}$$

Mathematically, we can say

$$\begin{aligned}
E|_{t+\Delta t} - E|_t &= -\left(E_{flux\ out} - E_{flux\ in}\right), && (1.72)\\
\underbrace{\rho A \Delta x}_{\text{kg}} \underbrace{\left(\overline{e}|_{t+\Delta t} - \overline{e}|_t\right)}_{\text{J/kg}} &= -\underbrace{\left(q_x|_{x_1+\Delta x} - q_x|_{x_1}\right)}_{\text{J/m}^2\text{/s}} \underbrace{A \Delta t}_{(\text{m}^2\ \text{s})}, && (1.73)\\
\rho \frac{\overline{e}|_{t+\Delta t} - \overline{e}|_t}{\Delta t} &= -\left(\frac{q_x|_{x_1+\Delta x} - q_x|_{x_1}}{\Delta x}\right). && (1.74)
\end{aligned}$$

Now, let $\Delta x \to 0$ and $\Delta t \to 0$ so as to induce mean values to be local values, and finite differences to be derivatives, yielding a differential representation of the first law of thermodynamics of

$$\rho \frac{\partial e}{\partial t} = -\frac{\partial q_x}{\partial x}. \tag{1.75}$$

Now, let us invoke some relationships known from experiment. First, the specific internal energy of many materials is well modeled by a so-called calorically perfect state equation:

$$e = cT + e_0. \tag{1.76}$$

Here c is the constant specific heat with units J/kg/K, T is the temperature with units of K, and e_0 is a constant with units of J/kg whose value is unimportant, as for nonreactive materials, it is only energy differences which have physical importance. The caloric state equation simply states the specific internal energy of a material is proportional to its temperature. Next, experiment reveals that Fourier's[3] law is a good model for the heat flux in many materials:

$$q_x = -k \frac{\partial T}{\partial x}. \tag{1.77}$$

Here k is the so-called thermal conductivity of a material. It has units J/s/m/K. It is sometimes dependent on T, but we will take it as a constant here. For agreement with experiment, we must have $k \geq 0$. The equation reflects the fact that heat flow in the positive x direction is often detected to be proportional to a field in which temperature is decreasing as x increases. In short, thermal energy flows from regions of high temperature to low temperature.

Equation (1.77) along with the caloric state equation, Eq. (1.76) when substituted into Eq. (1.75) yields

$$\begin{aligned}
\rho \frac{\partial}{\partial t}\left(cT + e_0\right) &= -\frac{\partial}{\partial x}\left(-k \frac{\partial T}{\partial x}\right), && (1.78)\\
\rho c \frac{\partial T}{\partial t} &= k \frac{\partial^2 T}{\partial x^2}, && (1.79)\\
\frac{\partial T}{\partial t} &= \underbrace{\frac{k}{\rho c}}_{\alpha} \frac{\partial^2 T}{\partial x^2}. && (1.80)
\end{aligned}$$

[3] Jean-Baptiste Joseph Fourier, 1768-1830, French mathematician and physicist.

Here we have defined the thermal diffusivity $\alpha = k/\rho/c$. Thermal diffusivity has units of m^2/s. In final form we have

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \tag{1.81}$$

Equation (1.81) is known as the *heat equation*.

Let us consider a particularly simple solution to the heat equation, Eq. (1.81). The solution will rely on a judicious guess, which will later be systematized, and will help develop physical intuition. Let us assume a solution of the form

$$T(x,t) = T_0 + A(t)\sin\left(\frac{2\pi x}{\lambda}\right). \tag{1.82}$$

Here we have presumed a sinusoidal form to capture the x variation of the solution with a single sine function of constant wavelength λ. We also have the constant T_0. We allow for a time-dependent amplitude $A(t)$. Let us seek to solve for $A(t)$ by substituting our assumed solution form into the heat equation, Eq. (1.81). Doing so yields

$$\begin{aligned}
\frac{\partial}{\partial t}\left(T_0 + A(t)\sin\left(\frac{2\pi x}{\lambda}\right)\right) &= \alpha\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(T_0 + A(t)\sin\left(\frac{2\pi x}{\lambda}\right)\right)\right), && (1.83)\\
\sin\left(\frac{2\pi x}{\lambda}\right)\frac{dA}{dt} &= \alpha A(t)\frac{2\pi}{\lambda}\frac{\partial}{\partial x}\left(\cos\left(\frac{2\pi x}{\lambda}\right)\right), && (1.84)\\
\cancel{\sin\left(\frac{2\pi x}{\lambda}\right)}\frac{dA}{dt} &= -\alpha A(t)\frac{4\pi^2}{\lambda^2}\cancel{\sin\left(\frac{2\pi x}{\lambda}\right)}, && (1.85)\\
\frac{dA}{dt} &= -\frac{4\pi^2\alpha}{\lambda^2}A(t). && (1.86)
\end{aligned}$$

Remarkably, the sine function cancels on both sides of the equation leaving us with a first order linear ordinary differential equation for the time-dependent amplitude $A(t)$. Solving yields

$$A(t) = C\exp\left(-\frac{4\pi^2\alpha}{\lambda^2}t\right). \tag{1.87}$$

Thus recombining to form $T(x,t)$, we get

$$T(x,t) = T_0 + C\exp\left(-\frac{4\pi^2\alpha}{\lambda^2}t\right)\sin\left(\frac{2\pi x}{\lambda}\right). \tag{1.88}$$

The solution describes a temperature field with an isothermal value of $T = T_0$ for $x = 0$ and $x = \lambda$. The initial value of the temperature field is

$$T(x,0) = T_0 + C\sin\left(\frac{2\pi x}{\lambda}\right). \tag{1.89}$$

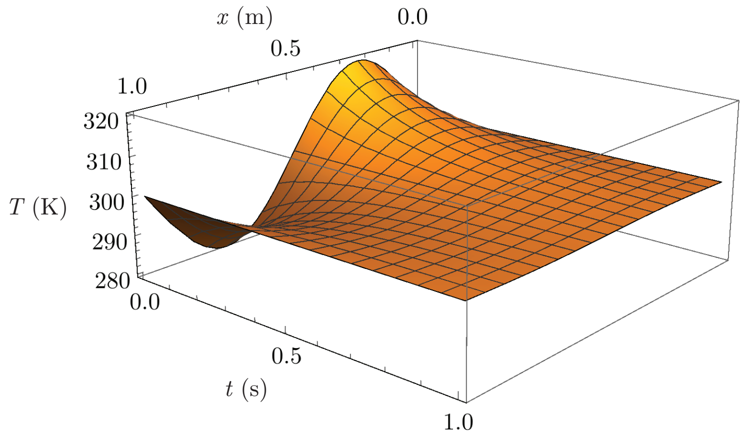


Figure 1.4: Plot of $T(x,t)$ for one-dimensional unsteady energy diffusion problem.

As $t \to \infty$, we find that $T(x,t) \to T_0$, a constant. The time constant τ of amplitude decay is by inspection

$$\tau = \frac{\lambda^2}{4\pi^2\alpha}. \tag{1.90}$$

With λ^2 having units of m^2 and thermal diffusivity having units of m^2/s, it is clear that the time constant has units of s. Importantly, we learn that rapid decay is induced by

- small wavelength λ, and
- high diffusivity α.

We plot results for $T_0 = 300$ K, $C = 20$ K, $\alpha = 0.1$ m^2/s, $\lambda = 1$ m in Fig. 1.4. For this case, the time constant of relaxation is

$$\tau = \frac{(1 \text{ m})^2}{4\pi^2 \left(0.1 \ \frac{\text{m}^2}{\text{s}}\right)} = 0.253 \text{ s}. \tag{1.91}$$

The figure clearly displays the initial sinusoidal temperature distribution along with its decay as $t \sim \tau$.

1.3 Two-dimensional steady energy diffusion

We can perform a similar analysis for two-dimensional steady energy diffusion, such as depicted in Fig. 1.5. A key difference is the presence of y variation. We shall assume a differential element in the y direction with dimension Δy; while it will be less important because we neglect variation in the z direction, we also take the differential element in the

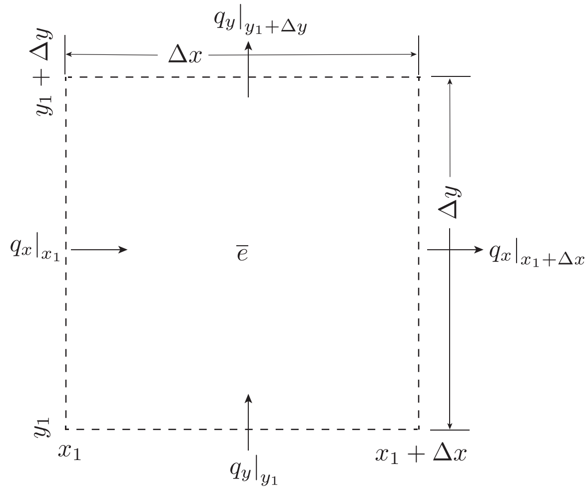


Figure 1.5: Sketch of two-dimensional energy diffusion.

z direction to have value Δz. We begin with Eq. (1.72) and analyze.

$$E|_{t+\Delta t} - E|_t = -\left(E_{flux\ out} - E_{flux\ in}\right), \tag{1.92}$$

$$\rho\Delta x\Delta y\Delta z\left(\overline{e}|_{t+\Delta t} - \overline{e}|_t\right) = -\left(q_x|_{x_1+\Delta x} - q_x|_{x_1}\right)\Delta y\Delta z\Delta t - \left(q_y|_{y_1+\Delta y} - q_y|_{y_1}\right)\Delta x\Delta z\Delta t, \tag{1.93}$$

$$\rho\left(\frac{\overline{e}|_{t+\Delta t} - \overline{e}|_t}{\Delta t}\right) = -\left(\frac{q_x|_{x_1+\Delta x} - q_x|_{x_1}}{\Delta x}\right) - \left(\frac{q_y|_{y_1+\Delta y} - q_y|_{y_1}}{\Delta y}\right). \tag{1.94}$$

Now, let $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta t \to 0$, yielding

$$\rho\frac{\partial e}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y}. \tag{1.95}$$

Defining the heat flux vector $\mathbf{q}$ as

$$\mathbf{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}, \tag{1.96}$$

and the differential operator ∇ for Cartesian[4] coordinates as

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}, \tag{1.97}$$

the two-dimensional energy diffusion equation, Eq. (1.95) can be rewritten[5] as

$$\rho\frac{\partial e}{\partial t} = -\nabla^T \cdot \mathbf{q}. \tag{1.98}$$

[4]René Descartes, 1596-1650, French philosopher and mathematician.
[5]We use the unusual notation ∇^T here, which formally only applies to Cartesian geometries.

In two dimensions, Fourier's law, Eq. (1.77), extends to the vector form

$$\mathbf{q} = -k\nabla T. \tag{1.99}$$

As an aside, we note that because the heat flux vector is expressed as the gradient of a scalar, T, we can say

- the scalar field T can be considered to be a potential field with energy diffusing in the direction of decreasing potential T, and
- the vector field $\mathbf{q}$ is curl-free, $\nabla\times\mathbf{q} = \mathbf{0}$. That is because any vector that is the gradient of a potential is guaranteed curl-free: $\nabla\times\nabla T \equiv \mathbf{0}$.
- These conclusions hold for both the steady, two-dimensional fields $T(x, y)$, $\mathbf{q}(x, y)$ of this chapter as well as three-dimensional unsteady fields $T(x, y, z, t)$, $\mathbf{q}(x, y, z, t)$. In both cases, T is a potential field and $\mathbf{q}$ is a curl-free field.

In terms of scalar components we can say $q_x = -k\ \partial T/\partial x$ and $q_y = -k\ \partial T/\partial y$. Substituting the caloric state equation, Eq. (1.75), and the multi-dimensional Fourier's law, Eq. (1.99), into our energy diffusion equation, Eq. (1.95), we get

$$\rho c\frac{\partial T}{\partial t} = -\nabla^T\cdot\underbrace{(-k\nabla T)}_{\mathbf{q}}. \tag{1.100}$$

Again, while k may be a function of T for some materials, we will take it to be a constant yielding

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T, \tag{1.101}$$

where we have once again employed the definition of thermal diffusivity, $\alpha = k/\rho/c$.[6] We have also defined the *Laplacian* operator as $\nabla^2 = \nabla^T\cdot\nabla$. We expand this important operator for a two-dimensional Cartesian system as

$$\nabla^2 = \nabla^T\cdot\nabla = \left(\begin{array}{cc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{array}\right)\left(\begin{array}{c}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{array}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.102}$$

For the important case of a steady state temperature distribution, we have T with no variation with t. In this case Eq. (1.101) reduces to the so-called *Laplace's*[7] *equation*:

$$\nabla^2 T = 0. \tag{1.103}$$

[6] When extended to multi-dimensional materials with linear anisotropy, Fourier's law takes on a vector form $\mathbf{q} = -\mathbf{K}\cdot\nabla T$, where $\mathbf{K}$ is a positive definite symmetric tensor, embodying the material's anisotropy. In such cases, the heat equation becomes $\rho c\ \partial T/\partial t = \nabla^T\cdot(\mathbf{K}\cdot\nabla T)$.

[7] Pierre-Simon Laplace, 1749-1827, French mathematician and physicist.

Remarkably, the diffusivity does not affect the temperature distribution in the steady state limit. In two-dimensions, this can be written as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{1.104}$$

In a similar fashion as for the previous section, let us consider a particularly simple solution to Laplace's equation, Eq. (1.104):

$$T(x,y) = T_0 + g(y)\sin\left(\frac{2\pi x}{\lambda}\right). \tag{1.105}$$

Once again, T_0 is a constant with units of K, and λ is a constant with units of m which can be interpreted as the wavelength of the disturbance in the x direction. We seek the function $g(y)$ which allows Laplace's equation to be satisfied. Let us substitute Eq. (1.105) into Eq. (1.104):

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(T_0 + g(y)\sin\left(\frac{2\pi x}{\lambda}\right)\right) + \frac{\partial^2}{\partial y^2}\left(T_0 + g(y)\sin\left(\frac{2\pi x}{\lambda}\right)\right) &= 0, &(1.106)\\
-g(y)\frac{4\pi^2}{\lambda^2}\cancel{\sin\left(\frac{2\pi x}{\lambda}\right)} + \frac{d^2 g}{dy^2}\cancel{\sin\left(\frac{2\pi x}{\lambda}\right)} &= 0, &(1.107)\\
\frac{d^2 g}{dy^2} - \frac{4\pi^2}{\lambda^2} g = 0. & &(1.108)
\end{aligned}$$

This is a second order linear differential equation. We recall such equations may be solved by assuming solutions of the form $g(y) = Ce^{ry}$. Substituting the assumed form into the differential equation gives $Cr^2e^{ry} - (4\pi^2/\lambda^2)Ce^{ry} = 0$. We cancel terms to get the *characteristic polynomial*: $r^2 - 4\pi^2/\lambda^2 = 0$. We solve to get $r = \pm 2\pi/\lambda$. Thus, there are two functions that satisfy. Because the original equation is linear, linear combinations also satisfy; thus $g(y) = K_1 e^{2\pi y/\lambda} + K_2 e^{-2\pi y/\lambda}$, where K_1 and K_2 are constants. The exponentials may be cast in terms of hyperbolic functions as we recall $\sinh y = (e^y - e^{-y})/2$ and $\cosh y = (e^y + e^{-y})/2$. This yields the general solution

$$g(y) = C_1 \sinh\left(\frac{2\pi y}{\lambda}\right) + C_2 \cosh\left(\frac{2\pi y}{\lambda}\right), \tag{1.109}$$

where C_1 and C_2 are arbitrary constants. Let us select $C_2 = 0$ so as to yield a solution for the temperature field of

$$T(x,y) = T_0 + C_1 \sinh\left(\frac{2\pi y}{\lambda}\right)\sin\left(\frac{2\pi x}{\lambda}\right). \tag{1.110}$$

We note that $T = T_0$ wherever $x = 0$, $x = \lambda$, or $y = 0$. The heat flux vector is $\mathbf{q} = -k\nabla T$. This gives

$$\mathbf{q} = -k\left(\frac{2C_1\pi}{\lambda}\right)\begin{pmatrix} \cos\left(\frac{2\pi x}{\lambda}\right)\sinh\left(\frac{2\pi y}{\lambda}\right) \\ \sin\left(\frac{2\pi x}{\lambda}\right)\cosh\left(\frac{2\pi y}{\lambda}\right)\end{pmatrix}. \tag{1.111}$$

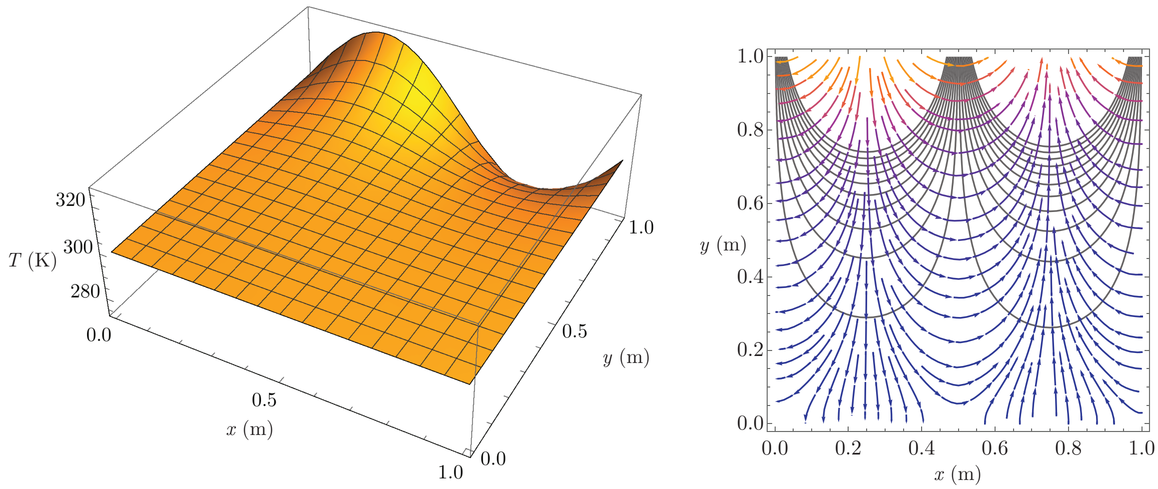


Figure 1.6: Plot of $T(x,y)$ and vector field $\mathbf{q}$ for steady two-dimensional energy diffusion problem.

We plot results for $T_0 = 300$ K, $C_1 = 0.1$ K, $\lambda = 1$ m, $k = 1$ W/m/K along with the heat flux vector field in Fig. 1.6.

Lastly, we consider global energy for our system. We integrate Eq. (1.98) over a fixed volume V bounded by surface S with unit outer normal $\mathbf{n}$. First apply the volume integration operator to both sides of Eq. (1.98):

$$\int_V \rho \frac{\partial e}{\partial t}\, dV = -\int_V \nabla^T \cdot \mathbf{q}\, dV. \tag{1.112}$$

Apply Leibniz's rule to the left side and Gauss'[8] theorem to the right side to obtain

$$\frac{d}{dt}\int_V \rho e\, dV = -\int_S \mathbf{q}^T \cdot \mathbf{n}\, dS. \tag{1.113}$$

The time rate of change of energy within V can be attributed solely to the net flux of energy crossing the boundary S. In the steady state limit, we have

$$\int_S \mathbf{q}^T \cdot \mathbf{n}\, dS = 0. \tag{1.114}$$

That is to say, in order for there to be no change in the energy within the volume, the net energy entering must be zero.

Problems

1. Consider the solution of the linear advection equation

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0.$$

[8]Johann Carl Friedrich Gauss, 1777-1855, German mathematician.

For $a = 2$, $x \in [0, 5]$, $t \in [0, 5]$, plot contours and three-dimensional surfaces of $\rho(x, t)$ for the following initial conditions:

(a) $\rho(x, 0) = \sin(\pi x)$,

(b) $\rho(x, 0) = H(x) - H(x - 1)$, where $H(x)$ is the Heaviside[9] unit step function.

[9] Oliver Heaviside, 1850-1925, English electrical engineer.

Chapter 2

Fundamental partial differential equations

see Mei, Chapter 2,

Here we consider some fundamental features of basic linear partial differential equations. We show how to classify linear partial differential equations and identify some fundamental solutions.

2.1 General method of classification

Many important partial differential equations can be cast in the so-called quasi-linear form of a system of first order partial differential equations

$$A_{ij}\frac{\partial u_j}{\partial t} + B_{ij}\frac{\partial u_j}{\partial x} = c_i, \qquad i = 1, \ldots, N;\ j = 1, \ldots, N. \tag{2.1}$$

In Gibbs[1] notation, we could say

$$\mathbf{A}\cdot\frac{\partial \mathbf{u}}{\partial t} + \mathbf{B}\cdot\frac{\partial \mathbf{u}}{\partial x} = \mathbf{c}. \tag{2.2}$$

Here we have N dependent variables u_j with $j = 1, \ldots, N$. The independent variables are x and t. The terms A_{ij} and B_{ij} may be a functions of x, t and any of the u_js. Both A_{ij} and B_{ij} are elements of $N \times N$ nonconstant matrices. The term c_i can be a function of x, t and any of the u_js; it is an element of a $N \times 1$ column matrix.

As described by Whitham,[2] there is a general technique to analyze such equations. First pre-multiply both sides of the equation by a yet-to-be-determined row of variables ℓ_i:

$$\ell_i A_{ij}\frac{\partial u_j}{\partial t} + \ell_i B_{ij}\frac{\partial u_j}{\partial x} \quad = \quad \ell_i c_i, \tag{2.3}$$

[1]Josiah Willard Gibbs, 1839-1903, American engineer and scientist.

[2]Gerald Beresford Whitham, 1927-2014, applied mathematician and developer of theory for nonlinear wave propagation.

$$\boldsymbol{\ell}^T \cdot \mathbf{A} \cdot \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\ell}^T \cdot \mathbf{B} \cdot \frac{\partial \mathbf{u}}{\partial x} = \boldsymbol{\ell}^T \cdot \mathbf{c}. \tag{2.4}$$

The method hinges upon choosing ℓ_i to render the left side of Eq. (2.4) to be of the form similar to $\partial/\partial t + \lambda(\partial/\partial x)$, where λ is a scalar that may be a variable. This is similar to the form analyzed in Eq. (1.14), $\partial\rho/\partial t + a\ \partial\rho/\partial x = 0$, except here we are allowing λ to be a variable.

Let us define the variable m_j such that

$$\ell_i A_{ij} \frac{\partial u_j}{\partial t} + \ell_i B_{ij} \frac{\partial u_j}{\partial x} = m_j \left(\frac{\partial u_j}{\partial t} + \lambda \frac{\partial u_j}{\partial x} \right), \tag{2.5}$$

$$= m_j \frac{du_j}{dt} \qquad \text{on} \qquad \frac{dx}{dt} = \lambda. \tag{2.6}$$

Reorganizing Eq. (2.5), we get

$$\underbrace{(\ell_i A_{ij} - m_j)}_{=0} \frac{\partial u_j}{\partial t} + \underbrace{(\ell_i B_{ij} - m_j \lambda)}_{=0} \frac{\partial u_j}{\partial x} = 0. \tag{2.7}$$

Because we expect $\partial u_j/\partial t$ to be linearly independent of $\partial u_j/\partial x$, we insist

$$\ell_i A_{ij} = m_j, \tag{2.8}$$

$$\ell_i B_{ij} = \lambda m_j. \tag{2.9}$$

Scaling Eq. (2.8) by λ gives

$$\lambda \ell_i A_{ij} = \lambda m_j. \tag{2.10}$$

Subtracting Eq. (2.9) from Eq. (2.10) to eliminate λm_j gives

$$\ell_i \left(\lambda A_{ij} - B_{ij} \right) = 0, \tag{2.11}$$

$$\boldsymbol{\ell}^T \cdot (\lambda \mathbf{A} - \mathbf{B}) = \mathbf{0}^T. \tag{2.12}$$

This is a generalized left eigenvalue problem, where λ is known as a generalized eigenvalue in the second sense and $\boldsymbol{\ell}$ is a generalized left eigenvector. One has nontrivial $\boldsymbol{\ell}$ when

$$\det (\lambda \mathbf{A} - \mathbf{B}) = 0. \tag{2.13}$$

If $\mathbf{A}$ is invertible, we can post-multiply Eq. (2.12) by $\mathbf{A}^{-1}$ to recover an ordinary left eigenvalue problem:

$$\boldsymbol{\ell}^T \cdot \left(\lambda \mathbf{I} - \mathbf{B} \cdot \mathbf{A}^{-1} \right) = \mathbf{0}^T. \tag{2.14}$$

We adopt the following classification nomenclature, following Zauderer, p. 135:

- *hyperbolic*: All generalized eigenvalues λ are real and there exist N linearly independent left generalized eigenvectors $\boldsymbol{\ell}$.
- *parabolic*: All generalized eigenvalues λ that exist are real, but there exist fewer than N linearly independent generalized left eigenvectors $\boldsymbol{\ell}$.
- *elliptic*: All the generalized eigenvalues λ are complex.
- *mixed*: Some generalized eigenvalues λ may be real, others complex, and there may or may not be N linearly independent generalized left eigenvectors $\boldsymbol{\ell}$.

2.2 Fundamental solutions

2.2.1 Wave equation: hyperbolic

By inspection the linear advection equation, Eq. (1.14) is already in the appropriate form. Let us examine a common extension of the linear advection equation, the so-called *wave equation*:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \tag{2.15}$$

We need to convert this second order equation into a system of first order equations. To enable this, let us define two new variables, v and w:

$$v \equiv \frac{\partial y}{\partial t}, \tag{2.16}$$

$$w \equiv \frac{\partial y}{\partial x}. \tag{2.17}$$

Substituting Eqs. (2.16) and (2.17) into the wave equation, Eq. (2.15), we get our first first order partial differential equation:

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial w}{\partial x}. \tag{2.18}$$

We can next differentiate Eq. (2.16) with respect to x and Eq. (2.17) with respect to t to get

$$\frac{\partial v}{\partial x} = \frac{\partial^2 y}{\partial x \partial t}, \tag{2.19}$$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 y}{\partial t \partial x}. \tag{2.20}$$

Now as long as y is sufficiently continuous and differentiable, the order of differentiation does not matter, so we can take $\partial^2 y/\partial x \partial t = \partial^2 y/\partial t \partial x$. This enables us to equate Eqs. (2.19) and (2.20), yielding our second first order partial differential equation:

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial t}. \tag{2.21}$$

We recast our two first order equations, Eqs. (2.18) and (2.21) as

$$\begin{aligned}\frac{\partial v}{\partial t} - a^2\frac{\partial w}{\partial x} &= 0, &(2.22)\\ \frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} &= 0. &(2.23)\end{aligned}$$

We next cast our two first order equations, Eqs. (2.22) and (2.23) in the general form of Eq. (2.1) to get

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial t}} + \underbrace{\begin{pmatrix} 0 & -a^2 \\ -1 & 0 \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathbf{c}}. \tag{2.24}$$

Here our vector of dependent variables is

$$\mathbf{u} = \begin{pmatrix} v \\ w \end{pmatrix}. \tag{2.25}$$

The associated eigenvalue problem is

$$\det(\lambda \mathbf{A} - \mathbf{B}) = \begin{vmatrix} \lambda & a^2 \\ 1 & \lambda \end{vmatrix} = 0. \tag{2.26}$$

Solving gives

$$\begin{aligned}\lambda^2 - a^2 &= 0, &(2.27)\\ \lambda &= \pm a. &(2.28)\end{aligned}$$

We have two real and distinct eigenvalues. Let us find the eigenvectors.

$$\begin{aligned}\boldsymbol{\ell}^T \cdot (\lambda \mathbf{A} - \mathbf{B}) &= \mathbf{0}^T, &(2.29)\\ (\,\ell_1 \quad \ell_2\,)\begin{pmatrix} \lambda & a^2 \\ 1 & \lambda \end{pmatrix} &= (\,0 \quad 0\,), &(2.30)\\ (\,\ell_1 \quad \ell_2\,)\begin{pmatrix} \pm a & a^2 \\ 1 & \pm a \end{pmatrix} &= (\,0 \quad 0\,). &(2.31)\end{aligned}$$

This yields two linearly dependent equations:

$$\begin{aligned}\pm a\ell_1 + \ell_2 &= 0, &(2.32)\\ a^2\ell_1 \pm a\ell_2 &= 0. &(2.33)\end{aligned}$$

If we multiply the first by $\pm a$, we get the second. It is obvious the solution is not unique. If we take $\ell_1 = s$, where s is any constant, then $\ell_2 = \mp as$. Let us take $s = 1$, and thus take the eigenvectors to be

$$\boldsymbol{\ell} = \begin{pmatrix} 1 \\ \mp a \end{pmatrix}. \tag{2.34}$$

Importantly, not only do we have two distinct and real eigenvalues, but we also have two linearly independent eigenvectors. Thus our wave equation is hyperbolic.

We lastly use the eigenvalues and eigenvectors to recast our original system. Multiplying both sides of Eq. (2.24) by $\boldsymbol{\ell}^T$, we get

$$\underbrace{\begin{pmatrix} 1 & \mp a \end{pmatrix}}_{\boldsymbol{\ell}^T} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial t}} + \underbrace{\begin{pmatrix} 1 & \mp a \end{pmatrix}}_{\boldsymbol{\ell}^T} \underbrace{\begin{pmatrix} 0 & -a^2 \\ -1 & 0 \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial x}} = \underbrace{\begin{pmatrix} 1 & \mp a \end{pmatrix}}_{\boldsymbol{\ell}^T} \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathbf{c}}, \tag{2.35}$$

$$\begin{pmatrix} 1 & \mp a \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{pmatrix} + \begin{pmatrix} \pm a & -a^2 \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix} = 0, \tag{2.36}$$

$$\frac{\partial v}{\partial t} \mp a\frac{\partial w}{\partial t} \pm a\frac{\partial v}{\partial x} - a^2\frac{\partial w}{\partial x} = 0, \tag{2.37}$$

$$\left(\frac{\partial v}{\partial t} \pm a\frac{\partial v}{\partial x}\right) \mp a\left(\frac{\partial w}{\partial t} \pm a\frac{\partial w}{\partial x}\right) = 0. \tag{2.38}$$

This reduces to two sets of differential equations valid on two different sets of characteristic lines:

$$\frac{dv}{dt} - a\frac{dw}{dt} = 0 \quad \text{on} \quad x = at + x_0, \tag{2.39}$$

$$\frac{dv}{dt} + a\frac{dw}{dt} = 0 \quad \text{on} \quad x = -at + x_0. \tag{2.40}$$

These combine to form

$$\frac{d}{dt}(v - aw) = 0 \quad \text{on} \quad x = at + x_0, \tag{2.41}$$

$$\frac{d}{dt}(v + aw) = 0 \quad \text{on} \quad x = -at + x_0. \tag{2.42}$$

Integrating, we find

$$v - aw = C_1 \quad \text{on} \quad x = at + x_0, \tag{2.43}$$

$$v + aw = C_2 \quad \text{on} \quad x = -at + x_0. \tag{2.44}$$

That is to say, the combinations of $v \mp aw$ are preserved on lines for which $x = \pm at + x_0$. In this solution signals are propagated in two distinct directions, and those signals are preserved as they propagate. The constants C_1 and C_2 are known as *Riemann*[3] *invariants* for the system. The Riemann invariants are only invariant on a given characteristic and may vary from one characteristic to another.

The just-completed analysis is common, and is often described as converting the partial differential equation to ordinary differential equations valid along so-called characteristic lines in $x - t$ space. This is somewhat unsatisfying as the variation of C_1 to C_2 reflects the fact

[3]Bernhard Riemann, 1826-1866, German mathematician.

that we really are considering partial differential equations. Motivated by the existence of characteristic lines on which linear combinations of v and w must retain a constant value, let us seek a coordinate transformation to clarify this. More importantly, the formal coordinate transform will show us how we really are considering a partial differential equation in a more easily analyzed space.

We take

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \tag{2.45}$$

Thus,

$$\xi(x,t) = x - at, \tag{2.46}$$
$$\eta(x,t) = x + at. \tag{2.47}$$

Inverting, we get

$$\begin{pmatrix} x \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2a} & \frac{1}{2a} \end{pmatrix}}_{\mathbf{J}} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{2.48}$$

Here the Jacobian $\mathbf{J}$ of the transformation is

$$\mathbf{J} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2a} & \frac{1}{2a} \end{pmatrix}. \tag{2.49}$$

Here we find

$$J = |\mathbf{J}| = \det \mathbf{J} = \frac{1}{2a}, \tag{2.50}$$

we see the transformation is only area-preserving when $a = \pm 1/2$, and is orientation-preserving when $a > 0$.

We need rules for how the partial derivatives transform. The chain rule tells us

$$\begin{pmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial t}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial t}{\partial \eta} \end{pmatrix}}_{\mathbf{J}^T} \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} \end{pmatrix}, \tag{2.51}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2a} \\ \frac{1}{2} & \frac{1}{2a} \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} \end{pmatrix}. \tag{2.52}$$

Inverting, we find

$$\begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -a & a \end{pmatrix}}_{\mathbf{J}^{T^{-1}}} \begin{pmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{pmatrix}. \tag{2.53}$$

This is in short

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \tag{2.54}$$

$$\frac{\partial}{\partial t} = -a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta}. \tag{2.55}$$

Employing these transformed operators on our original wave equation, Eq. (2.15), we get

$$\left(-a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta}\right)\left(-a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta}\right) y = a^2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) y, \tag{2.56}$$

$$a^2\left(\frac{\partial^2 y}{\partial \xi^2} - 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2}\right) = a^2\left(\frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2}\right), \tag{2.57}$$

$$-2a^2\frac{\partial^2 y}{\partial \xi \partial \eta} = 2a^2\frac{\partial^2 y}{\partial \xi \partial \eta}, \tag{2.58}$$

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = 0. \tag{2.59}$$

We integrate this equation first with respect to ξ to get

$$\frac{\partial y}{\partial \eta} = h(\eta). \tag{2.60}$$

Note that when we integrate homogeneous partial differential equations, we must include an arbitrary function rather than the arbitrary constant we get for ordinary differential equations. We next integrate with respect to η to get

$$y = \underbrace{\int_0^\eta h(\hat{\eta})\ d\hat{\eta}}_{=f(\eta)} + g(\xi). \tag{2.61}$$

The integral of $h(\eta)$ simply yields another function of η, which we call $f(\eta)$. Thus the general solution to $\partial^2 y/\partial\xi\partial\eta = 0$ is

$$y(\xi, \eta) = f(\eta) + g(\xi). \tag{2.62}$$

We might say that we have separated the solution into two functions of two independent variables. Here the separated functions were combined as a sum. In other problems, the separated functions will combine as a product. In terms of our original coordinates, we can say

$$y(x, t) = f(x + at) + g(x - at). \tag{2.63}$$

We note that f and g are completely arbitrary functions. This is known as the *d'Alembert*[4] *solution*. Compared to the related solution of the linear advection equation, we see that

[4]Jean le Rond d'Alembert, 1717-1783, French mathematician.

two independent modes are admitted for signal propagation. One travels in the direction of increasing x, the other in the direction of decreasing x. Both have speed a. The functional forms of f and g admit discontinuous solutions, and the forms are preserved as t advances.

Example 2.1

If $\partial^2 y/\partial t^2 = a^2 \partial^2 y/\partial x^2$, find $y(t)$ if

$$y(x,0) = 2e^{-x^2}, \qquad \left.\frac{\partial y}{\partial t}\right|_{x,t=0} = 0. \tag{2.64}$$

First, we have the d'Alembert solution, Eq. (2.63),

$$y(x,t) = f(x+at) + g(x-at). \tag{2.65}$$

Taking a time derivative, we get

$$\frac{\partial y}{\partial t} = af'(x+at) - ag'(x-at). \tag{2.66}$$

At the initial state, $t = 0$, we must have

$$y(x,0) = 2e^{-x^2} = f(x) + g(x), \tag{2.67}$$

$$\left.\frac{\partial y}{\partial t}\right|_{x,t=0} = 0 = af'(x) - ag'(x). \tag{2.68}$$

From the condition on the time derivative, we get

$$f'(x) = g'(x). \tag{2.69}$$

Thus $f(x) = g(x) + C$. So the initial condition gives

$$2e^{-x^2} = 2g(x) + C. \tag{2.70}$$

This is satisfied iff $C = 0$ and $f(x) = g(x) = e^{-x^2}$. Thus, the general solution is

$$y(x,t) = e^{-(x+at)^2} + e^{-(x-at)^2}. \tag{2.71}$$

Direct substitution of the solution into the original wave equation and initial condition verifies both are satisfied for all wave speeds a. Let us consider one of these solutions for $a = 1$:

$$y(x,t) = e^{-(x+t)^2} + e^{-(x-t)^2}. \tag{2.72}$$

This solution is shown for $t = 0$ and $t = 3$ in Fig. 2.1. At $t = 3$, the peak of the left-propagating mode is at $x = -3$, and the peak of the right-propagating mode is at $x = 3$. This is consistent with a wave speed of $a = 1$. The amplitude of the peaks does not decay with time, as is typical for a wave equation.

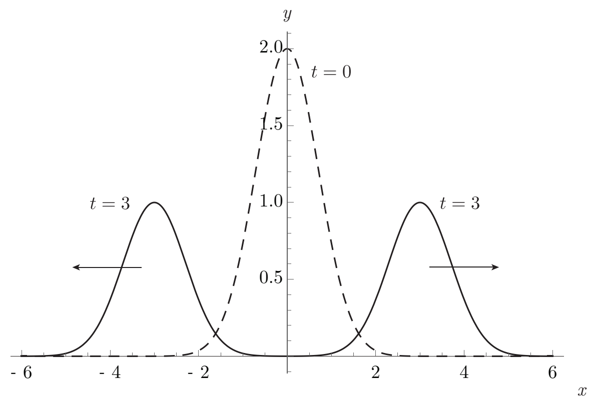


Figure 2.1: Solution to wave equation with $a = 1$, $y(x, 0) = 2e^{-x^2}$.

2.2.2 Heat equation: parabolic

Let us analyze the heat equation, Eq. (1.81), as a system of first order partial differential equations. Our earlier analysis has already assisted in this. Equation (1.81) can be considered a combination of the energy conservation principle, a caloric state equation, and Fourier's law. Taking the first of our equations as the combination of energy conservation, Eq. (1.75) and the caloric state equation, Eq. (1.76) and the second as Fourier's law, Eq. (1.77), we write

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x}, \tag{2.73}$$

$$q_x = -k \frac{\partial T}{\partial x}. \tag{2.74}$$

This can be recast as

$$\underbrace{\begin{pmatrix} \rho c & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \frac{\partial T}{\partial t} \\ \frac{\partial q_x}{\partial t} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial t}} + \underbrace{\begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial q_x}{\partial x} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial x}} = \underbrace{\begin{pmatrix} 0 \\ q_x \end{pmatrix}}_{\mathbf{c}}. \tag{2.75}$$

Here our vector $\mathbf{u}$ is

$$\mathbf{u} = \begin{pmatrix} T \\ q_x \end{pmatrix}. \tag{2.76}$$

The associated eigenvalue problem is

$$\det(\lambda \mathbf{A} - \mathbf{B}) = \begin{vmatrix} \lambda \rho c & -1 \\ k & 0 \end{vmatrix} = 0. \tag{2.77}$$

Solving gives

$$\lambda \rho c(0) + k = 0, \tag{2.78}$$

$$\lambda \to \infty. \tag{2.79}$$

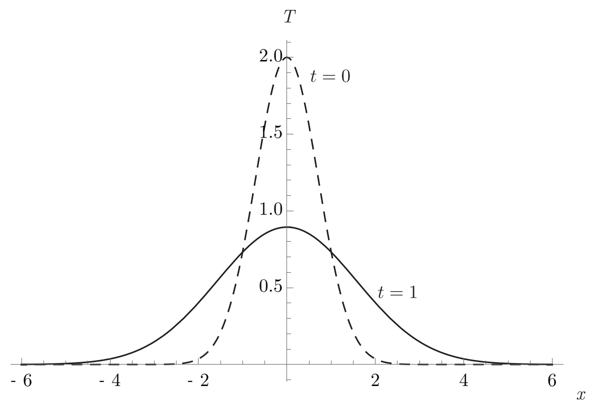


Figure 2.2: Solution to heat equation with $\alpha = 1$, $T(x, 0) = 2e^{-x^2}$.

One cannot find any associated eigenvectors $\boldsymbol{\ell}$. Because there are an insufficient number of eigenvectors on which to project our system, the heat equation is parabolic.

Just as there exists a general solution to the wave equation, there exists a general solution to the heat equation. It will be shown in Sec. 9.1 that the solution of

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad T(x \to \pm\infty, t) \to 0, \qquad T(x, 0) = f(x), \tag{2.80}$$

is

$$T(x, t) = \frac{1}{2\sqrt{\pi \alpha t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4\alpha t}} \; d\xi. \tag{2.81}$$

Example 2.2

Find $T(x, t)$ if

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad T(x \to \pm\infty, t) \to 0, \qquad T(x, 0) = 2e^{-x^2}. \tag{2.82}$$

Direct substitution of $f(x) = 2e^{-x^2}$ into Eq. (2.81) gives

$$T(x, t) = \frac{1}{2\sqrt{\pi \alpha t}} \int_{-\infty}^{\infty} 2e^{-\xi^2} e^{-\frac{(x-\xi)^2}{4\alpha t}} \; d\xi. \tag{2.83}$$

Evaluation of the integral yields

$$T(x, t) = \frac{2e^{-\frac{x^2}{1+4\alpha t}}}{\sqrt{1 + 4\alpha t}}. \tag{2.84}$$

Direct substitution reveals the solution satisfies both the heat equation, the initial condition, and the boundary conditions. This solution for $\alpha = 1$ is shown for $t = 0$ and $t = 1$ in Fig. 2.2. At $t = 1$, the peak has decayed from its initial value of 2 to a value just less than 1.

Example 2.3

Find $T(x,t)$ if the initial condition is a finite strength pulse

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad T(x \to \pm\infty, t) \to 0, \qquad T(x,0) = \delta(x). \tag{2.85}$$

The pulse, also called a point source, is embodied in δ, the Dirac[5] delta function. It has the properties

$$\begin{aligned} \delta(x-a) &= 0, \qquad x \neq a, && (2.86)\\ \delta(x-a) &\to \infty, \qquad x = a, && (2.87)\\ \int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx &= f(a), && (2.88)\\ \int_{-\infty}^{\infty} \delta(x)\,dx &= 1. && (2.89) \end{aligned}$$

Direct substitution of $f(x) = \delta(x)$ into Eq. (2.81) gives

$$T(x,t) = \frac{1}{2\sqrt{\pi\alpha t}} \int_{-\infty}^{\infty} \delta(\xi) e^{-\frac{(x-\xi)^2}{4\alpha t}}\,d\xi. \tag{2.90}$$

Evaluation of the integral yields

$$T(x,t) = \frac{1}{2\sqrt{\pi\alpha t}} e^{-\frac{x^2}{4\alpha t}}. \tag{2.91}$$

Direct substitution reveals the solution satisfies both the heat equation, the initial condition, and the boundary conditions. This solution for $\alpha = 1$ is shown for $t = 1/10, 1/2, 1, 2$ in Fig. 2.3.

2.2.3 Laplace's equation: elliptic

Let us next analyze in a similar fashion Laplace's equation, Eq. (1.104), $\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 = 0$. Here the independent variables are x and y, rather than x and t. Now our Laplace's equation arose from the two-dimensional time-independent form of Eq. (1.98), which is

$$\begin{aligned} \underbrace{\rho \frac{\partial e}{\partial t}}_{=0} &= -\nabla^T \cdot \mathbf{q}, && (2.92)\\ \nabla^T \cdot \mathbf{q} &= 0, && (2.93)\\ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} &= 0. && (2.94) \end{aligned}$$

[5]Paul Adrien Maurice Dirac, 1902-1984, English mathematician and winner of the 1933 Nobel prize for physics.

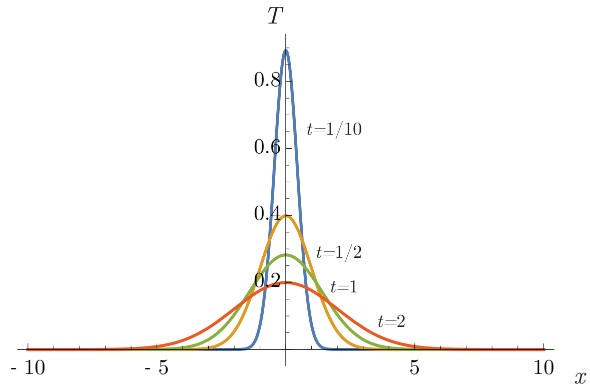


Figure 2.3: Solution to heat equation with $\alpha = 1$, $T(x,0) = \delta(x)$.

This is our first first order partial differential equation. To aid this analysis, let us recall from Eq. (1.99) that

$$q_x = -k\frac{\partial T}{\partial x}, \tag{2.95}$$

$$q_y = -k\frac{\partial T}{\partial y}. \tag{2.96}$$

We then see that

$$\frac{\partial q_x}{\partial y} = -k\frac{\partial^2 T}{\partial y \partial x}, \tag{2.97}$$

$$\frac{\partial q_y}{\partial x} = -k\frac{\partial^2 T}{\partial x \partial y}. \tag{2.98}$$

Equating the mixed second partial derivatives, we get our second first order partial differential equation:

$$\frac{\partial q_x}{\partial y} = \frac{\partial q_y}{\partial x}. \tag{2.99}$$

Equations (2.94) and (2.99) form the system

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0, \tag{2.100}$$

$$\frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} = 0. \tag{2.101}$$

Equations (2.100) and (2.101) can be recast as

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \frac{\partial q_x}{\partial x} \\ \frac{\partial q_y}{\partial x} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial x}} + \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} \frac{\partial q_x}{\partial y} \\ \frac{\partial q_y}{\partial y} \end{pmatrix}}_{\frac{\partial \mathbf{u}}{\partial y}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mathbf{c}}. \tag{2.102}$$

Here our vector $\mathbf{u}$ is

$$\mathbf{u} = \begin{pmatrix} q_x \\ q_y \end{pmatrix}. \tag{2.103}$$

The associated eigenvalue problem is

$$\det(\lambda \mathbf{A} - \mathbf{B}) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0. \tag{2.104}$$

Solving gives

$$\lambda^2 + 1 = 0, \tag{2.105}$$
$$\lambda = \pm i. \tag{2.106}$$

The eigenvalues are distinct but not real. Presence of complex eigenvalues indicates the equation cannot be written in characteristic form, and that finite speed signaling phenomena are not present in the solution. Because its eigenvalues are imaginary, Laplace's equation is *elliptic*.

We have spoken of general solutions for the wave equation and the heat equation. General solutions for Laplace's equation may be considered, but they are not straightforward, unless one is willing to consider complex variables. These methods to solve Laplace's equation will be considered in earnest in Ch. 8. That said, if we recognize the complex number $i = \sqrt{-1}$, and consider $x \in \mathbb{R}$, $y \in \mathbb{R}$, we find an equivalent to the d'Alembert solution, Eq. (2.63):

$$T(x, y) = f(x + iy) + g(x - iy). \tag{2.107}$$

Here f and g are arbitrary functions. We can induce this from a similar logic used for the wave equation in Sec. 2.2.1. On lines on which

$$\frac{dy}{dx} = \lambda = \pm i, \tag{2.108}$$

we can expect $T(x, y)$ to be a constant. This will be true then on lines such that

$$y = \pm ix + C, \tag{2.109}$$
$$\pm iy = -x \pm iC, \tag{2.110}$$
$$x \pm iy = \hat{C}, \qquad \text{with} \qquad \hat{C} = \pm iC. \tag{2.111}$$

This induces $T(x, y) = f(x + iy) + g(x - iy)$.

Let us show this satisfies Laplace's equation. We have the following partial derivatives:

$$\frac{\partial T}{\partial x} = f' + g', \qquad \frac{\partial^2 T}{\partial x^2} = f'' + g'', \qquad \frac{\partial T}{\partial y} = if' - ig', \qquad \frac{\partial^2 T}{\partial y^2} = -f'' - g''. \tag{2.112}$$

Obviously Laplace's equation is satisfied for arbitrary f and g:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f'' + g'' - f'' - g'' = 0. \tag{2.113}$$

Note that while we have identified general solutions, it is not yet obvious how these can be constructed to satisfy specified boundary conditions.

We have developed Laplace's equation loosely in the context of two-dimensional steady heat transfer in an isotropic material. Later, in Ch. 8, we will consider Laplace's equation in the context of two-dimensional, irrotational, incompressible, inviscid, steady fluid mechanics. There are clear analogs between the two physically motivating problems. They are as follows. We can consider temperature $T(x, y)$ to be a potential field. In fluid mechanics, we can consider $\phi(x, y)$ to be the equivalent velocity potential field. For isotropic thermal conductivity $k = 1$, the negative of the gradient of the temperature field yields the heat flux vector $\mathbf{q} = -\nabla T$. In fluid mechanics, the equivalent is that the velocity vector is the gradient of the velocity potential $\mathbf{u} = \nabla\phi$. Both the heat flux and velocity vector fields are curl-free because they are gradients of a scalar field: $\nabla \times \mathbf{q} = \mathbf{0}$, $\nabla \times \mathbf{u} = \mathbf{0}$. The vector fields of both $\mathbf{q}$ and $\mathbf{u}$ are divergence-free because of the first law of thermodynamics, and incompressibility, respectively: $\nabla^T \cdot \mathbf{q} = 0$, $\nabla^T \cdot \mathbf{u} = 0$. Both potentials satisfy Laplace's equation: $\nabla^2 T = 0$, $\nabla^2\phi = 0$. Functions that satisfy Laplace's equation are often defined as *harmonic functions*. Lastly Eq. (2.101) implies the definition of what we will call a *thermal stream function*, ψ_T, defined such that

$$q_x = \frac{\partial \psi_T}{\partial y}, \qquad q_y = -\frac{\partial \psi_T}{\partial x}. \tag{2.114}$$

Substituting these into Eq. (2.101) yields

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi_T}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi_T}{\partial y}\right) = 0, \tag{2.115}$$

$$\frac{\partial^2 \psi_T}{\partial x^2} + \frac{\partial^2 \psi_T}{\partial y^2} = 0, \tag{2.116}$$

$$\nabla^2 \psi_T = 0. \tag{2.117}$$

Example 2.4

Examine the solution of Laplace's equation $T(x, y) = f(x+iy) = (x+iy)^2$. Consider the associated heat flux vector for $k = 1$, $\mathbf{q} = -\nabla T$.

Let us expand and examine our proposed solution:

$$T(x, y) = x^2 - y^2 + 2xyi. \tag{2.118}$$

Now Laplace's equation is linear so the method of superposition applies. Because of linearity, we can separately examine the solutions. Let us consider

$$T(x, y) = x^2 - y^2. \tag{2.119}$$

Clearly, it satisfies Laplace's equation as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 2 - 2 = 0. \tag{2.120}$$

For this field, the heat flux vector is

$$\mathbf{q} = -\begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x \\ 2y \end{pmatrix}. \tag{2.121}$$

Remarkably, it will be shown in Ch. 8 that curves on which the imaginary part of our T are constant:

$$2xy = C, \tag{2.122}$$

are parallel to $\mathbf{q}$. In fact, it is easy to show that the imaginary part represents the thermal stream function:

$$\psi_T = 2xy. \tag{2.123}$$

Parameterizing this curve, we can say

$$x = s, \qquad y = \frac{C}{2s}. \tag{2.124}$$

So

$$\frac{dx}{ds} = 1, \qquad \frac{dy}{ds} = -\frac{C}{2s^2}. \tag{2.125}$$

The tangent vector $\mathbf{t}$ is then

$$\mathbf{t} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} = \mathbf{i} - \frac{C}{2s^2}\mathbf{j} = \mathbf{i} - \frac{y}{x}\mathbf{j}. \tag{2.126}$$

If $\mathbf{t}$ is parallel to $\mathbf{q}$, then the cross product of the two must be zero. Let us check.

$$\mathbf{t} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -\frac{y}{x} & 0 \\ -2x & 2y & 0 \end{vmatrix} = (2y - 2y)\mathbf{k} = \mathbf{0}. \tag{2.127}$$

Curves of constant T along with the heat flux vector field $\mathbf{q}$ for $k = 1$ are shown in Fig. 2.4. This general method can be used to easily generate some remarkable plots of harmonic functions. We give a few in Fig. 2.5.

Example 2.5

Show that

$$T(x, y) = C_1 \ln\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right) + C_2, \tag{2.128}$$

is a solution to Laplace's equation.

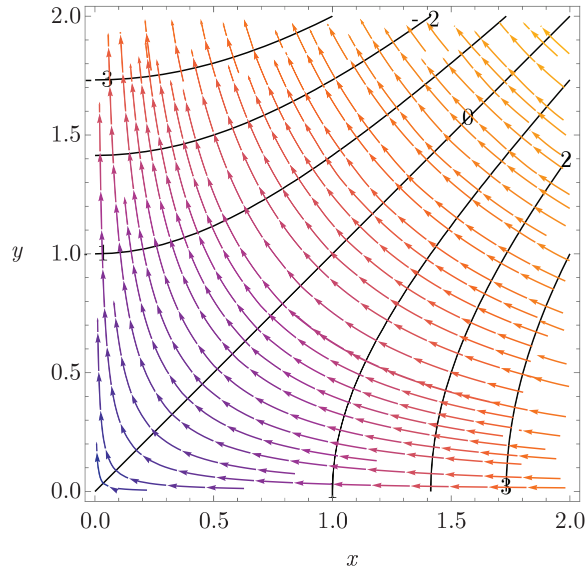


Figure 2.4: A fundamental solution, $T(x,y) = x^2 - y^2$, to Laplace's equation along with heat flux vector field for $k = 1$.

Taking the partial derivatives, we get

$$\frac{\partial^2 T}{\partial x^2} = C_1 \left(\frac{1}{(x-x_0)^2 + (y-y_0)^2} - \frac{2(x-x_0)^2}{\left((x-x_0)^2 + (y-y_0)^2\right)^2} \right), \tag{2.129}$$

$$\frac{\partial^2 T}{\partial y^2} = C_1 \left(\frac{1}{(x-x_0)^2 + (y-y_0)^2} - \frac{2(y-y_0)^2}{\left((x-x_0)^2 + (y-y_0)^2\right)^2} \right). \tag{2.130}$$

Thus,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{2.131}$$

and Laplace's equation is satisfied for arbitrary x_0, y_0. We could also superpose fundamental solutions at various points. If $x_0 = y_0 = 0$, and transforming to polar coordinates with $r^2 = x^2 + y^2$, our fundamental solution becomes

$$T(r,\theta) = C_1 \ln r + C_2. \tag{2.132}$$

One can show that Laplace's equation in polar coordinates is represented as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0. \tag{2.133}$$

The fundamental solution obviously satisfies this as

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{C_1}{r} \right)}_{=0} + \frac{1}{r^2}\underbrace{\frac{\partial^2}{\partial \theta^2}\left(C_1 \ln r + C_2\right)}_{=0} = 0. \tag{2.134}$$

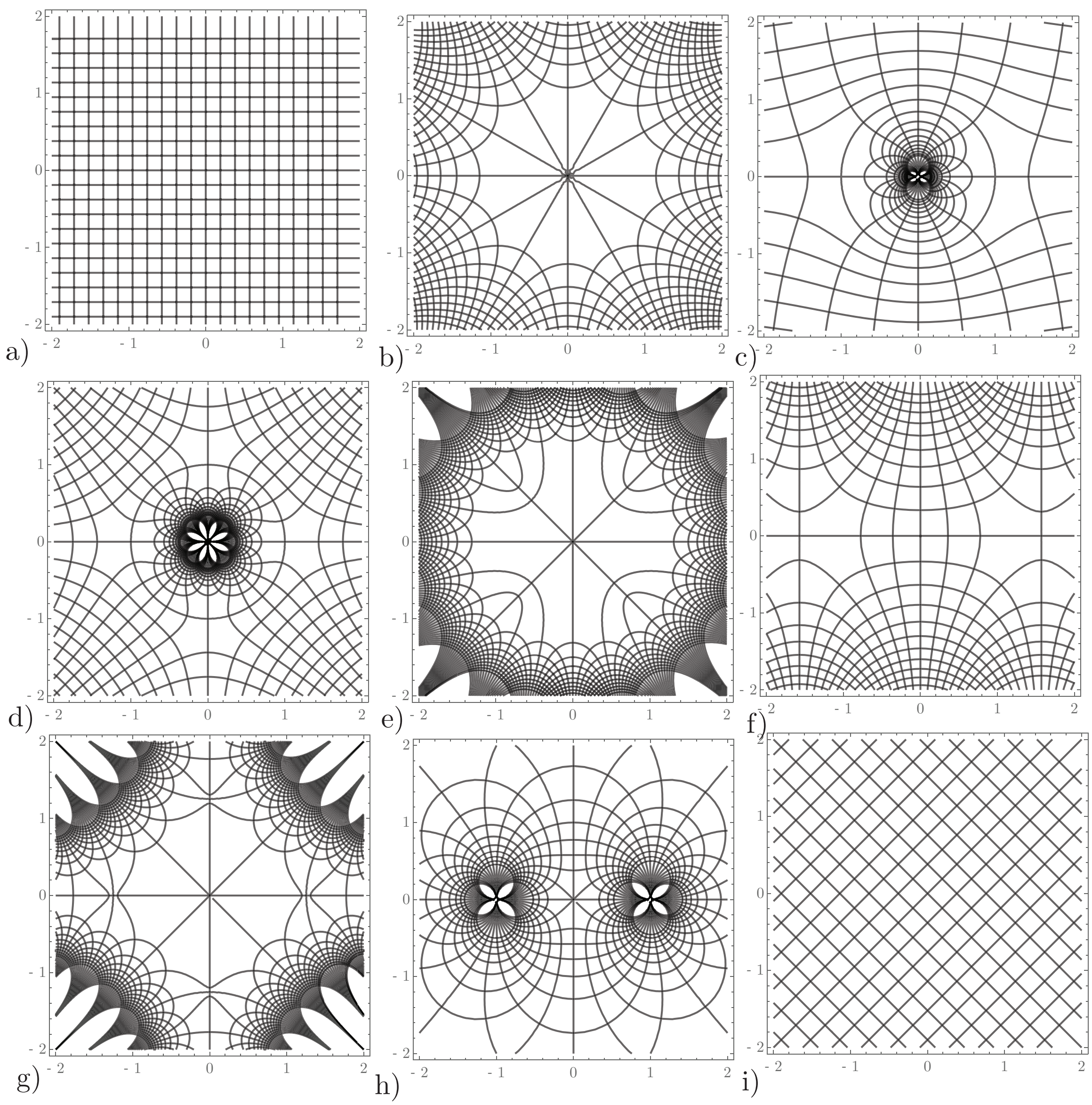


Figure 2.5: Harmonic functions: isotherms and adiabats for a) $T = z$, b) $T = z^3$, c) $T = z+1/z$, d) $T = z^2+1/z^2$, e) $T = z^6+z^2$, f) $T = \sin z$, g) $T = \sin z^2$, h) $T = 1/(z+1)-1/(z-1)$, i) $T = (1+i)z$.

In three-dimensional systems, the fundamental solution to Laplace's equation is

$$T(x,y,z) = \frac{C_1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + C_2. \tag{2.135}$$

Direct substitution reveals it satisfies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{2.136}$$

One can also sum solutions centered at different values of x_0, y_0, z_0. If $x_0 = y_0 = z_0 = 0$, transformation to spherical coordinates reveals the fundamental solution is

$$T(r,\theta,\phi) = \frac{C_1}{r} + C_2, \tag{2.137}$$

which satisfies Laplace's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial T}{\partial \phi}\right) = 0. \tag{2.138}$$

Extending to higher dimensions, we find

$$T(w,x,y,z) = \frac{C_1}{(w-w_0)^2+(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} + C_2, \tag{2.139}$$

satisfies

$$\frac{\partial^2 T}{\partial w^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{2.140}$$

And for general N-dimensional systems with $N \geq 3$ we find

$$T(x_1,\ldots,x_N) = C_1 \left(\sum_{n=1}^{N} (x_n - x_{on})^2\right)^{\frac{-(N-2)}{2}} + C_2, \tag{2.141}$$

satisfies

$$\sum_{n=1}^{N} \frac{\partial^2 T}{\partial x_n^2} = 0. \tag{2.142}$$

With

$$r = \left(\sum_{n=1}^{N} (x_n - x_{on})^2\right)^{1/2}, \tag{2.143}$$

we can say

$$T(r) = \frac{C_1}{r^{N-2}} + C_2, \qquad N \geq 3, \tag{2.144}$$

satisfies the N-dimensional Laplace's equation.

This is consistent with solutions to the one-dimensional Laplace's equation for hyperspherical coordinates:

$$\frac{1}{r^{N-1}}\frac{d}{dr}\left(r^{N-1}\frac{dT}{dr}\right) = 0. \tag{2.145}$$

For various values of N, we get fundamental solutions

$$\begin{aligned} N=1, \quad \text{planar}, \quad T(r) &= C_1 r + C_2, && (2.146)\\ N=2, \quad \text{polar}, \quad T(r) &= C_1 \ln r + C_2, && (2.147)\\ N=3, \quad \text{spherical}, \quad T(r) &= \frac{C_1}{r} + C_2, && (2.148)\\ N\geq 3, \quad \text{hyperspherical}, \quad T(r) &= \frac{C_1}{r^{N-2}} + C_2. && (2.149) \end{aligned}$$

Example 2.6

Show in one dimension that the fundamental solution

$$T(x) = \frac{1}{2}\left(1 - |x|\right), \tag{2.150}$$

satisfies

$$-\nabla^T \cdot \nabla T = \delta(x), \qquad T(1) = T(-1) = 0. \tag{2.151}$$

Away from the origin, this is Laplace's equation. Formally it is what is known as a Poisson[6] equation because of the presence of a source term. In general, the Poisson equation takes the form

$$-\nabla^T \cdot \nabla T = f(\mathbf{x}). \tag{2.152}$$

In this case the source term f is a point source. In one dimension, our equation becomes

$$-\frac{d^2T}{dx^2} = \delta(x), \qquad T(-1) = T(1) = 0. \tag{2.153}$$

Let us integrate both sides, giving

$$\begin{aligned} -\int_{-1}^{1} \frac{d^2T}{dx^2}\, dx &= \int_{-1}^{1} \delta(x)\, dx, && (2.154)\\ -\left.\frac{dT}{dx}\right|_{-1}^{1} &= 1, && (2.155)\\ \left.\frac{dT}{dx}\right|_{x=-1} - \left.\frac{dT}{dx}\right|_{x=1} &= 1. && (2.156) \end{aligned}$$

Now our fundamental solution has

$$T(-1) = \frac{1}{2}\left(1 - |-1|\right) = 0, \qquad T(1) = \frac{1}{2}\left(1 - |1|\right) = 0, \tag{2.157}$$

so the boundary conditions are satisfied. And for $x \in [-1, 0]$, the fundamental solution is

$$\begin{aligned} T(x) &= \frac{1}{2}\left(1 + x\right), && (2.158)\\ \frac{dT}{dx} &= \frac{1}{2}. && (2.159) \end{aligned}$$

[6] Siméon Denis Poisson, 1781-1840, French mathematician.

For $x \in [0, 1]$, the fundamental solution is

$$T(x) = \frac{1}{2}(1 - x), \tag{2.160}$$

$$\frac{dT}{dx} = -\frac{1}{2}. \tag{2.161}$$

Thus, we have

$$\left.\frac{dT}{dx}\right|_{x=-1} - \left.\frac{dT}{dx}\right|_{x=1} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1, \tag{2.162}$$

so our fundamental solution is a solution to the Poisson equation.

Example 2.7

Show in two dimensions that the fundamental solution

$$T(r) = -\frac{1}{2\pi}\ln r, \tag{2.163}$$

satisfies

$$-\nabla^T \cdot \nabla T = \delta(x)\delta(y). \tag{2.164}$$

Away from the origin, this is Laplace's equation. Formally it is a Poisson equation because of the presence of a source term. In this case the source term is a point source. The point source is embodied in δ, the Dirac delta function. Let us integrate both sides over a circular region that encloses the origin and use the two-dimensional divergence theorem, giving

$$-\int_A \nabla^T \cdot \nabla T \; dA = \int_A \delta(x)\delta(y) \; dA, \tag{2.165}$$

$$-\oint_C \nabla T \cdot \mathbf{n} \; ds = \int_x \int_y \delta(x)\delta(y) \; dy \; dx, \tag{2.166}$$

$$= \int_x \delta(x) \underbrace{\int_y \delta(y) \; dy}_{=1} \; dx, \tag{2.167}$$

$$= \int_x \delta(x) \; dx, \tag{2.168}$$

$$-\int_0^{2\pi} \frac{\partial T}{\partial r} r \; d\theta = 1, \tag{2.169}$$

$$-\int_0^{2\pi} \left(-\frac{1}{2\pi r}\right) r \; d\theta = 1, \tag{2.170}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta = 1, \tag{2.171}$$

$$1 = 1. \tag{2.172}$$

Example 2.8

Show in three dimensions that the fundamental solution

$$T(r) = \frac{1}{4\pi r}, \tag{2.173}$$

satisfies

$$-\nabla^T \cdot \nabla T = \delta(x)\delta(y)\delta(z). \tag{2.174}$$

Away from the origin, this is Laplace's equation. Formally it is a Poisson equation because of the presence of a source term. In this case the source term is a point source. Let us integrate both sides over a spherical volume that encloses the origin and use the divergence theorem, giving

$$\begin{aligned}
-\int_V \nabla^T \cdot \nabla T \; dV &= \int_V \delta(x)\delta(y)\delta(z)\; dV, && (2.175)\\
-\int_A \nabla T \cdot \mathbf{n}\; dA &= \int_x \int_y \int_z \delta(x)\delta(y)\delta(z)\; dz\; dy\; dx, && (2.176)\\
-\int_0^{2\pi}\int_0^{\pi} \frac{\partial T}{\partial r} r^2 \sin\phi\; d\phi\; d\theta &= 1, && (2.177)\\
-\int_0^{2\pi}\int_0^{\pi} \left(\frac{-1}{4\pi r^2}\right) r^2 \sin\phi\; d\phi\; d\theta &= 1, && (2.178)\\
\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\phi\; d\phi\; d\theta &= 1, && (2.179)\\
\frac{1}{4\pi}\int_0^{2\pi} \left.(-\cos\phi)\right|_0^{\pi}\; d\theta &= 1, && (2.180)\\
\frac{1}{4\pi}\int_0^{2\pi} 2\; d\theta &= 1, && (2.181)\\
1 &= 1. && (2.182)
\end{aligned}$$

Plots for the fundamental solutions for planar, cylindrical, and spherical coordinate systems are shown in Fig. 2.6.

Example 2.9

Consider a two-dimensional domain with a point source of thermal energy at $(x, y) = (1, 1)$ and a point sink of thermal energy at $(x, y) = (-1, -1)$. Find the temperature field and the heat flux vector field.

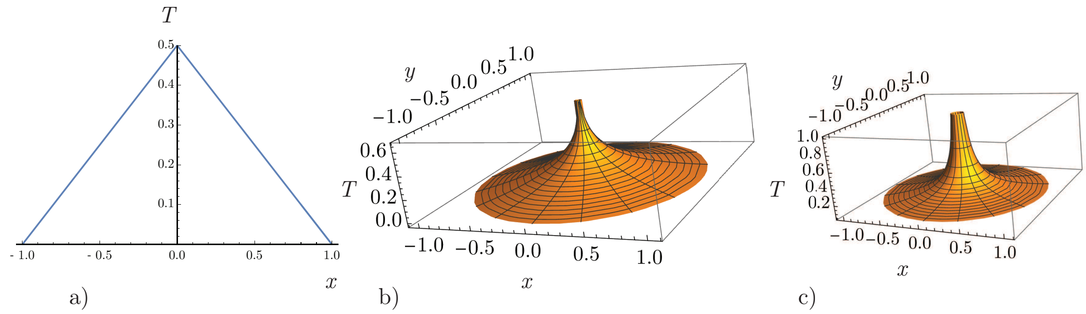


Figure 2.6: Fundamental solutions to Laplace's equation for a) planar, b) cylindrical, and c) spherical systems.

A point source centered at the origin of a two-dimensional field has

$$T = -\frac{1}{2\pi}\ln r = -\frac{1}{2\pi}\ln\sqrt{x^2+y^2}. \tag{2.183}$$

We have seen that this satisfies, Eq. (2.164), $-\nabla^T \cdot \nabla T = \delta(x)\delta(y)$, with the Dirac delta functions each of unit strength. Now a sink has the opposite sign. A source centered at $(1, 1)$ and a sink centered at $(-1, -1)$ has the temperature field that arises from superposition of two fields:

$$T = \underbrace{-\frac{1}{2\pi}\ln\sqrt{(x-1)^2+(y-1)^2}}_{\text{source}} + \underbrace{\frac{1}{2\pi}\ln\sqrt{(x+1)^2+(y+1)^2}}_{\text{sink}}. \tag{2.184}$$

It satisfies

$$-\nabla^T \cdot \nabla T = -\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \delta(x-1)\delta(y-1) - \delta(x+1)\delta(y+1). \tag{2.185}$$

Direct substitution of the temperature field into Laplace's equation reveals that as long as $(x, y) \neq (1, 1)$, $(x, y) \neq (-1, -1)$, the Laplacian is zero. So it is a solution. The heat flux vector is

$$\mathbf{q} = -\nabla T = \begin{pmatrix} \frac{x^2+2xy-y^2-2}{\pi(x^4+2x^2y^2-8xy+y^4+4)} \\ \frac{-x^2+2xy+y^2-2}{\pi(x^4+2x^2y^2-8xy+y^4+4)} \end{pmatrix}. \tag{2.186}$$

A plot of the temperature and heat flux vector fields is shown in Fig. 2.7. We clearly see the heat flux emanating from the source at $(x, y) = (1, 1)$ and flowing into the sink at $(x, y) = (-1, -1)$. It is easy to add sources and sinks. A plot of the temperature and heat flux vector fields for point sources at $(\pm 1, 0)$ and point sinks at $(0, \pm 1)$ is given in Fig. 2.8.

Example 2.10

Consider a three-dimensional domain with a point source of thermal energy at $(x, y, z) = (1, 1, 0)$ and a point sink of thermal energy at $(x, y, z) = (-1, -1, 0)$. Find the temperature field and the heat flux vector field.

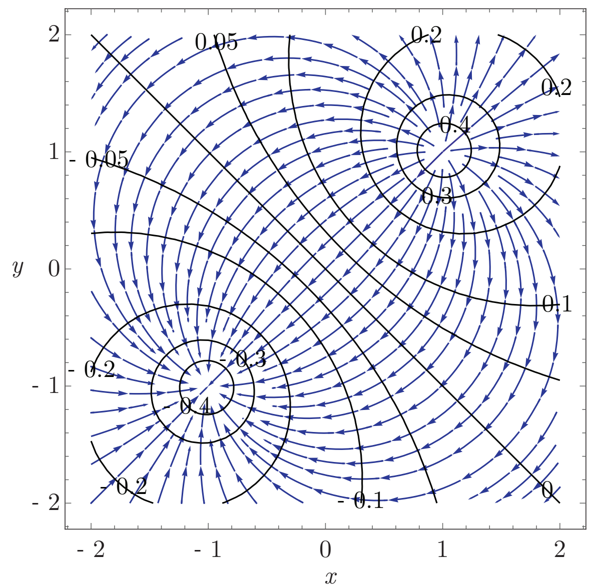


Figure 2.7: Temperature contours, $T(x, y)$, and heat flux vector field, $\mathbf{q}(x, y)$, for the superposition of a point source and sink of thermal energy in a two-dimensional geometry.

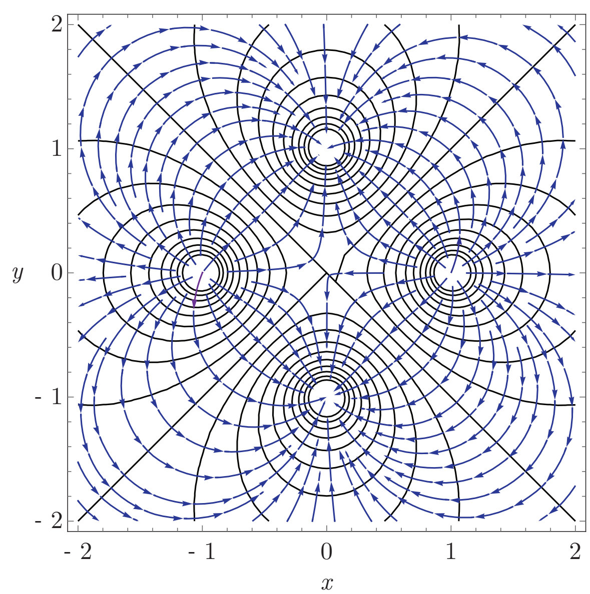


Figure 2.8: Temperature contours, $T(x, y)$, and heat flux vector field, $\mathbf{q}(x, y)$, for the superposition of two point sources and two point sinks in a two-dimensional geometry.

A point source centered at the origin of a three-dimensional field has

$$T = \frac{1}{4\pi r} = \frac{1}{4\pi\sqrt{x^2+y^2+z^2}}. \tag{2.187}$$

We have seen that this satisfies, Eq. (2.174), $-\nabla^T \cdot \nabla T = \delta(x)\delta(y)\delta(z)$, with the Dirac delta functions each of unit strength. Now a sink has the opposite sign. A source centered at $(1,1,0)$ and a sink centered at $(-1,-1,0)$ has the temperature field that arises from superposition of two fields:

$$T = \underbrace{\frac{1}{4\pi\sqrt{(x-1)^2+(y-1)^2+z^2}}}_{\text{source}} - \underbrace{\frac{1}{4\pi\sqrt{(x+1)^2+(y+1)^2+z^2}}}_{\text{sink}}. \tag{2.188}$$

It satisfies

$$-\nabla^T \cdot \nabla T = -\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \delta(x-1)\delta(y-1)\delta(z) - \delta(x+1)\delta(y+1)\delta(z). \tag{2.189}$$

Direct substitution of the temperature field into Laplace's equation reveals that as long as $(x,y) \neq (1,1)$, $(x,y) \neq (-1,-1)$, the Laplacian is zero. So it is a solution. The heat flux vector is

$$\mathbf{q} = -\nabla T = \begin{pmatrix} \frac{x-1}{4\pi((x-1)^2+(y-1)^2+z^2)^{3/2}} - \frac{x+1}{4\pi((x+1)^2+(y+1)^2+z^2)^{3/2}} \\ \frac{y-1}{4\pi((x-1)^2+(y-1)^2+z^2)^{3/2}} - \frac{y+1}{4\pi((x+1)^2+(y+1)^2+z^2)^{3/2}} \\ \frac{z}{4\pi((x-1)^2+(y-1)^2+z^2)^{3/2}} - \frac{z}{4\pi((x+1)^2+(y+1)^2+z^2)^{3/2}} \end{pmatrix}. \tag{2.190}$$

Confining attention to the plane of $z = 0$, a plot of the temperature and heat flux vector fields is shown in Fig. 2.9a. We show in Fig. 2.9b the three-dimensional vector field of heat flux. We clearly see the heat flux emanating from the source at $(x,y,z) = (1,1,0)$ and flowing into the sink at $(x,y,z) = (-1,-1,0)$.

Problems

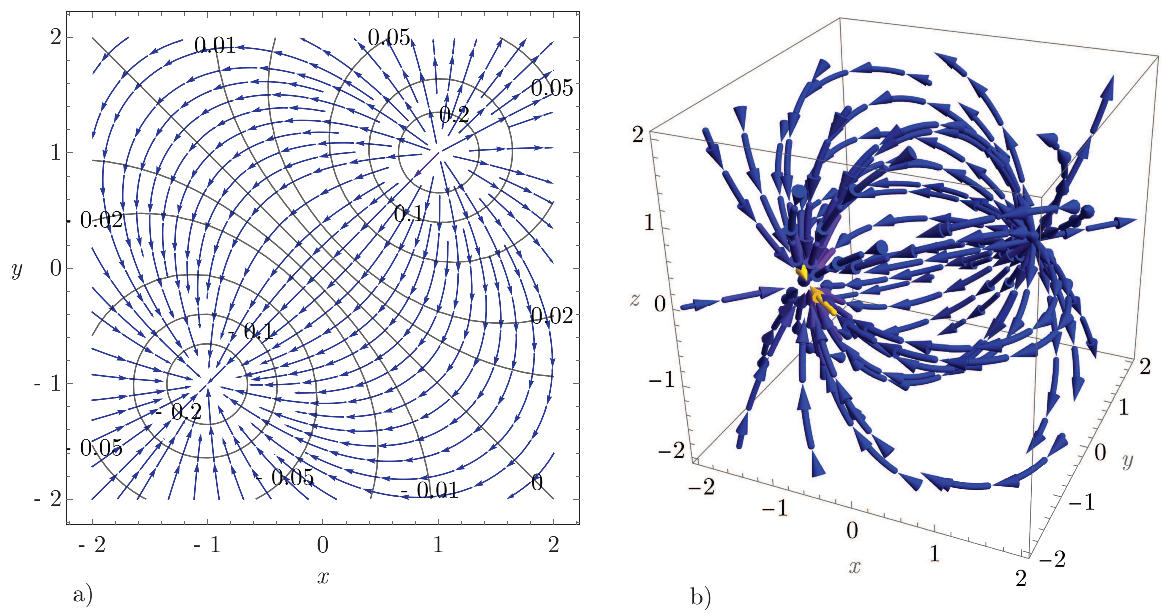


Figure 2.9: Superposition of a point source and sink of thermal energy in a three-dimensional geometry: a) temperature contours, $T(x, y, z = 0)$, and heat flux vector field, $\mathbf{q}(x, y, z = 0)$, b) heat flux vector field $\mathbf{q}(x, y, z)$.

Chapter 3

Separation of variables

see Mei, Chapter 4,5.

Here we consider the technique of separation of variables. This method is appropriate for a wide variety of linear partial differential equations.

3.1 Well-posedness

An important philosophical notion permeates the literature of partial differential equations, that being that a partial differential equation should be accompanied by a set of initial and/or boundary conditions that render its solutions consistent with those observed in nature for systems it is intended to model. This idea was developed most notably by Hadamard,[1] who established three criteria of a problem to be *well-posed*. The solution must

- exist,
- be uniquely determined, and
- depend continuously on the initial and/or boundary data.

Using these criteria, one can, for example, show that Laplace's equation, Eq. (1.104), is well-posed if values of T are specified on the full boundary of a given domain. His famous counterexample shows how Laplace's equation is ill-posed if values of T and its derivatives are simultaneously imposed on a portion of the boundary. Let us consider this counterexample, which will also serve as a vehicle to introduce the main topic of this chapter: *separation of variables* as a means to solve partial differential equations.

Example 3.1

Analyze

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad y > 0, \tag{3.1}$$

[1] Jacques Hadamard, 1865-1963, French mathematician.

with boundary conditions

$$T(x,0) = 0, \qquad \frac{\partial T}{\partial y}(x,0) = \frac{\sin(nx)}{n}. \tag{3.2}$$

A sketch of this scenario is shown in Fig. 3.1.

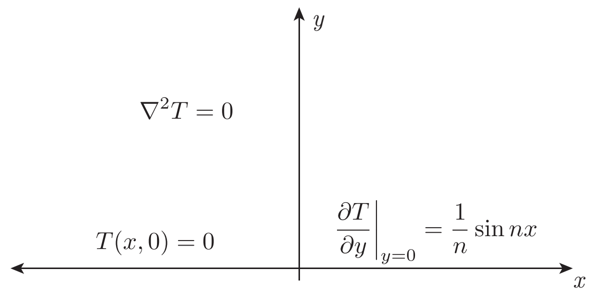


Figure 3.1: Configuration for the counterexample of Hadamard to assess the well-posedness of Laplace's equation.

Thus we have specified both T and its derivative on the boundary $y = 0$. Note that as $n \to \infty$, that $\partial T/\partial y \to 0$. Let us assume the solution $T(x,y)$ can be separated into the following forms:

$$T(x,y) = A(x)B(y). \tag{3.3}$$

In contrast to the d'Alembert solution, whose separated functions combine as a sum, here we have the separated functions combine as a product. We shall examine if this assumption leads to a viable solution. With our assumption, we get the following expressions for various derivatives:

$$\frac{\partial T}{\partial x} = B(y)\frac{dA}{dx}, \qquad \frac{\partial^2 T}{\partial x^2} = B(y)\frac{d^2A}{dx^2}, \tag{3.4}$$

$$\frac{\partial T}{\partial y} = A(x)\frac{dB}{dy}, \qquad \frac{\partial^2 T}{\partial y^2} = A(x)\frac{d^2B}{dy^2}. \tag{3.5}$$

We substitute these into Eq. (3.1) to get

$$B(y)\frac{d^2A}{dx^2} + A(x)\frac{d^2B}{dy^2} = 0, \tag{3.6}$$

$$\underbrace{\frac{1}{A(x)}\frac{d^2A}{dx^2}}_{\text{function of } x \text{ only}} = -\underbrace{\frac{1}{B(y)}\frac{d^2B}{dy^2}}_{\text{function of } y \text{ only}} \tag{3.7}$$

The left side is a function only of x, while the right side is function only of y. This can only happen if both sides are equal to the same constant. Let us choose the constant to be $-\lambda^2$:

$$\frac{1}{A(x)}\frac{d^2A}{dx^2} = -\frac{1}{B(y)}\frac{d^2B}{dy^2} = -\lambda^2. \tag{3.8}$$

This choice is non-intuitive. It is guided by the boundary conditions for this particular problem. Had we made more general choices, we would be led precisely to the same destination as this useful choice. This induces two linear second order ordinary differential equations:

$$\frac{d^2A}{dx^2} + \lambda^2 A = 0, \tag{3.9}$$

$$\frac{d^2B}{dy^2} - \lambda^2 B = 0. \tag{3.10}$$

We focus first on the second, Eq. (3.10). By inspection, it has a solution composed of a linear combination of two linearly independent complementary functions, and is

$$B(y) = C_1 \cosh(\lambda y) + C_2 \sinh(\lambda y). \tag{3.11}$$

Here C_1 and C_2 are arbitrary constants. Because $T(x,0) = 0$, we must insist that $B(0) = 0$, giving

$$B(0) = 0 = C_1 \cosh 0 + \cancel{C_2 \sinh 0}. \tag{3.12}$$

We thus learn that $C_1 = 0$, giving

$$B(y) = C_2 \sinh \lambda y. \tag{3.13}$$

We return to the first, Eq. (3.9), which has solution

$$A(x) = C_3 \sin \lambda x + C_4 \cos \lambda x. \tag{3.14}$$

Combining, we get

$$T(x,y) = A(x)B(y) = \sinh \lambda y \left(\hat{C}_3 \sin \lambda x + \hat{C}_4 \cos \lambda x \right). \tag{3.15}$$

Here we have defined $\hat{C}_3 = C_2 C_3$ and $\hat{C}_4 = C_2 C_4$. Now let us satisfy the second condition at $y = 0$, that on $\partial T/\partial y$:

$$\frac{\partial T}{\partial y} = \lambda \cosh \lambda y \left(\hat{C}_3 \sin \lambda x + \hat{C}_4 \cos \lambda x \right), \tag{3.16}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = \lambda \cosh 0 \left(\hat{C}_3 \sin \lambda x + \hat{C}_4 \cos \lambda x \right) = \frac{\sin nx}{n}, \tag{3.17}$$

$$= \lambda \left(\hat{C}_3 \sin \lambda x + \hat{C}_4 \cos \lambda x \right) = \frac{\sin nx}{n}. \tag{3.18}$$

This is achieved if we take $\lambda = n$, $\hat{C}_3 = 1/n^2$, and $\hat{C}_4 = 0$, giving

$$T(x,y) = \frac{1}{n^2} \sinh ny \sin nx. \tag{3.19}$$

Now we must admit that Eq. (3.19) satisfies the original partial differential equation and both conditions at $y = 0$. Because the original equation is linear, we are inclined to believe that this is the unique solution which does so.

Let us examine the properties of our solution. We can consider T at a small, fixed, positive value of y: $y = \hat{y} > 0$ and study this in the limit as $n \to \infty$. Now, we recognize that it is the inhomogeneous boundary condition, $\sin(nx)/n$ that entirely drives the solution for $T(x,y)$ to be nontrivial. As $n \to \infty$, the sole driving impetus becomes a low amplitude, high frequency driver. At $y = \hat{y}$, we have

$$T(x,\hat{y}) = \frac{1}{n^2} \sinh n\hat{y} \sin nx, \tag{3.20}$$

$$= \frac{1}{n^2} \left(\frac{e^{n\hat{y}} - e^{-n\hat{y}}}{2} \right) \sin nx. \tag{3.21}$$

For large n and $\hat{y} > 0$, the first term dominates yielding

$$T(x,\hat{y}) \approx \frac{e^{n\hat{y}}}{n^2}\sin nx. \tag{3.22}$$

Because as $n \to \infty$ the $e^{n\hat{y}}$ approaches infinity faster than n^2, we have the amplitude of T,

$$\lim_{n\to\infty}\frac{e^{n\hat{y}}}{n^2} \to \infty. \tag{3.23}$$

Paradoxically then, *as the amplitude of the driver at the boundary is reduced to zero by increasing n, the amplitude of the response nearby the boundary is simultaneously driven to infinity.* This despite the fact that T at the boundary is in fact zero. Clearly as $n \to \infty$, the solution loses its continuity property at the boundary. This famous counter-example problem is thus *not well-posed*, as such behavior is not observed in nature.

3.2 Cartesian geometries

Let us consider a series of example problems posed on Cartesian geometries.

3.2.1 Wave equation

Example 3.2

Solve the wave equation, Eq. (2.15),

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, \tag{3.24}$$

subject to boundary and initial conditions

$$y(0,t) = y(L,t) = 0, \qquad y(x,0) = f(x), \qquad \frac{\partial y}{\partial t}(x,0) = 0. \tag{3.25}$$

Generate solutions for four sets of initial conditions:

$$f(x) = y_0 \sin\frac{\pi x}{L}, \qquad \text{mono-modal}, \tag{3.26}$$

$$= y_0\left(\sin\frac{\pi x}{L} + \frac{1}{10}\sin\frac{10\pi x}{L}\right), \qquad \text{bi-modal}, \tag{3.27}$$

$$= y_0\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right), \qquad \text{poly-modal}, \tag{3.28}$$

$$= y_0\left(H\left(\frac{x}{L}-\frac{2}{5}\right) - H\left(\frac{x}{L}-\frac{3}{5}\right)\right), \qquad \text{poly-modal}. \tag{3.29}$$

We assume solutions of the type

$$y(x,t) = A(x)B(t). \tag{3.30}$$

With this assumption, Eq. (3.24) becomes

$$A(x)\frac{d^2B}{dt^2} = a^2B(t)\frac{d^2A}{dx^2}, \tag{3.31}$$

$$-\frac{1}{a^2B(t)}\frac{d^2B}{dt^2} = -\frac{1}{A(x)}\frac{d^2A}{dx^2} = \lambda^2. \tag{3.32}$$

Once again, for an arbitrary function of t to be equal to an arbitrary function of x, both functions must be the same constant, which we have selected to be λ^2. This induces two second order linear ordinary differential equations:

$$\frac{d^2A}{dx^2} + \lambda^2 A = 0, \tag{3.33}$$

$$\frac{d^2B}{dt^2} + a^2\lambda^2 B = 0. \tag{3.34}$$

Consider the equation for A first. In order to satisfy the boundary conditions $y(0,t) = y(L,t) = 0$, we must have $A(0) = A(L) = 0$. As is done here, if y is specified on a boundary it is known as a *Dirichlet*[2] boundary condition. Had the derivative $\partial y/\partial x$ been specified, the boundary condition would have been called a *Neumann*[3] boundary condition. When a linear combination of y and $\partial y/\partial x$ is specified on a boundary, it is known as a *Robin*[4] boundary condition. Along with these boundary conditions, Eq. (3.33) can be recast as an eigenvalue problem:

$$-\frac{d^2}{dx^2}A = \lambda^2 A, \qquad A(0) = A(L) = 0. \tag{3.35}$$

With $\mathbf{L} = -d^2/dx^2$, a self-adjoint positive definite linear operator, this takes the form

$$\mathbf{L}A = \lambda^2 A. \tag{3.36}$$

We recall that *self-adjoint* operators have *orthogonal* eigenfunctions and real eigenvalues. Because it can be shown that our $\mathbf{L}$ is positive definite, the eigenvalues are also positive, which is why we describe the eigenvalue as λ^2.

Solving Eq. (3.33), we see that

$$A(x) = C_1 \sin \lambda x + C_2 \cos \lambda x. \tag{3.37}$$

For $A(0) = 0$, we get

$$A(0) = 0 = C_1 \cancel{\sin 0} + C_2 \cos 0 = C_2. \tag{3.38}$$

Thus,

$$A(x) = C_1 \sin \lambda x. \tag{3.39}$$

Now at $x = L$, we have

$$A(L) = 0 = C_1 \sin \lambda L. \tag{3.40}$$

To guarantee this condition is satisfied, we must require that

$$\lambda L = n\pi, \qquad n = 1, 2, \ldots, \tag{3.41}$$

$$\lambda = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots. \tag{3.42}$$

[2]Peter Gustav Lejeune Dirichlet, 1805-1859, German mathematician.
[3]Carl Gottfried Neumann, 1832-1925, German mathematician.
[4]Victor Gustave Robin, 1855-1897, French mathematician.

With this, we have

$$A(x) = C_1 \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \dots \tag{3.43}$$

We note that,

- the eigenvalues λ^2 are real and positive,
- the eigenfunctions $C_n \sin n\pi x$ have an arbitrary amplitude.

It will also soon be useful to employ the orthogonality property, $\int_0^L (\sin m\pi x/L)(\sin n\pi x/L)\, dx = 0$ if $m \neq n$ when m and n are integers, and that the integral is nonzero if $n = m$.

We now turn to solution of Eq. (3.34), which is now restated as

$$\frac{d^2B}{dt^2} + \left(\frac{n\pi a}{L}\right)^2 B = 0. \tag{3.44}$$

This has solution

$$B(t) = C_3 \sin \frac{n\pi at}{L} + C_4 \cos \frac{n\pi at}{L}. \tag{3.45}$$

Now for $\partial y/\partial t$ to be everywhere 0 at $t = 0$, we must insist that $dB/dt(0) = 0$. Enforcing this gives

$$\frac{dB}{dt} = \frac{C_3 n\pi a}{L} \cos \frac{n\pi at}{L} - \frac{C_4 n\pi a}{L} \sin \frac{n\pi at}{L}, \tag{3.46}$$

$$\frac{dB}{dt}(t=0) = \frac{C_3 n\pi a}{L} \cos 0 - \frac{C_4 n\pi a}{L} \sin 0 = 0, \tag{3.47}$$

$$= \frac{C_3 n\pi a}{L} = 0. \tag{3.48}$$

We thus insist that $C_3 = 0$. Taking $\hat{C}_4 = C_1 C_4$, our solution combines to form

$$y(x,t) = \hat{C}_4 \cos \frac{n\pi at}{L} \sin \frac{n\pi x}{L}. \tag{3.49}$$

We next recognize that this solution is valid for arbitrary positive integer n; moreover, because the original equation is linear, the principle of superposition applies and arbitrary linear combinations also are valid solutions. We can express this by generalizing to

$$y(x,t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi at}{L} \sin \frac{n\pi x}{L}. \tag{3.50}$$

We can use standard trigonometric reductions to recast Eq. (3.50) as

$$y(x,t) = \sum_{n=1}^{\infty} \frac{C_n}{2} \left(\sin\left(\frac{n\pi}{L}(x + at)\right) + \sin\left(\frac{n\pi}{L}(x - at)\right)\right). \tag{3.51}$$

Importantly, we note that

- The solution can be thought of an infinite sum of left- and right-propagating waves.
- All modes travel at the same velocity magnitude a; formally such waves are *non-dispersive*.
- The amplitude of each mode does not decay with time; formally, such waves are *non-diffusive*.

We can fix the various values of C_n by applying the initial condition for $y(x, 0) = f(x)$:

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L}. \tag{3.52}$$

This amounts to finding the Fourier sine series expansion of $f(x)$. We get this by taking advantage of the orthogonality properties of $\sin n\pi x/L$ on the domain $x \in [0, L]$ by the following series of operations.

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L}, && (3.53)\\
\sin \frac{m\pi x}{L} f(x) &= \sum_{n=1}^{\infty} C_n \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}, && (3.54)\\
\int_0^L \sin \frac{m\pi x}{L} f(x)\ dx &= \int_0^L \sum_{n=1}^{\infty} C_n \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\ dx, && (3.55)\\
&= \sum_{n=1}^{\infty} C_n \underbrace{\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\ dx}_{=L\delta_{mn}/2}, && (3.56)
\end{aligned}$$

Because of orthogonality, the integral has value of 0 for $n \neq m$ and $L/2$ for $n = m$. Employing the Kronecker[5] delta notation,

$$\delta_{nm} = \left\{ \begin{array}{cc} 0 & n \neq m \\ 1 & n = m \end{array} \right. , \tag{3.57}$$

we get

$$\begin{aligned}
\int_0^L \sin \frac{m\pi x}{L} f(x)\ dx &= \frac{L}{2} \sum_{n=1}^{\infty} C_n \delta_{nm}, && (3.58)\\
&= \frac{L}{2} C_m, && (3.59)\\
C_n &= \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} f(x)\ dx. && (3.60)
\end{aligned}$$

This combined with Eq. (3.50) forms the solution for arbitrary $f(x)$.

If we have the mono-modal $f(x) = y_0 \sin(\pi x/L)$, the full solution is particularly simple. In this case the initial condition has exactly the functional form of the eigenfunction, and there is thus only a one-term Fourier series. The solution is, by inspection,

$$y(x,t) = y_0 \cos \frac{\pi a t}{L} \sin \frac{\pi x}{L}. \tag{3.61}$$

The solution is a single fundamental mode, given by half of a sine wave pinned at $x = 0$ and $x = L$. At any given point x, the position y oscillates. For example at $x = L/2$, we have

$$y(L/2,t) = y_0 \cos \frac{\pi a t}{L}. \tag{3.62}$$

We call this a *standing wave*. Because there is only one Fourier mode, it is also known as *mono-modal*.

[5] Leopold Kronecker, 1823-1891, German mathematician.

Trigonometric expansion shows that Eq. (3.61) can be expanded as

$$y(x,t) = \frac{y_0}{2}\left(\sin\left(\frac{\pi}{L}(x-at)\right) + \sin\left(\frac{\pi}{L}(x+at)\right)\right). \tag{3.63}$$

This form illustrates that the standing wave can be considered as a sum of two propagating signals, one moving to the left with speed a, the other moving to the right at speed a. This is consistent with the d'Alembert solution, Eq. (2.63). A plot of the single mode standing wave is shown in Fig. 3.2a for parameter values shown in the caption.

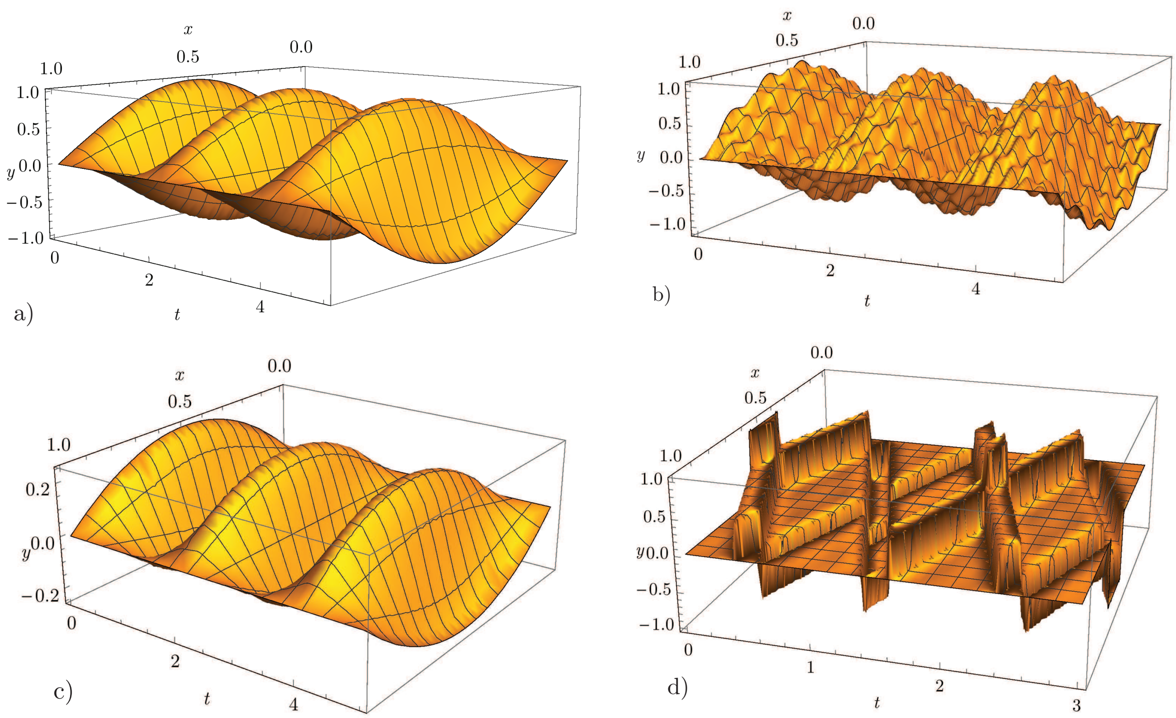


Figure 3.2: Response $y(x,t)$ for a solution to the wave equation with a) a single Fourier mode (mono-modal), b) two Fourier modes (bi-modal), c) multiple Fourier modes (poly-modal), $f(x) = y_0(x/L)(1-x/L)$, and d) a poly-modal top hat initial condition, $f(x) = y_0(H((x/L-2/5)-H(x/L-3/5))$, all with $y_0 = 1$, $a = 1$, $L = 1$.

The solution is almost as simple for the bi-modal second initial condition. We must have

$$y(x,0) = y_0 \sin\frac{\pi x}{L} + \frac{y_0}{10}\sin\frac{10\pi x}{L}. \tag{3.64}$$

By inspection again the solution is

$$y(x,t) = y_0 \cos\frac{\pi a t}{L}\sin\frac{\pi x}{L} + \frac{y_0}{10}\cos\frac{10\pi a t}{L}\sin\frac{10\pi x}{L}. \tag{3.65}$$

A plot of the bi-modal standing wave is shown in Fig. 3.2c for parameter values shown in the caption.

Next let us consider the poly-modal third initial distribution:

$$y(x,0) = f(x) = y_0\frac{x}{L}\left(1-\frac{x}{L}\right). \tag{3.66}$$

For this $f(x)$, evaluation of C_n via Eq. (3.60) gives the set of C_ns as

$$C_n = \frac{8y_0}{\pi^3}\left\{1, 0, \frac{1}{27}, 0, \frac{1}{125}, 0\dots\right\}, \qquad n = 1, \dots, \infty. \tag{3.67}$$

Obviously every odd term in the series is zero. This is a result of $f(x)$ having symmetry about $x = L/2$. It is possible to get a simple expression for C_n:

$$C_n = \begin{cases} \frac{8y_0}{n^3\pi^3}, & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases} \tag{3.68}$$

It is then easy to show that the solution can be expressed as the infinite series

$$y(x,t) = \frac{8y_0}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\cos\frac{(2m-1)\pi a t}{L}\sin\frac{(2m-1)\pi x}{L}. \tag{3.69}$$

A plot of the poly-modal standing wave is shown in Fig. 3.2c for parameter values shown in the caption. The plot looks similar to that for the mono-modal initial condition; this is because the quadratic polynomial initial condition is well modeled by a single Fourier mode. Recognize however that an infinite number smaller amplitude modes are present across infinite spectrum of frequencies. Note also that the frequencies of the modes are discretely separated. This is known as a *discrete spectrum* of frequencies. This feature is a consequence of the fact that for each sine wave to fit within the finite domain and still match the boundary conditions, only discretely separated frequencies are admitted. If we were to remove the boundary conditions at $x = 0$ and $x = L$, we would find instead a *continuous spectrum*.

We lastly consider the poly-modal initial condition which is a top hat function:

$$f(x) = y_0\left(H\left(\frac{x}{L} - \frac{2}{5}\right) - H\left(\frac{x}{L} - \frac{3}{5}\right)\right). \tag{3.70}$$

Evaluation of C_n via Eq. (3.60) gives the set of C_ns as

$$C_n = y_0\left\{\frac{\sqrt{5}-1}{\pi}, 0, -\frac{1+\sqrt{5}}{3\pi}, 0, \frac{4}{5\pi}, 0, -\frac{1+\sqrt{5}}{7\pi}, 0, \frac{\sqrt{5}-1}{9\pi}, 0, \dots\right\}. \tag{3.71}$$

A plot of the solution is shown in Fig. 3.2c for parameter values shown in the caption. Here fifty nonzero terms have been retained in the series. We note several important features of Fig. 3.2c:

- The initial top hat signal immediately breaks into two distinct waveforms. One propagates to the right and the other to the left. This is consistent with the d'Alembert nature of the solution to the wave equation.
- When either waveform strikes the boundary at either $x = 0$ or $x = L$, there is a reflection, with the sign of y changing.
- After a second reflection, both waves recombine to recover the initial waveform at a particular time.
- The pattern repeats, and there is no loss of information in the signal.
- Due to the finite number of terms in the series, there is a choppiness in the solution.

A so-called $x - t$ diagram can be useful in understanding wave phenomena. In such a diagram either contours or shading is used to show how the dependent variable varies in the $x - t$ plane. Fig. 3.3 gives such a diagram for solution to the wave equation with the top hat function as an initial condition. Here dark and light regions correspond to small and large y, respectively. Clearly signals are propagating

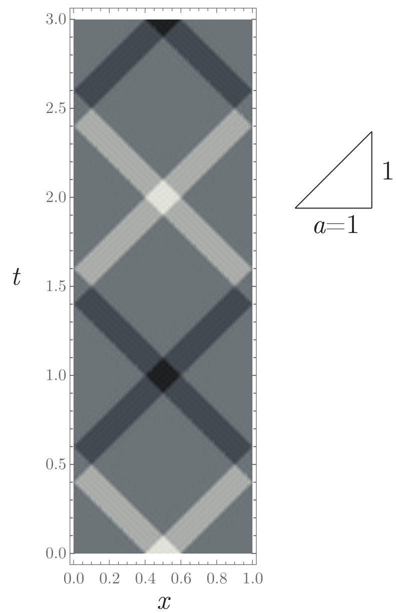


Figure 3.3: $x-t$ diagram for solution to the wave equation with a top hat initial condition, $f(x) = y_0(H((x/L - 2/5) - H(x/L - 3/5))$, all with $y_0 = 1$, $a = 1$, $L = 1$.
.

with slope $\pm\pi/4$ in this plane, which corresponds to a wave speed of $a = 1$. We also clearly see the reflection process at $x = 0$ and $x = 1$.

Lastly we examine the variation of the amplitudes $|C_n|$ with n for each of the four cases. A plot is shown in Fig. 3.4. Figures such as this are related to the so-called power spectral density of a signal; in other contexts it is known as the energy spectral density. One can easily see how energy is partitioned into various modes of oscillation. One can associate a frequency of oscillation ν with n via insisting that $2\pi\nu t = n\pi a t/L$; thus,

$$n = \frac{2\nu L}{a}, \qquad \nu = \frac{na}{2L}. \tag{3.72}$$

As an aside, let us recall some common notation for oscillatory systems. The *wavelength* is often named λ, not to be confused with an eigenvalue. It has units of length and can be defined as the distance for which a sine wave executes a full oscillation. We might say then that if our function $f(x)$ is

$$f(x) = \sin\frac{2\pi x}{\lambda}, \tag{3.73}$$

then the wavelength is indeed λ. When $x = \lambda/2$, we get the first half of the sine wave for which $f > 0$, and we get the complete sine wave when $x = \lambda$. Sometimes a sine wave is

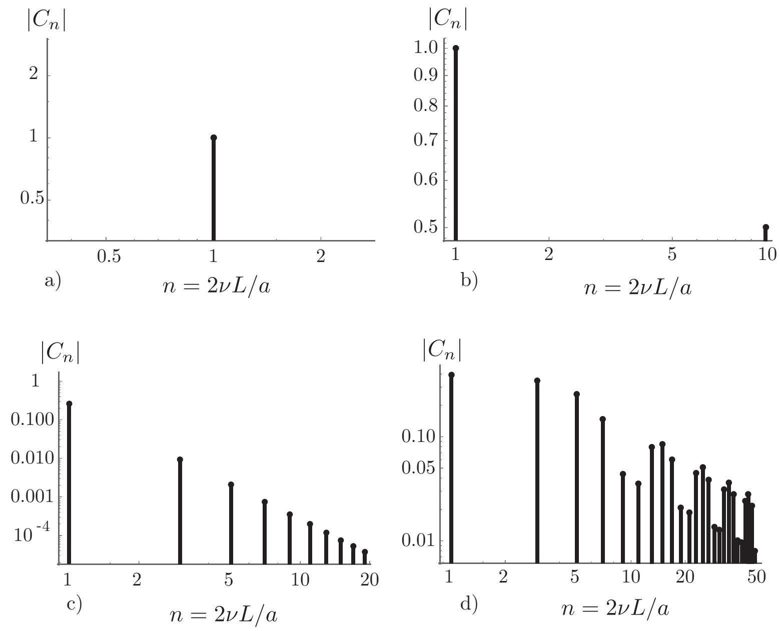


Figure 3.4: Variation of Fourier mode amplitude $|C_n|$ with n for solutions to the wave equation with a) mono-modal signal (a single Fourier mode), b) a bi-modal signal (two Fourier modes), c) a poly-modal signal (multiple Fourier modes), $f(x) = y_0(x/L)(1 - x/L)$, and d) a poly-modal top hat initial condition, $f(x) = y_0(H((x/L - 2/5) - H(x/L - 3/5))$, all with $y_0 = 1$, $a = 1$, $L = 1$.
.

expressed as

$$f(x) = \sin kx. \tag{3.74}$$

Here k is known as the *wave number* which has units of the reciprocal of length. We see that

$$k = \frac{2\pi}{\lambda}. \tag{3.75}$$

Similarly the period T, not to be confused with temperature, is defined as the time for which a sine wave undergoes a single complete cycle. We might imagine then

$$f(t) = \sin \frac{2\pi t}{T}. \tag{3.76}$$

When $t = T$, the sine wave has underwent a complete cycle. Sometimes the sine wave is expressed as

$$f(t) = \sin \omega t. \tag{3.77}$$

Here ω is the *angular frequency* and has units of the reciprocal of time. We see that

$$\omega = \frac{2\pi}{T}. \tag{3.78}$$

We also often express the sine wave as

$$f(t) = \sin 2\pi\nu t. \tag{3.79}$$

Here ν is the *frequency* with units of the reciprocal of time. We see that

$$\nu = \frac{1}{T}. \tag{3.80}$$

For waves with the form suggested by Eq. (3.51), we can consider

$$\begin{aligned} f(x,t) &= \frac{1}{2}\left(\sin\left(\frac{n\pi}{L}(x+at)\right) + \sin\left(\frac{n\pi}{L}(x-at)\right)\right), & (3.81)\\ &= \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}. & (3.82)\end{aligned}$$

Comparing to Eqs. (3.73,3.76), we see that

$$\begin{aligned} \frac{2\pi x}{\lambda} &= \frac{n\pi x}{L}, & (3.83)\\ \lambda &= \frac{2L}{n}, & (3.84)\end{aligned}$$

and

$$\begin{aligned} \frac{2\pi t}{T} &= \frac{n\pi at}{L}, & (3.85)\\ T &= \frac{2L}{na}. & (3.86)\end{aligned}$$

Then we also see the wave number is

$$k = \frac{2\pi}{\lambda} = \frac{2n\pi}{2L} = \frac{n\pi}{L}. \tag{3.87}$$

We also see the frequency is

$$\nu = \frac{1}{T} = \frac{na}{2L}. \tag{3.88}$$

And the angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi na}{2L} = \frac{n\pi a}{L}. \tag{3.89}$$

We could then cast our $f(x,t)$, Eq. (3.82), as

$$\begin{aligned} f(x,t) &= \cos\omega t \sin kx = \cos\frac{2\pi t}{T}\sin\frac{2\pi x}{\lambda} = \cos 2\pi\nu t \sin\frac{2\pi x}{\lambda}, && (3.90)\\ &= \frac{1}{2}\left(\sin\left(\frac{n\pi}{L}(x+at)\right) + \sin\left(\frac{n\pi}{L}(x-at)\right)\right), && (3.91)\\ &= \frac{1}{2}\left(\sin\left(k(x+at)\right) + \sin\left(k(x-at)\right)\right), && (3.92)\\ &= \frac{1}{2}\left(\sin\left(kx+\omega t\right) + \sin\left(kx-\omega t\right)\right). && (3.93) \end{aligned}$$

3.2.2 Heat equation

Example 3.3

Solve the heat equation, Eq. (1.81)

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}, \tag{3.94}$$

subject to boundary and initial conditions

$$T(0,t) = T(L,t) = 0, \qquad T(x,0) = f(x). \tag{3.95}$$

Generate solutions for four sets of initial conditions:

$$\begin{aligned} f(x) &= T_0 \sin\left(\frac{\pi x}{L}\right), && (3.96)\\ &= T_0\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{10}\sin\left(\frac{10\pi x}{L}\right)\right), && (3.97)\\ &= T_0\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right), && (3.98)\\ &= T_0\left(H\left(\frac{x}{L}-\frac{2}{5}\right) - H\left(\frac{x}{L}-\frac{3}{5}\right)\right). && (3.99) \end{aligned}$$

Once again, we assume solutions of the form

$$T(x,t) = A(x)B(t). \tag{3.100}$$

With this assumption, Eq. (3.94) becomes

$$\begin{aligned} A(x)\frac{dB}{dt} &= \alpha B(t)\frac{d^2A}{dx^2}, && (3.101)\\ -\frac{1}{\alpha B(t)}\frac{dB}{dt} &= -\frac{1}{A(x)}\frac{d^2A}{dx^2} = \lambda^2. && (3.102) \end{aligned}$$

This induces

$$\begin{aligned} \frac{d^2A}{dx^2} + \lambda^2 A &= 0, && (3.103)\\ \frac{dB}{dt} + \alpha\lambda^2 B &= 0. && (3.104) \end{aligned}$$

Consider the equation for A first. In order to satisfy the boundary conditions $T(0,t) = T(L,t) = 0$, we must have $A(0) = A(L) = 0$. Along with these boundary conditions, Eq. (3.103) can be recast as an eigenvalue problem:

$$-\frac{d^2}{dx^2}A = \lambda^2 A, \qquad A(0) = A(L) = 0. \tag{3.105}$$

With $\mathbf{L} = -d^2/dx^2$, a self-adjoint positive definite linear operator, this takes the form

$$\mathbf{L}A = \lambda^2 A. \tag{3.106}$$

Solving Eq. (3.103), we see that

$$A(x) = C_1 \sin \lambda x + C_2 \cos \lambda x. \tag{3.107}$$

For $A(0) = 0$, we get

$$A(0) = 0 = C_1 \cancel{\sin 0} + C_2 \cos 0 = C_2. \tag{3.108}$$

Thus,

$$A(x) = C_1 \sin \lambda x. \tag{3.109}$$

Now at $x = L$, we have

$$A(L) = 0 = C_1 \sin \lambda L. \tag{3.110}$$

To guarantee this condition is satisfied, we must require that

$$\lambda L = n\pi, \qquad n = 1, 2, \ldots, \tag{3.111}$$
$$\lambda = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots. \tag{3.112}$$

With this, we have

$$A(x) = C_1 \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots \tag{3.113}$$

We note that,

- the eigenvalues λ^2 are real and positive,
- the eigenfunctions $C_n \sin n\pi x$ have an arbitrary amplitude.

It will once again soon be useful to employ the orthogonality property, $\int_0^L (\sin m\pi x/L)(\sin n\pi x/L)\,dx = 0$ if $m \neq n$ when m and n are integers, and that the integral is nonzero if $n = m$.

We now cast Eq. (3.104) as

$$\frac{dB}{dt} + \frac{n^2\pi^2\alpha}{L^2}B = 0. \tag{3.114}$$

This has solution

$$B(t) = C_3 \exp\left(\frac{-n^2\pi^2\alpha t}{L^2}\right). \tag{3.115}$$

We note that for $\alpha > 0$ $L > 0$, that

$$\lim_{t\to\infty} B(t) = 0. \tag{3.116}$$

That is to say, any amplitude of a given mode of $T(x,t)$ decays to zero. By inspection the time scale of decay of one of these modes is

$$\tau = \frac{L^2}{n^2\pi^2\alpha}. \tag{3.117}$$

Thus fast decay is induced by

- small domain length L,
- high frequency of a given Fourier mode, where n is proportional to the frequency,
- large diffusivity, α.

Taking $\hat{C}_3 = C_1C_3$, our solution combines to form

$$T(x,t) = \hat{C}_3 \exp\left(\frac{-n^2\pi^2\alpha t}{L^2}\right)\sin\frac{n\pi x}{L}. \tag{3.118}$$

By the principle of superposition, we can admit arbitrary linear combinations thus giving the general solution

$$T(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(\frac{-n^2\pi^2\alpha t}{L^2}\right)\sin\frac{n\pi x}{L}. \tag{3.119}$$

Here we note

- The amplitude of each Fourier mode decays with time; this is characteristic of *diffusive* phenomena.
- There is no wave propagation phenomena.

Once again the initial conditions fix the values of C_n after application of the initial condition $T(x,0) = f(x)$:

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}. \tag{3.120}$$

Once again, this amounts to finding the Fourier sine series expansion of $f(x)$. The first expansions are the same as those from the previous example problem, so we will not repeat the analysis. We report plots of $T(x,t)$ for the four initial conditions given by $f(x)$. Plots of the solutions are shown in Fig. 3.5 for parameter values shown in the caption. In Fig. 3.5b, we see that the high frequency mode present in the initial condition decays much more rapidly than the low frequency mode.

3.2.3 Laplace's equation

Example 3.4

Solve Laplace's equation, Eq. (1.104)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{3.121}$$

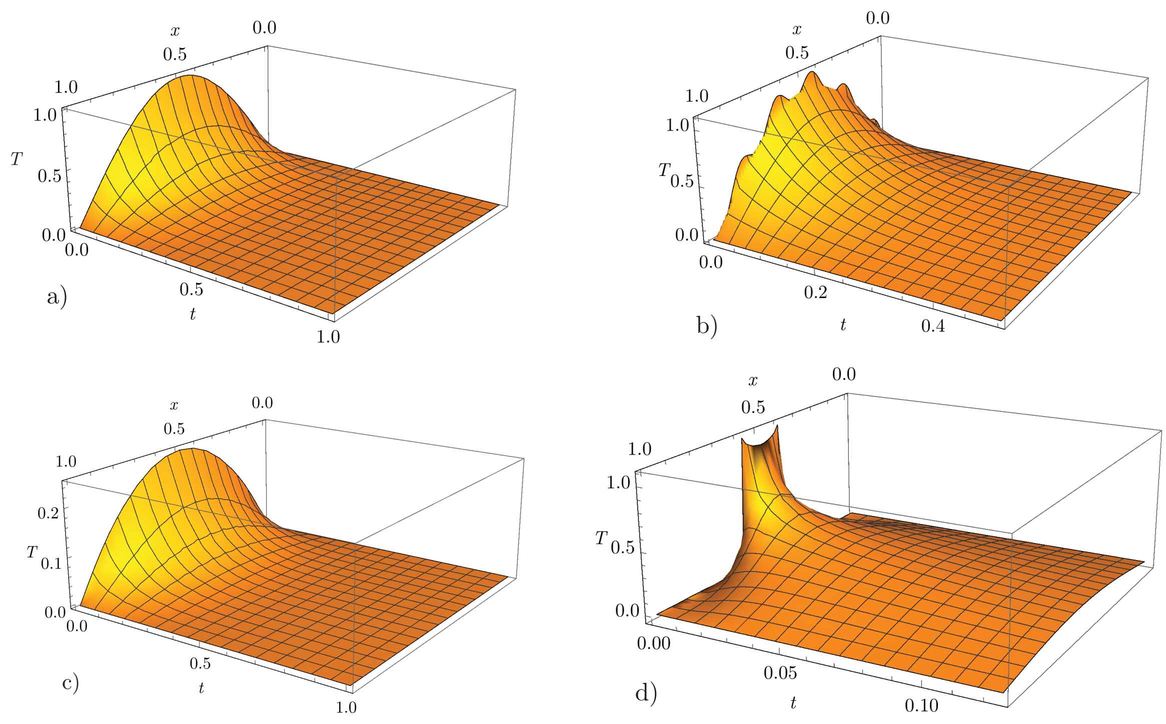


Figure 3.5: Response $T(x,t)$ for a solution to the heat equation with a) a single Fourier mode, b) two Fourier modes, c) multiple Fourier modes, $f(x) = T_0(x/L)(1 - x/L)$, d) a top hat initial condition, $f(x) = T_0(H((x/L - 2/5) - H(x/L - 3/5))$, all with $T_0 = 1$, $L = 1$, and $\alpha = 1$.

subject to boundary conditions

$$T(0,y) = T(L,y) = T(x,0) = 0, \qquad T(x,L) = f(x). \tag{3.122}$$

Generate solutions for four sets of boundary conditions:

$$\begin{aligned} f(x) &= T_0 \sin\left(\frac{\pi x}{L}\right), & (3.123)\\ &= T_0 \left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{10}\sin\left(\frac{10\pi x}{L}\right)\right), & (3.124)\\ &= T_0 \left(\frac{x}{L}\right)\left(1 - \frac{x}{L}\right), & (3.125)\\ &= T_0 \left(H\left(\frac{x}{L} - \frac{2}{5}\right) - H\left(\frac{x}{L} - \frac{3}{5}\right)\right). & (3.126) \end{aligned}$$

We assume solutions of the form

$$T(x,y) = A(x)B(y). \tag{3.127}$$

With this assumption, Eq. (3.121) becomes

$$B(y)\frac{d^2A}{dx^2} + A(x)\frac{d^2B}{dy^2} = 0, \tag{3.128}$$

$$\frac{1}{B(y)}\frac{d^2B}{dy^2} = -\frac{1}{A(x)}\frac{d^2A}{dx^2} = \lambda^2. \tag{3.129}$$

This gives

$$\frac{d^2A}{dx^2} + \lambda^2 A = 0, \tag{3.130}$$
$$\frac{d^2B}{dy^2} - \lambda^2 B = 0. \tag{3.131}$$

Solving the first equation, we find

$$A(x) = C_1 \sin \lambda x + C_2 \cos \lambda x. \tag{3.132}$$

To satisfy the boundary conditions at $x = 0$ and $x = L$, we will need $A(x) = A(L) = 0$. Thus, we have

$$A(0) = 0 = C_1 \cancel{\sin 0} + C_2 \cos 0, \tag{3.133}$$
$$= C_2. \tag{3.134}$$

Thus

$$A(x) = C_1 \sin \lambda x. \tag{3.135}$$

At $x = L$, we then have

$$A(L) = 0 = C_1 \sin \lambda L. \tag{3.136}$$

We can thus take

$$\lambda L = n\pi, \qquad n = 1, 2, \ldots, \tag{3.137}$$

and

$$A(x) = C_1 \sin \frac{n\pi x}{L}. \tag{3.138}$$

Then Eq. (3.131) becomes

$$\frac{d^2B}{dy^2} - \frac{n^2\pi^2}{L^2} B = 0. \tag{3.139}$$

This has solution

$$B(y) = C_3 \sinh \frac{n\pi y}{L} + C_4 \cosh \frac{n\pi y}{L}. \tag{3.140}$$

At $y = 0$ we must have $B(y) = 0$, so that

$$B(y) = 0 = C_3 \cancel{\sinh 0} + C_4 \cosh 0. \tag{3.141}$$

We learn then that $C_4 = 0$, so that

$$B(y) = C_3 \sinh \frac{n\pi y}{L}, \tag{3.142}$$

and with $\hat{C}_3 = C_3 C_1$,

$$T(x, y) = \hat{C}_3 \sinh \frac{n\pi y}{L} \sin \frac{n\pi x}{L}. \tag{3.143}$$

The principle of superposition holds here, so we can say

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi y}{L} \sin \frac{n\pi x}{L}. \tag{3.144}$$

Our boundary condition then gives

$$f(x) = \sum_{n=1}^{\infty} \underbrace{C_n \sinh n\pi}_{\tilde{C}_n} \sin \frac{n\pi x}{L}. \tag{3.145}$$

Now if $\tilde{C}_n$ are the Fourier sine series coefficients of $f(x)$, we have

$$C_n = \frac{\tilde{C}_n}{\sinh n\pi}. \tag{3.146}$$

Plots of the solutions are shown in Fig. 3.6 for parameter values shown in the caption. The heat

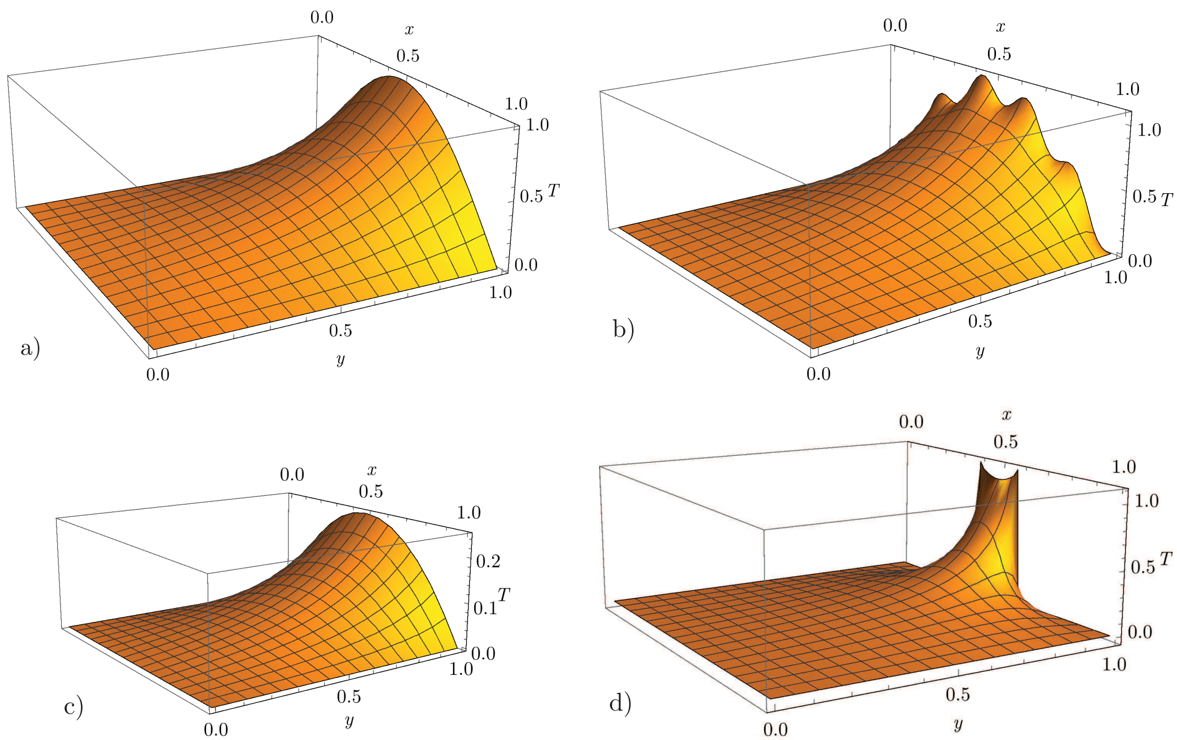


Figure 3.6: Response $T(x,y)$ for a solution to Laplace's equation with a) a single Fourier mode, b) two Fourier modes, c) multiple Fourier modes, $f(x) = T_0(x/L)(1 - x/L)$, and d) a top hat boundary condition, $f(x) = T_0(H((x/L - 2/5) - H(x/L - 3/5))$, all with $T_0 = 1$, $L = 1$.

flux vectors may be determined using $\mathbf{q} = -\nabla T$. We show a plot of the isotherms and adiabats for the top hat boundary condition is shown in Fig. 3.7.

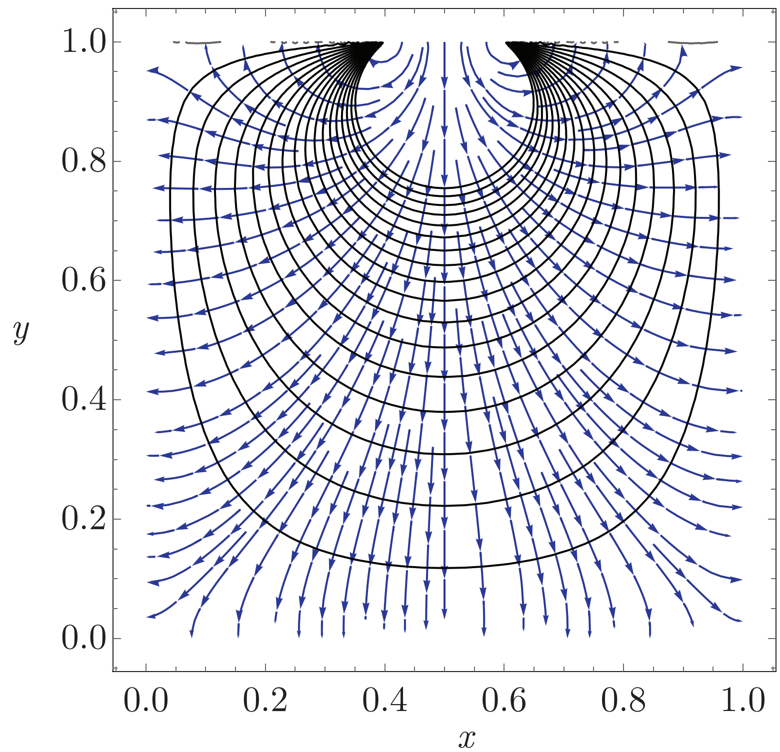


Figure 3.7: Adiabats and isotherms for a solution to Laplace's equation with a top hat boundary condition, $T(x,1) = f(x) = (H((x-2/5) - H(x-3/5))$.

3.2.4 Heat equation: inhomogeneous boundary conditions

There are many types of inhomogeneities that can be introduced in either the initial conditions or the boundary conditions. Many times they can be simplified. In the next example, we introduce a simple inhomogeneity into the initial condition and another simple inhomogeneity into a pair of Dirichlet boundary conditions.

Example 3.5

Consider the heat equation with initial and boundary conditions of

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad T(x,0) = T_0,\ T(0,t) = T_1,\ T(L,t) = T_1. \tag{3.147}$$

Find $T(x,t)$.

Physically, one might imagine this as a rod of length L, initially at uniform temperature T_0, whose ends are suddenly heated to T_1 and held there. Let us scale the problem. We can take

$$T_* = \frac{T - T_0}{T_1 - T_0}, \qquad x_* = \frac{x}{L}, \quad t_* = \frac{t}{t_c}. \tag{3.148}$$

Our choice of T_* maps $T \in [T_0, T_1]$ to $T_* \in [0, 1]$. Our choice of x_* maps $x \in [0, L]$ to $x_* \in [0, 1]$. We need to choose t_c, and will do so following a simple analysis. With our choices, our system becomes

$$\frac{1}{t_c}\frac{\partial}{\partial t_*}\left((T_1 - T_0)T_* + T_0\right) = \frac{\alpha}{L^2}\frac{\partial^2}{\partial x_*^2}\left((T_1 - T_0)T_* + T_0\right), \quad T_*(x_*, 0) = 0, \; T_*(0, t_*) = 1, \; T_*(1, t_*) = 1, \tag{3.149}$$

$$\frac{\partial T_*}{\partial t_*} = \frac{\alpha t_c}{L^2}\frac{\partial^2 T_*}{\partial x_*^2}, \qquad T_*(x_*, 0) = 0, \; T_*(0, t_*) = 1, \; T_*(1, t_*) = 1. \tag{3.150}$$

Let us select t_c to remove the effect of the parameter, giving

$$t_c = \frac{L^2}{\alpha}, \tag{3.151}$$

Thus, our system is scaled to be parameter-free:

$$\frac{\partial T_*}{\partial t_*} = \frac{\partial^2 T_*}{\partial x_*^2}, \qquad T_*(x_*, 0) = 0, \; T_*(0, t_*) = 1, \; T_*(1, t_*) = 1. \tag{3.152}$$

We anticipate a Sturm[6]-Liouville[7] problem of a second order nature in x_*. But we will likely need homogeneous boundary conditions in order to pose an eigenvalue problem. Let us redefine T_* to achieve this. In the limit of a steady state, our heat equation will have a solution $T_{*s}(x_*)$ which satisfies the time-independent version of Eq. (3.152):

$$0 = \frac{d^2 T_{*s}}{dx_*^2}, \qquad T_{*s}(0) = T_{*s}(1) = 1. \tag{3.153}$$

This has solution $T_{*s}(x_*) = C_1 + C_2 x_*$. To satisfy the boundary conditions, we must have $C_1 = 1$ and $C_2 = 0$, so the steady state solution is

$$T_{*s}(x_*) = 1. \tag{3.154}$$

Let us now define a deviation from the steady state solution $\tilde{T}_*$:

$$\tilde{T}(x_*, t_*) = T_*(x_*, t_*) - T_{*s}(x_*) = T_*(x_*, t_*) - 1. \tag{3.155}$$

We then recast our system in terms of $\tilde{T}$:

$$\frac{\partial}{\partial t_*}(\tilde{T} + 1) = \frac{\partial^2}{\partial x_*^2}(\tilde{T} + 1), \qquad \tilde{T}(x_*, 0) + 1 = 0, \; \tilde{T}(0, t_*) + 1 = 1, \; \tilde{T}(1, t_*) + 1 = 1, \tag{3.156}$$

$$\frac{\partial \tilde{T}}{\partial t_*} = \frac{\partial^2 \tilde{T}}{\partial x_*^2}, \qquad \tilde{T}(x_*, 0) = -1, \; \tilde{T}(0, t_*) = 0, \; \tilde{T}(1, t_*) = 0. \tag{3.157}$$

Our change of variables has moved the inhomogeneity from the boundary condition to the initial condition.

We can now separate variables and proceed much as before. First take

$$\tilde{T}(x_*, t_*) = A(x_*)B(t_*). \tag{3.158}$$

[6]Jacques Charles François Sturm (1803–1855, French mathematician.

[7]Joseph Liouville, 1809-1882, French mathematician and engineer.

This gives

$$A\frac{dB}{dt_*} = B\frac{d^2A}{dx_*^2}, \tag{3.159}$$

$$-\frac{1}{B}\frac{dB}{dt_*} = -\frac{1}{A}\frac{d^2A}{dx_*^2} = \lambda^2. \tag{3.160}$$

This gives two ordinary differential equations:

$$\frac{d^2A}{dx_*^2} + \lambda^2 A = 0, \tag{3.161}$$

$$\frac{dB}{dt_*} + \lambda^2 B = 0. \tag{3.162}$$

Solving the first gives

$$A(x_*) = C_1 \sin \lambda x_* + C_2 \cos \lambda x_*. \tag{3.163}$$

Here is where the homogenous boundary conditions are important. We need $A(0) = 0$, so

$$A(0) = 0 = C_1(0) + C_2. \tag{3.164}$$

Thus $C_2 = 0$ and

$$A(x_*) = C_1 \sin \lambda x_*. \tag{3.165}$$

We also need $A(1) = 0$, so

$$A(1) = 0 = C_1 \sin \lambda. \tag{3.166}$$

For this, we insist that

$$\lambda = n\pi, \qquad n = 1, 2, \ldots \tag{3.167}$$

Thus

$$A(x_*) = C_1 \sin n\pi x_*. \tag{3.168}$$

Then for B, we get

$$\frac{dB}{dt_*} + n^2\pi^2 B = 0, \tag{3.169}$$

$$B(t_*) = C_3 \exp\left(-n^2\pi^2 t_*\right). \tag{3.170}$$

Taking then our solution to be a linear combination of the various modes, we can assert

$$\tilde{T}(x_*, t_*) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t_*} \sin n\pi x_*. \tag{3.171}$$

Enforcing the initial condition, we get

$$\tilde{T}(x_*, 0) = -1 = \sum_{n=1}^{\infty} C_n \sin n\pi x_*. \tag{3.172}$$

We need the Fourier sine coefficients for -1. We operate as usual to get

$$
\begin{aligned}
-\sin m\pi x_* &= \sum_{n=1}^{\infty} C_n \sin m\pi x_* \sin n\pi x_*, && (3.173)\\
-\int_0^1 \sin m\pi x_*\, dx_* &= \sum_{n=1}^{\infty} C_n \underbrace{\int_0^1 \sin m\pi x_* \sin n\pi x_*\, dx_*}_{=\delta_{mn}/2}, && (3.174)\\
\left\{ \begin{array}{cc} -\frac{2}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{array} \right. &= \sum_{n=1}^{\infty} C_n \frac{\delta_{mn}}{2}, && (3.175)\\
&= \frac{C_m}{2}, && (3.176)\\
C_n &= \left\{ \begin{array}{cc} -\frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{array} \right. . && (3.177)
\end{aligned}
$$

Thus

$$\tilde{T}(x_*, t_*) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-(2n-1)^2\pi^2 t_*} \sin(2n-1)\pi x_*. \tag{3.178}$$

In terms of T_*, we can then say

$$T_*(x_*, t_*) = 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-(2n-1)^2\pi^2 t_*} \sin(2n-1)\pi x_*. \tag{3.179}$$

We note also that

- High frequency modes decay rapidly.
- Low frequency modes decay slowly.
- All modes decay to zero leaving the long time solution $T_* = 1$.

The slowest decaying mode has $n = 1$, for which the approximate solution is at $t_* \to \infty$,

$$T_* \approx 1 - \frac{4}{\pi} e^{-\pi^2 t_*} \sin \pi x_*. \tag{3.180}$$

The time constant of decay of the slowest mode is τ and is by inspection

$$\tau = \frac{1}{\pi^2}. \tag{3.181}$$

The dimensional decay time τ_d is thus

$$\tau_d = \tau t_c = \frac{L^2}{\pi^2 \alpha}. \tag{3.182}$$

Thus rapid decay is induced by short length scales L and high diffusivity α. A plot of the solutions is shown in Fig. 3.8. Here we have incorporated the original dimensional variables into the scaled axes of Fig. 3.8.

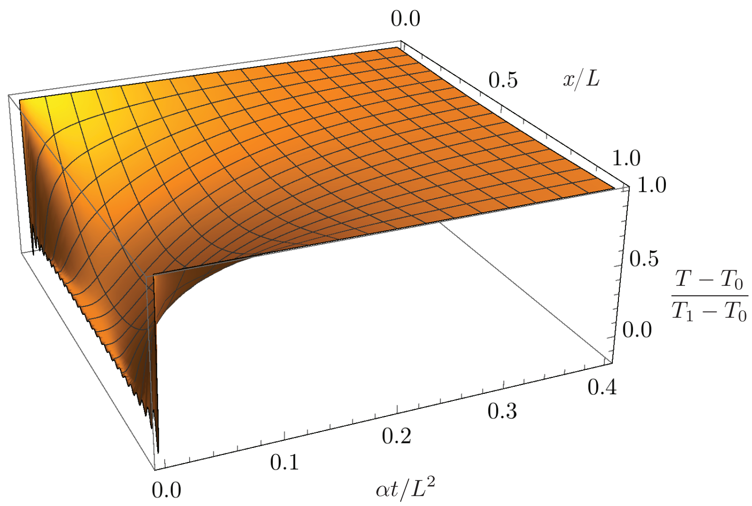


Figure 3.8: Response $T(x,t)$ for a solution to the heat equation with suddenly imposed inhomogeneous boundary condition.

3.2.5 Heat equation: homogeneous Robin boundary conditions

Example 3.6

Consider the heat equation with a general initial condition and general homogeneous boundary conditions (i.e. Robin boundary conditions):

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{\partial^2 T}{\partial x^2}, && (3.183)\\ T(x,0) &= f(x), && (3.184)\\ \alpha_1 T(0,t) + \alpha_2 \frac{\partial T}{\partial x}(0,t) &= 0, && (3.185)\\ \beta_1 T(1,t) + \beta_2 \frac{\partial T}{\partial x}(1,t) &= 0. && (3.186) \end{aligned}$$

Find a general expression for $T(x,t)$.

Let us take

$$T(x,t) = A(x)B(t). \tag{3.187}$$

Thus

$$\begin{aligned} A(x)\frac{dB}{dt} &= B(t)\frac{d^2 A}{dx^2}, && (3.188)\\ -\frac{1}{B(t)}\frac{dB}{dt} &= -\frac{1}{A(x)}\frac{d^2 A}{dx^2} = \lambda^2. && (3.189) \end{aligned}$$

This yields

$$\frac{d^2A}{dx^2} + \lambda^2 A = 0, \tag{3.190}$$

$$\frac{dB}{dt} + \lambda^2 B = 0. \tag{3.191}$$

The first has general solution

$$A(x) = C_1 \cos \lambda x + C_2 \sin \lambda x. \tag{3.192}$$

We also see

$$\frac{dA}{dx} = -C_1 \lambda \sin \lambda x + C_2 \lambda \cos \lambda x. \tag{3.193}$$

Enforcing the boundary conditions at $x = 0$ and $x = 1$ gives

$$\alpha_1 C_1 + \alpha_2 \lambda C_2 = 0, \tag{3.194}$$

$$C_1(\beta_1 \cos \lambda - \beta_2 \lambda \sin \lambda) + C_2(\beta_1 \sin \lambda + \beta_2 \lambda \cos \lambda) = 0. \tag{3.195}$$

In matrix form, this becomes

$$\begin{pmatrix} \alpha_1 & \alpha_2 \lambda \\ \beta_1 \cos \lambda - \beta_2 \lambda \sin \lambda & \beta_1 \sin \lambda + \beta_2 \lambda \cos \lambda \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.196}$$

For a nontrivial solution, the determinant of the coefficient matrix must be zero, yielding

$$\alpha_1(\beta_1 \sin \lambda + \beta_2 \lambda \cos \lambda) - \alpha_2 \lambda (\beta_1 \cos \lambda - \beta_2 \lambda \sin \lambda) = 0. \tag{3.197}$$

Assuming $\alpha_1 \neq 0$ and $\beta_1 \neq 0$, we can scale to get

$$\left(\sin \lambda + \frac{\beta_2}{\beta_1} \lambda \cos \lambda \right) - \frac{\alpha_2}{\alpha_1} \lambda \left(\cos \lambda - \frac{\beta_2}{\beta_1} \lambda \sin \lambda \right) = 0. \tag{3.198}$$

In general, for a given α_2/α_1 and β_2/β_1, this is a transcendental equation which must be solved numerically for λ. In special cases, there is an exact solution. For the Dirichlet conditions found when $\alpha_2 = \beta_2 = 0$, we get $\sin \lambda = 0$, yielding $\lambda = n\pi$, and $A(x) = C_2 \sin(n\pi x)$. For the Neumann conditions when $\alpha_1 = \beta_1 = 0$, we get $\lambda^2 \sin \lambda = 0$. This gives $\lambda = n\pi$ and $A(x) = C_1 \cos n\pi x$. For the Robin conditions, we must find a numerical solution, and we still expect an infinite number of eigenvalues λ. Consider the case when $\alpha_1 = \beta_1 = \beta_2 = 1$ and $\alpha_2 = 0$. Thus our Robin conditions are

$$T(0, t) = 0, \tag{3.199}$$

$$T(1, t) + \frac{\partial T}{\partial x}(1, t) = 0. \tag{3.200}$$

Our expression for the eigenvalues, Eq. (3.198) reduces to

$$\sin \lambda + \lambda \cos \lambda = 0, \tag{3.201}$$

$$\tan \lambda = -\lambda. \tag{3.202}$$

To understand how the roots are distributed, we plot λ and $\tan \lambda$ in Fig. 3.9. Numerical solution reveals the eigenvalues are given by

$$\lambda = \{0, \pm 2.02876, \pm 4.91318, \pm 7.97867, \ldots\} \tag{3.203}$$

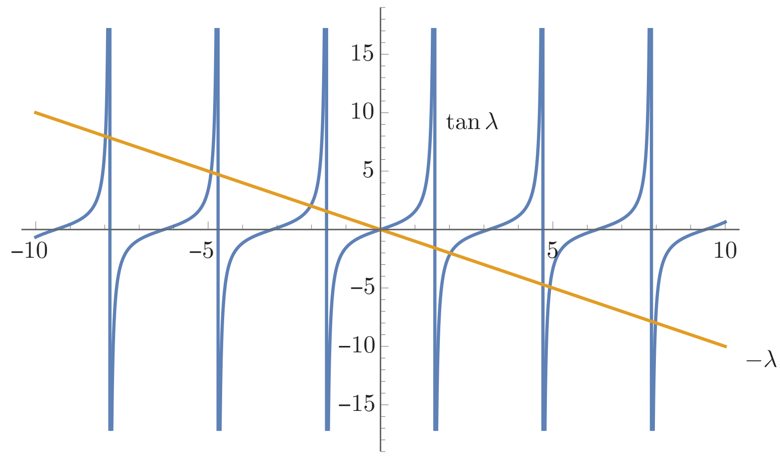


Figure 3.9: Curves whose intersection gives roots of $\tan\lambda = -\lambda$, eigenvalues of a problem with Robin boundary conditions.

For our purposes, it suffices to only consider the nonzero positive eigenvalues so we take

$$\lambda = \{2.02876, 4.91318, 7.97867, \ldots\} \tag{3.204}$$

For large $|\lambda|$, the eigenvalues are given where $\tan\lambda$ is singular, which is where $\cos\lambda = 0$:

$$\lambda \approx \left(n + \frac{1}{2}\right)\pi. \tag{3.205}$$

For this problem, the boundary condition $T(0,t) = 0$ gives us $C_1 = 0$, so $A(x) = C_2 \sin\lambda x$. We solve for B to get

$$B(t) = C \exp\left(-\lambda^2 t\right). \tag{3.206}$$

Then forming linear combinations of the solutions, we have as a general solution

$$T(x,t) = \sum_{n=0}^{\infty} C_n e^{-\lambda_n^2 t} \sin\lambda_n x. \tag{3.207}$$

At $t = 0$, we have

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\lambda_n x. \tag{3.208}$$

Our eigenfunctions, $\sin\lambda_n x$, are guaranteed orthogonal, but are not orthonormal. Let us use orthogonality to find C_n for a given $f(x)$:

$$f(x)\sin\lambda_m x = \sum_{n=1}^{\infty} C_n \sin\lambda_m x \sin\lambda_n x, \tag{3.209}$$

$$\int_0^1 f(x)\sin\lambda_m x \, dx = \sum_{n=1}^{\infty} C_n \int_0^1 \sin\lambda_m x \sin\lambda_n x \, dx, \tag{3.210}$$

$$= C_m \int_0^1 \sin\lambda_m x \sin\lambda_m x \, dx, \tag{3.211}$$

$$C_n = \frac{\int_0^1 f(x)\sin\lambda_n x \, dx}{\int_0^1 \sin\lambda_n x \sin\lambda_n x \, dx}. \tag{3.212}$$

For other problems, we may need a more general Fourier expansion of the form

$$f(x) = \sum_{n=0}^{\infty} C_n \cos \lambda_n x + B_n \sin \lambda_n x. \tag{3.213}$$

Calculation of the Fourier coefficients C_n and B_n is aided greatly by the orthogonality of the eigenfunctions; details can be found in Powers and Sen[8].

3.2.6 Poisson equation

Example 3.7

Consider a Poisson equation, first introduced in Eq. (2.152). Recall that it is an inhomogeneous version of Laplace's equation:

$$-\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = f(x, y). \tag{3.214}$$

Solve if $f(x, y) = 100H(x - 1/2)H(y - 1/2)$, $T(x, 0) = T(x, 1) = T(0, y) = T(1, y) = 0$.

Here the boundary conditions are homogeneous, but there is an inhomogeneous source term. Physically, this might represent a heat transfer problem in a two-dimensional unit square plate whose boundaries are held at fixed temperature and whose interior is heated in a spatially inhomogeneous fashion. In this case, with H as the Heaviside step function, the unit square is heated in its upper quarter square; its other three quarters are unheated.

Let us make an intuitive guess for the solution based on our experience with Laplace's equation. Guess a separation of variables solution of the type

$$T(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} X_n(x) Y_m(y). \tag{3.215}$$

We have not yet specified the functional form of $X_n(x)$ or $Y_m(y)$, but will soon do so by making useful choices. Now substitute our guess, Eq. (3.215) into our Poisson equation, Eq. (3.214) to get

$$-\left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \frac{d^2 X_n}{dx^2} Y_m(y) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} X_n(x) \frac{d^2 Y_m}{dy^2}\right) = f(x, y), \tag{3.216}$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \frac{d^2 X_n}{dx^2} Y_m(y) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} X_n(x) \frac{d^2 Y_m}{dy^2} = -f(x, y). \tag{3.217}$$

For convenience, let us insist now that $X_n(x)$ and $Y_m(y)$ be eigenfunctions of the positive definite linear operators $-d^2/dx^2$ and $-d^2/dy^2$, respectively and that each satisfy homogeneous boundary conditions consistent with those of our Poisson equation:

$$-\frac{d^2 X_n}{dx^2} = \lambda_n^2 X_n(x), \qquad X_n(0) = X_n(1) = 0, \tag{3.218}$$

$$-\frac{d^2 Y_m}{dy^2} = \mu_m^2 Y_m(y), \qquad Y_m(0) = Y_m(1) = 0. \tag{3.219}$$

[8] J. M. Powers and M. Sen, *Mathematical Methods in Engineering*, Cambridge University Press, New York, 2015. See Section 6.5.

We have studied this problem before and know it is satisfied iff $\lambda_n = n\pi$, $\mu_m = m\pi$, where n and m must be integers. Thus, we have

$$-\frac{d^2X_n}{dx^2} = n^2\pi^2 X_n(x), \qquad X_n(0) = X_n(1) = 0, \tag{3.220}$$

$$-\frac{d^2Y_m}{dy^2} = m^2\pi^2 Y_m(y), \qquad Y_m(0) = Y_m(1) = 0. \tag{3.221}$$

The eigenfunctions, which are guaranteed to be orthogonal, are

$$X_n(x) = C_1 \sin n\pi x, \qquad Y_m(y) = C_2 \sin m\pi y, \qquad n = 1, 2, 3, \ldots, \quad m = 1, 2, 3, \ldots \tag{3.222}$$

We note that $X_n(x = 0) = X_n(x = 1) = 0$ for all integer values of n and that $Y_m(y = 0) = Y_m(y = 1) = 0$ for all integer values of m; thus, the homogeneous boundary conditions will be identically satisfied. If we select $C_1 = C_2 = \sqrt{2}$, the eigenfunctions will be

$$X_n(x) = \sqrt{2}\sin n\pi x, \qquad Y_m(y) = \sqrt{2}\sin m\pi y, \qquad n = 1, 2, 3, \ldots, \quad m = 1, 2, 3, \ldots, \tag{3.223}$$

and they will be orthonormal in their respective domains:

$$\int_0^1 X_n(x)X_k(x)\ dx = 2\int_0^1 \sin n\pi x \sin k\pi x\ dx = \delta_{nk}, \tag{3.224}$$

$$\int_0^1 Y_m(y)Y_j(y)\ dy = 2\int_0^1 \sin m\pi y \sin j\pi y\ dy = \delta_{mj}. \tag{3.225}$$

We next substitute our Eqs. (3.220, 3.221) into our Poisson equation, Eq. (3.217), to get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\left(n^2\pi^2 X_n(x)Y_m(y) + m^2\pi^2 X_n(x)Y_m(y)\right) = f(x, y). \tag{3.226}$$

Now multiply both sides by $X_k(x)$, integrate over the domain $x \in [0, 1]$, and take advantage of orthonormality to get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\int_0^1 \left(n^2\pi^2 X_n(x)X_k(x)Y_m(y) + m^2\pi^2 X_k(x)X_n(x)Y_m(y)\right)\ dx = \int_0^1 f(x, y)X_k(x)\ dx, \tag{3.227}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\left(n^2\pi^2\delta_{nk}Y_m(y) + m^2\pi^2\delta_{nk}Y_m(y)\right) = \int_0^1 f(x, y)X_k(x)\ dx, \tag{3.228}$$

$$\sum_{m=1}^{\infty} C_{km}\left(k^2\pi^2 Y_m(y) + m^2\pi^2 Y_m(y)\right) = \int_0^1 f(x, y)X_k(x)\ dx. \tag{3.229}$$

Now multiply both sides by $Y_j(y)$ and integrate over the domain $y \in [0, 1]$ and take advantage of orthonormality to get

$$\sum_{m=1}^{\infty} C_{km}\int_0^1 \left(k^2\pi^2 Y_j(y)Y_m(y) + m^2\pi^2 Y_j(y)Y_m(y)\right)\ dy = \int_0^1\int_0^1 f(x, y)X_k(x)Y_j(y)\ dx\ dy, \tag{3.230}$$

$$\sum_{m=1}^{\infty} C_{km}\left(k^2\pi^2\delta_{jm} + m^2\pi^2\delta_{jm}\right) = \int_0^1\int_0^1 f(x, y)X_k(x)Y_j(y)\ dx\ dy, \tag{3.231}$$

$$C_{kj}\left(k^2\pi^2 + j^2\pi^2\right) = \int_0^1\int_0^1 f(x, y)X_k(x)Y_j(y)\ dx\ dy. \tag{3.232}$$

Now trade k for n and j for m to get

$$C_{nm} = \frac{1}{n^2\pi^2 + m^2\pi^2} \int_0^1 \int_0^1 f(x,y) X_n(x) Y_m(y)\ dx\ dy. \tag{3.233}$$

In terms of the orthonormal eigenfunctions, we get then

$$C_{nm} = \frac{2}{n^2\pi^2 + m^2\pi^2} \int_0^1 \int_0^1 f(x,y) \sin n\pi x \sin m\pi y\ dx\ dy. \tag{3.234}$$

And for general $f(x,y)$, we get the solution on the unit square to be from Eq. (3.215)

$$T(x,y) = 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin n\pi x \sin m\pi y. \tag{3.235}$$

Fully substituting for C_{mn}, we could say

$$T(x,y) = 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{2}{n^2\pi^2 + m^2\pi^2} \int_0^1 \int_0^1 f(x',y') \sin n\pi x' \sin m\pi y'\ dx'\ dy' \right) \sin n\pi x \sin m\pi y. \tag{3.236}$$

For our function $f(x,y) = 100H(x-1/2)H(y-1/2)$, we get

$$C_{nm} = \frac{200}{n^2\pi^2 + m^2\pi^2} \int_{1/2}^1 \int_{1/2}^1 \sin n\pi x \sin m\pi y\ dx\ dy. \tag{3.237}$$

This is easily evaluated. Doing so and substituting into Eq. (3.215), we find the first few terms are

$$T(x,y) = \frac{200 \sin \pi x \sin \pi y}{\pi^4} - \frac{80 \sin 2\pi x \sin \pi y}{\pi^4} - \frac{80 \sin \pi x \sin 2\pi y}{\pi^4} + \frac{50 \sin 2\pi x \sin 2\pi y}{\pi^4} + \dots \tag{3.238}$$

A high accuracy plot of isotherms and adiabats, obtained with $\mathbf{q} = -\nabla T$, using $20 \times 20 = 400$ terms of the solution is shown in Fig. 3.10.

3.2.7 Method of manufactured solutions

Example 3.8

Find the Poisson equation along with boundary conditions on the unit square with vertices $(x,y) = (0,0), (1,0), (1,1), (0,1)$ whose solution is $T(x,y) = \sin \pi x \sin \pi y$.

This is an example of the so-called *method of manufactured solutions*.[9] [10] The method is useful in generating easy-to-code exact solutions that can be used to verify discrete computational solution methods. The essence of the method is to find the source term that renders the solution to be an exact

[9] Roache, P. J., 1998, *Verification and Validation in Computational Science and Engineering*, Hermosa.

[10] Oberkampf, W. L., and Roy, C. J., 2025, *Verification, Validation and Uncertainty Quantification in Scientific Computing*, Second Edition, Cambridge.

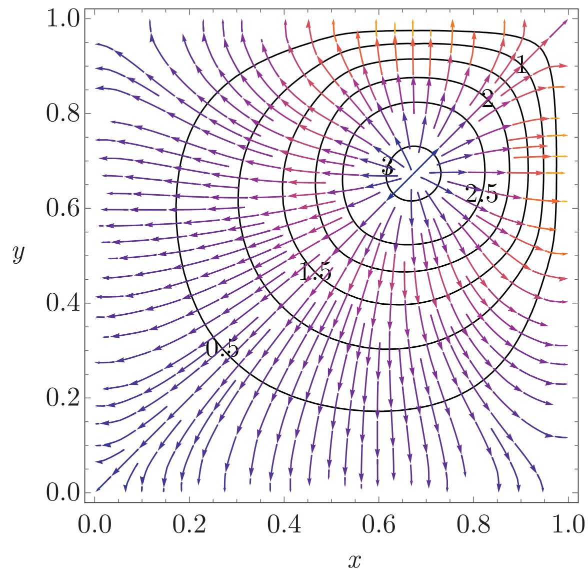


Figure 3.10: Solution contours for $T(x, y)$ using $20 \times 20 = 400$ terms of an eigenfunction expansion for a two-dimensional Poisson equation with homogeneous boundary conditions and $f(x, y) = 100H(x - 1/2)H(y - 1/2)$.

solution. Here we apply it to a linear equation. Its value is much greater for nonlinear problems for which exact solutions usually do not exist; see the upcoming Sec. 4.3.4.

First, by inspection, we see our $T(x, y) = \sin \pi x \sin \pi y$ satisfies homogeneous boundary conditions on the given unit square:

$$T(x, 0) = T(x, 1) = T(0, y) = T(1, y) = 0. \tag{3.239}$$

Now let us find the appropriate source term $f(x, y)$ for the Poisson equation

$$-\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = f(x, y). \tag{3.240}$$

Direct substitution shows that

$$-\left(-\pi^2 \sin \pi x \sin \pi y - \pi^2 \sin \pi x \sin \pi y\right) = f(x, y). \tag{3.241}$$

So

$$f(x, y) = 2\pi^2 \sin \pi x \sin \pi y. \tag{3.242}$$

A plot of the solution is shown in Fig. 3.11. Note it is easy to generate multiscale solutions. For example the solution

$$T(x, y) = \sin 4\pi x \sin 4\pi y, \tag{3.243}$$

is induced by the source term

$$f(x, y) = 32\pi^2 \sin 4\pi x \sin 4\pi y. \tag{3.244}$$

A plot of the solution is shown in Fig. 3.12.

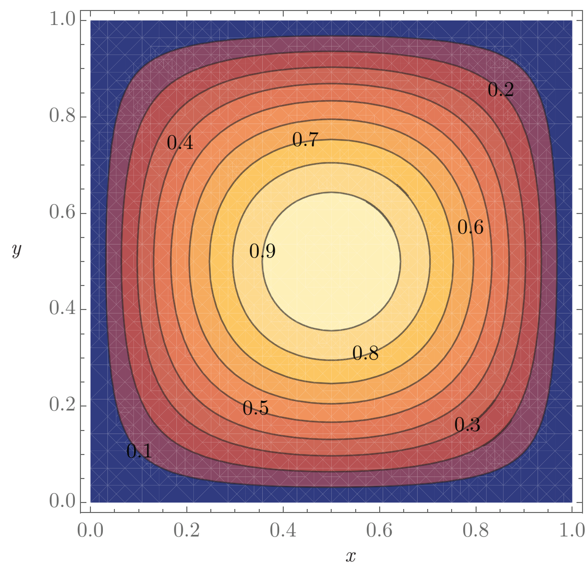


Figure 3.11: Manufactured solution contours for $T(x, y)$ for a two-dimensional Poisson equation with homogeneous boundary conditions and $f(x, y) = 2\pi^2 \sin \pi x \sin \pi y$.

3.3 Non-Cartesian geometries

Let us consider some common partial differential equations in non-Cartesian coordinate systems.

3.3.1 Cylindrical

One can transform from the Cartesian system with (x, y, z) as coordinates to the cylindrical system with $(r, \theta, \hat{z})$ as coordinates via

$$x = r \cos \theta, \tag{3.245}$$

$$y = r \sin \theta, \tag{3.246}$$

$$z = \hat{z}. \tag{3.247}$$

A sketch of the geometry is shown in Fig. 3.13. We will consider the domain $r \in [0, \infty)$, $\theta \in [0, 2\pi]$, $\hat{z} \in (-\infty, \infty)$. Then, with the exception of the origin $(x, y, z) = (0, 0, 0)$, every (x, y, z) will map to a unique $(r, \theta, \hat{z})$.

The Jacobian of the transformation is

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(r, \theta, \hat{z})}, \tag{3.248}$$

$$= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \hat{z}} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \hat{z}} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \hat{z}} \end{pmatrix}, \tag{3.249}$$

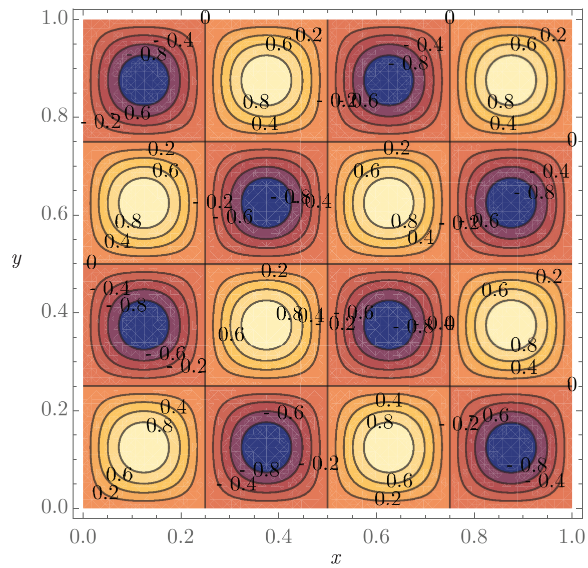


Figure 3.12: Manufactured solution contours for $T(x, y)$ for a two-dimensional Poisson equation with homogeneous boundary conditions and $f(x, y) = 32\pi^2 \sin 4\pi x \sin 4\pi y$.

$$= \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.250}$$

We have $J = |\mathbf{J}| = r$, so the transformation is singular and thus nonunique when $r = 0$. It is orientation-preserving for $r > 0$, and it is volume preserving only for $r = 1$; thus, in general it does not preserve volume.

The metric tensor $\mathbf{G}$ is

$$\mathbf{G} = \mathbf{J}^T \cdot \mathbf{J}, \tag{3.251}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.252}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.253}$$

Because $\mathbf{G}$ is diagonal, the new coordinates axes are also orthogonal.

Now it can be shown that the gradient operator in the Cartesian system is related to that of the cylindrical system via

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = (\mathbf{J}^T)^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}, \tag{3.254}$$

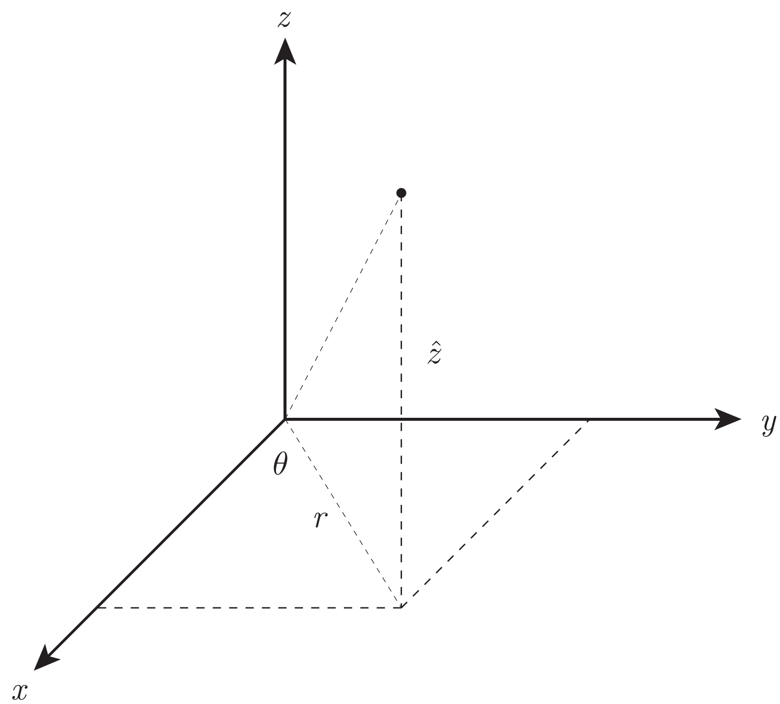


Figure 3.13: Cylindrical coordinate geometry.

$$= \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} & 0 \\ \sin\theta & \frac{\cos\theta}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}, \tag{3.255}$$

$$= \begin{pmatrix} \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}. \tag{3.256}$$

Consider then the Laplacian operator, $\nabla^2 = \nabla^T \cdot \nabla$, which is

$$\nabla^2 = \nabla^T \cdot \nabla, \tag{3.257}$$

$$= \begin{pmatrix} \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} & \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial \hat{z}} \end{pmatrix} \begin{pmatrix} \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \hat{z}} \end{pmatrix}. \tag{3.258}$$

Detailed expansion followed by extensive use of trigonometric identities reveals that this reduces to

$$\nabla^T \cdot \nabla = \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \hat{z}^2}. \tag{3.259}$$

Example 3.9

Consider the heat equation $\partial T/\partial t = \nabla^2 T$ which governs the distribution of T within a cylinder of unit radius. Assume there is no variation of T with respect to θ or $\hat{z}$. Thus, we have $T = T(r,t)$. Take $T(r,0) = f(r)$, $T(1,t) = 0$, and $T(0,t) < \infty$. Generate $T(r,t)$ if $f(r) = r^2(1-r)$.

Drawing upon Eq. (3.259) in the limits of this problem, our heat equation reduces to

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3.260}$$

$$= \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}. \tag{3.261}$$

Let us separate variables and see if we can find a solution. Take

$$T(r,t) = A(r)B(t). \tag{3.262}$$

Then we get

$$A(r)\frac{dB}{dt} = B(t)\frac{d^2A}{dr^2} + \frac{B(t)}{r}\frac{dA}{dr}, \tag{3.263}$$

$$-\frac{1}{B(t)}\frac{dB}{dt} = -\frac{1}{A(r)}\frac{d^2A}{dr^2} - \frac{1}{rA(r)}\frac{dA}{dr} = \lambda^2. \tag{3.264}$$

This yields two ordinary differential equations:

$$\frac{dB}{dt} + \lambda^2 B = 0, \tag{3.265}$$

$$\frac{d^2A}{dr^2} + \frac{1}{r}\frac{dA}{dr} + \lambda^2 A = 0. \tag{3.266}$$

Note that Eq. (3.266) can be rewritten in Sturm-Liouville form as $-(1/r)d/dr(rdA/dr) = \lambda^2 A$, and the self-adjoint positive definite Sturm-Liouville operator is $\mathbf{L}_s = -(1/r)d/dr(rd/dr)$. The solution to Eq. (3.266) is of the form

$$A(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r). \tag{3.267}$$

Here J_0 is a Bessel[11] function of order zero, and Y_0 is a Neumann function of order zero. The Neumann function is singular at $r = 0$; thus, we insist that $C_2 = 0$ to keep T bounded. Thus

$$A(r) = C_1 J_0(\lambda r). \tag{3.268}$$

We need $A(1)$ to be zero to satisfy the Dirichlet condition $T(1,t) = 0$. This gives

$$A(1) = 0 = C_1 J_0(\lambda). \tag{3.269}$$

For a nontrivial solution, we must select λ such that $J_0(\lambda) = 0$. These zeros must be found numerically. We get an idea of their distribution by plotting $J_0(\lambda)$ in Fig. 3.14. The first four are given by

$$\lambda = \{2.40483, 5.52008, 8.65373, 11.7915, \ldots\}. \tag{3.270}$$

Each of these eigenvalues is associated with an eigenfunction. The first four are

$$J_0(2.40483r),\ J_0(5.52008r),\ J_0(8.65373r),\ J_0(11.7915r). \tag{3.271}$$

We map these back to a Cartesian coordinate system and plot the first four eigenfunctions in Fig. 3.15.

[11]Friedrich Bessel, 1784-1846, German astronomer and mathematician.

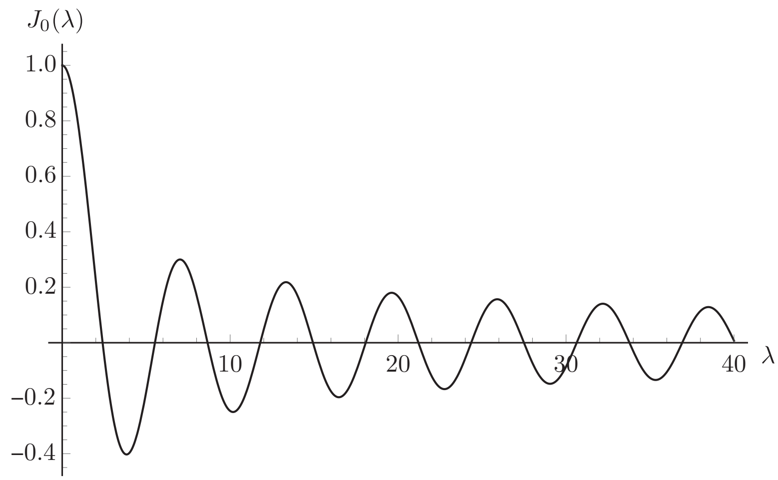


Figure 3.14: Plot of $J_0(\lambda)$.

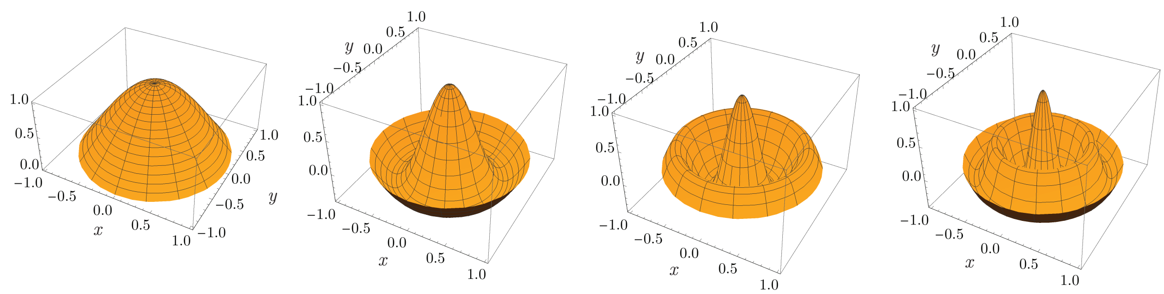


Figure 3.15: Plot of the first four eigenfunctions, $J_0(\lambda_n r)$, $n = 1, 2, 3, 4$, projected onto a Cartesian space.

Knowing λ, we can now integrate Eq. (3.265) to get

$$B(t) = C_3 \exp\left(-\lambda^2 t\right). \tag{3.272}$$

Combining with Eq. (3.268) and forming arbitrary linear combinations, we can say

$$T(r,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 t} J_0(\lambda_n r). \tag{3.273}$$

Here λ_n is the n^{th} term of Eq. (3.270). We use the initial condition to find the C_n values. Doing so we get

$$T(r,0) = f(r) = \sum_{n=1}^{\infty} C_n J_0(\lambda_n r). \tag{3.274}$$

We thus need to expand $f(r)$ in a Fourier-Bessel series. We do so via the following steps.

$$r J_0(\lambda_m r) f(r) \quad = \quad \sum_{n=1}^{\infty} C_n r J_0(\lambda_m r) J_0(\lambda_n r), \tag{3.275}$$

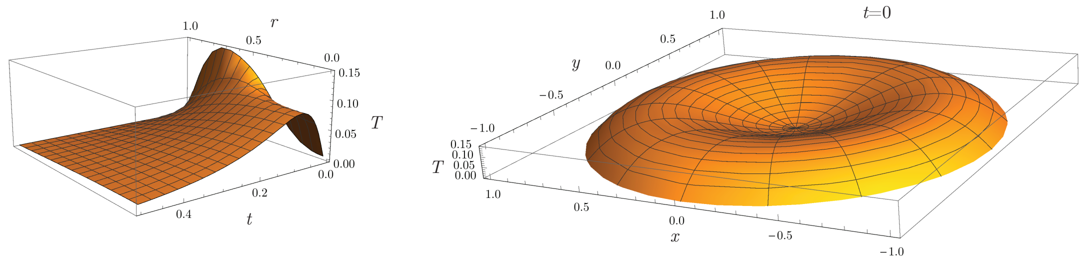


Figure 3.16: Solution $T(r,t)$ to heat equation within a cylindrical geometry with $T(r,0) = r^2(1-r)$ along with $T(x,y,t=0)$.

$$\int_0^1 rJ_0(\lambda_m r)f(r)\ dr = \sum_{n=1}^{\infty} C_n \int_0^1 rJ_0(\lambda_m r)J_0(\lambda_n r)\ dr. \tag{3.276}$$

Now the orthogonality of the Bessel functions is such that one can show

$$\int_0^1 rJ_0(\lambda_n r)J_0(\lambda_m r)\ dr = \frac{1}{2}(J_1(\lambda_n))^2\delta_{mn}. \tag{3.277}$$

Therefore, we have

$$\int_0^1 rJ_0(\lambda_m r)f(r)\ dr = \sum_{n=1}^{\infty} C_n \frac{1}{2}(J_1(\lambda_n))^2\delta_{mn}, \tag{3.278}$$

$$= \frac{C_m}{2}(J_1(\lambda_m))^2, \tag{3.279}$$

$$C_n = \frac{2}{(J_1(\lambda_n))^2}\int_0^1 rJ_0(\lambda_n r)f(r)\ dr. \tag{3.280}$$

Thus,

$$T(r,t) = \sum_{n=1}^{\infty}\left(\frac{2}{(J_1(\lambda_n))^2}\int_0^1 \hat{r}J_0(\lambda_n\hat{r})f(\hat{r})\ d\hat{r}\right)e^{-\lambda_n^2 t}J_0(\lambda_n r). \tag{3.281}$$

If $f(r) = r^2(1-r)$, we calculate that

$$C_n = \{0.164131, -0.19501, 0.0504623, -0.0273625, 0.0138598, \ldots\}. \tag{3.282}$$

For this $f(r)$, a plot of $T(r,t)$ along with $T(x,y,t=0)$ is shown in Fig. 3.16. We see the initial distribution loses some of its structure, then the entire solution relaxes to zero at t advances.

Example 3.10

Consider the wave equation $\partial^2\phi/\partial t^2 = a^2\nabla^2\phi$ which governs the distribution of ϕ within a two-dimensional circular domain with radius of unity. Assume there is no variation of ϕ with respect to θ or $\hat{z}$. Thus we have $\phi = \phi(r,t)$. Take $\partial\phi/\partial t(r,0) = 0$, $\phi(r,0) = f(r)$, $\phi(1,t) = 0$, and $\phi(0,t) < \infty$. Generate $\phi(r,t)$ if $f(r) = 1 - H(r - 1/4)$ and $a = 1$.

Again drawing upon Eq. (3.259) in the limits of our problem, our wave equation reduces to

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right), \tag{3.283}$$

$$\frac{1}{a^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}. \tag{3.284}$$

Let us separate variables:

$$\phi(r,t) = A(r)B(t). \tag{3.285}$$

Then we get

$$\frac{A(r)}{a^2}\frac{d^2B}{dt^2} = B(t)\frac{d^2A}{dr^2} + \frac{B(t)}{r}\frac{dA}{dr}, \tag{3.286}$$

$$-\frac{1}{a^2B(t)}\frac{d^2B}{dt^2} = -\frac{1}{A(r)}\frac{d^2A}{dr^2} - \frac{1}{rA(r)}\frac{dA}{dr} = \lambda^2. \tag{3.287}$$

This yields

$$\frac{d^2B}{dt^2} + a^2\lambda^2 B = 0, \tag{3.288}$$

$$\frac{d^2A}{dr^2} + \frac{1}{r}\frac{dA}{dr} + \lambda^2 A = 0. \tag{3.289}$$

As before, the solution to Eq. (3.289) is

$$A(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r), \tag{3.290}$$

and we choose $C_2 = 0$ to retain a bounded solution at $r = 0$, so

$$A(r) = C_1 J_0(\lambda r). \tag{3.291}$$

And as before in order that $\phi(1,t) = 0$, we must select λ so that $J_0(\lambda) = 0$ giving, as from Eq. (3.270),

$$\lambda = \{2.40483, 5.52008, 8.65373, 11.7915, \ldots\}. \tag{3.292}$$

Knowing λ, we can integrate Eq. (3.288) to get

$$B(t) = C_3 \sin a\lambda t + C_4 \cos a\lambda t. \tag{3.293}$$

Now to satisfy the initial condition that $\partial\phi/\partial t(r,0) = 0$, we must set $dB/dt(0) = 0$:

$$\frac{dB}{dt} = a\lambda C_3 \cos a\lambda t - a\lambda C_4 \sin a\lambda t, \tag{3.294}$$

$$\left.\frac{dB}{dt}\right|_{t=0} = a\lambda C_3 = 0. \tag{3.295}$$

Thus $C_3 = 0$ and we get

$$B(t) = C_4 \cos a\lambda t. \tag{3.296}$$

So our general solution is a linear combination of the various modes, yielding

$$\phi(r,t) = \sum_{n=1}^{\infty} C_n \cos(a\lambda_n t) J_0(\lambda_n r). \tag{3.297}$$

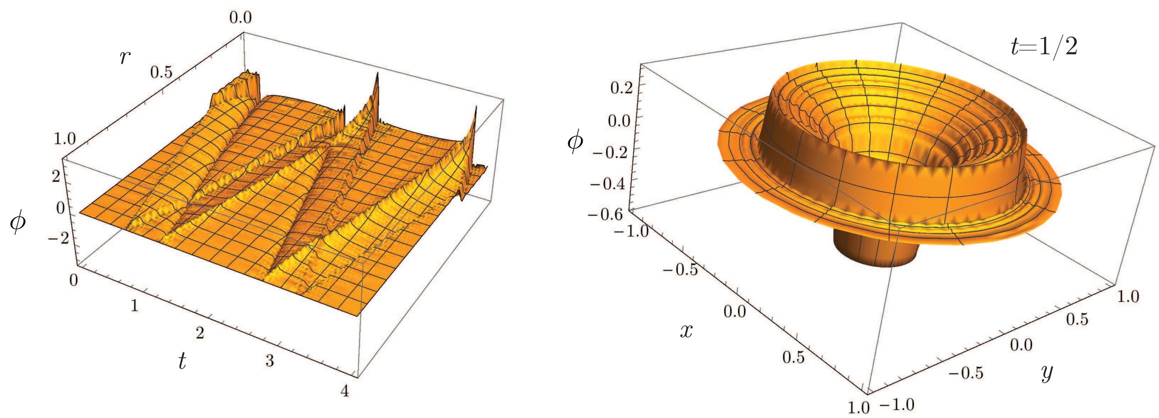


Figure 3.17: Solution $\phi(r,t)$ to wave equation within a cylindrical geometry with $\phi(r,0) = 1 - H(r-1/4)$ with $a=1$ along with $\phi(x,y,t=1/2)$.

The ratio of the frequencies of oscillation of the various modes do not come in integer multiples because of the nature of the cylindrical geometry. At the initial state, we require $\phi(r,0) = f(r)$, yielding, as before,

$$C_n = \frac{2}{(J_1(\lambda_n))^2} \int_0^1 r J_0(\lambda_n r) f(r)\; dr. \tag{3.298}$$

For $f(r) = 1 - H(r-1/4)$, we get

$$C_n = \{0.221578, 0.421112, 0.439822, 0.281122, \ldots\}. \tag{3.299}$$

Thus

$$\phi(r,t) = \sum_{n=1}^{\infty} \left(\frac{2}{(J_1(\lambda_n))^2} \int_0^1 \hat{r} J_0(\lambda_n \hat{r}) f(\hat{r})\; d\hat{r} \right) \cos(a\lambda_n t) J_0(\lambda_n r). \tag{3.300}$$

For this $f(r)$, a plot of $\phi(r,t)$ along with $\phi(x,y,t=1/2)$ is shown in Fig. 3.17. A plot of ϕ in $r-t$ space is shown in Fig. 3.18. From this figure, we see that all disturbances propagate with speed of unity.

One feature of particular interest is the early time behavior of the wave form. The initial jump breaks into two jumps. One moves in the direction of increasing r; the other moves towards $r=0$. The state between the two jumps varies with r. When the jump moving towards the center reaches the center, there is a reflection.

Example 3.11

Find the two-dimensional field T which satisfies $\nabla^2 T = 0$ with boundary conditions of $T = T_1$ on the upper half of a circle of radius a and $T = T_2$ on the lower half of the circle.

While we could do this problem in Cartesian coordinates, the specification of the boundary conditions on the circle renders the cylindrical coordinate system to be of greater utility. In general, we

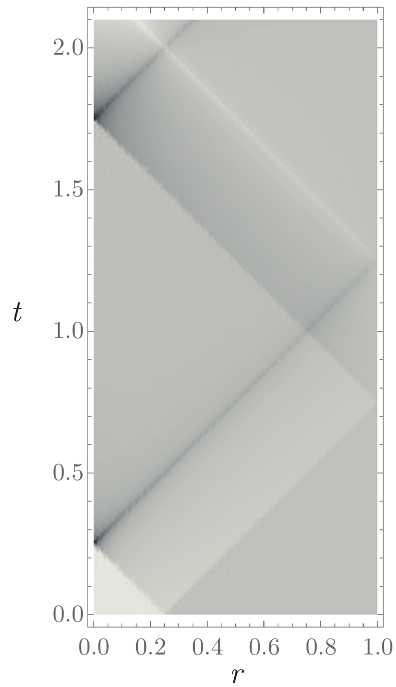


Figure 3.18: Solution $\phi(r,t)$ to wave equation within a cylindrical geometry with $\phi(r,0) = 1 - H(r - 1/4)$ with $a = 1$.

can expect $T = T(r,\theta,\hat{z})$. But due to the nature of the problem statement, we expect no variation in $\hat{z}$, and one can consider a variation of $T(r,\theta)$, which implies a polar coordinate system. For this two-dimensional polar geometry, $\nabla^2 T = 0$ is written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0. \tag{3.301}$$

We take as boundary conditions

$$T(a,\theta) = \begin{cases} T_1, & \theta \in [0,\pi], \\ T_2, & \theta \in [\pi, 2\pi]. \end{cases} \tag{3.302}$$

Often we can simplify analysis by scaling the equations in a convenient fashion. Scaling choices are not unique. We adopt the following guidelines to aid our choices:

- Try to render quantities to lie between zero and unity.
- Try to induce and take advantage of natural symmetry.
- Try to remove inhomogeneities, so as to have as many things be zero as possible.

Here we have some useful choices. Let us take

$$r_* \equiv \frac{r}{a}, \tag{3.303}$$

$$T_* \equiv 1 + 2\frac{T - T_1}{T_1 - T_2}. \tag{3.304}$$

With this choice the domain $r \in [0, a]$ is mapped to $r_* \in [0, 1]$. And when $T = T_1$, $T_* = 1$; when $T = T_2$, $T_* = -1$. This choice does introduce both ± 1 into T_*, but it introduces an anti-symmetry about $\theta = 0$ and $\theta = \pi$. By the chain rule, we see that

$$\frac{\partial}{\partial r} = \frac{dr_*}{dr}\frac{\partial}{\partial r_*} = \frac{1}{a}\frac{\partial}{\partial r_*}. \tag{3.305}$$

So Eq. (3.301) becomes

$$\frac{1}{r_* a^2}\frac{\partial}{\partial r_*}\left(a r_* \frac{1}{a}\frac{\partial}{\partial r_*}\left(\frac{(T_1 - T_2)(T_* - 1)}{2} + T_1\right)\right) + \frac{1}{{r_*}^2 a^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{(T_1 - T_2)(T_* - 1)}{2} + T_1\right) = 0. \tag{3.306}$$

This and the boundary conditions reduce to

$$\frac{1}{r_*}\frac{\partial}{\partial r_*}\left(r_*\frac{\partial T_*}{\partial r_*}\right) + \frac{1}{{r_*}^2}\frac{\partial^2 T_*}{\partial \theta^2} = 0, \qquad T_*(1, \theta) = f(\theta) = -1 + 2H(\pi - \theta) = \begin{cases} 1, & \theta \in [0, \pi], \\ -1, & \theta \in [\pi, 2\pi]. \end{cases} \tag{3.307}$$

Let us now separate variables and assume

$$T_*(r_*, \theta) = A(r_*)B(\theta). \tag{3.308}$$

Our Laplace's equation then becomes

$$\begin{aligned} \frac{B(\theta)}{r_*}\frac{d}{dr_*}\left(r_*\frac{dA}{dr_*}\right) + \frac{A(r_*)}{{r_*}^2}\frac{d^2 B}{d\theta^2} &= 0, && (3.309) \\ \frac{r_*}{A(r_*)}\frac{d}{dr_*}\left(r_*\frac{dA}{dr_*}\right) + \frac{1}{B(\theta)}\frac{d^2 B}{d\theta^2} &= 0, && (3.310) \\ \frac{r_*}{A(r_*)}\frac{d}{dr_*}\left(r_*\frac{dA}{dr_*}\right) &= -\frac{1}{B(\theta)}\frac{d^2 B}{d\theta^2} = \lambda^2. && (3.311) \end{aligned}$$

This gives us two ordinary differential equations

$$\begin{aligned} r_*\frac{d}{dr_*}\left(r_*\frac{dA}{dr_*}\right) - \lambda^2 A &= 0, && (3.312) \\ \frac{d^2 B}{d\theta^2} + \lambda^2 B &= 0. && (3.313) \end{aligned}$$

The second of these has solution

$$B(\theta) = C_1 \sin\lambda\theta + C_2\cos\lambda\theta. \tag{3.314}$$

Now we expect both T and its spatial derivative to be periodic in θ. So we expect $B(0) = B(2\pi)$ and $dB/d\theta(0) = dB/d\theta(2\pi)$. The condition $B(0) = B(2\pi)$ gives

$$C_2 = C_1\sin(2\pi\lambda) + C_2\cos(2\pi\lambda). \tag{3.315}$$

The condition $dB/d\theta(0) = dB/d\theta(2\pi)$ gives

$$\lambda C_1 = \lambda C_1\cos(2\pi\lambda) - \lambda C_2\sin(2\pi\lambda). \tag{3.316}$$

We write this as a linear system of equations as

$$\begin{pmatrix} \sin 2\pi\lambda & \cos 2\pi\lambda - 1 \\ \cos 2\pi\lambda - 1 & -\sin 2\pi\lambda \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.317}$$

For nontrivial C_1 and C_2, we must require the determinant of the coefficient matrix be zero, giving

$$\begin{aligned} -\sin^2 2\pi\lambda - (\cos 2\pi\lambda - 1)^2 &= 0, &(3.318)\\ \underbrace{-\sin^2 2\pi\lambda - \cos^2 2\pi\lambda}_{=-1} + 2\cos 2\pi\lambda - 1 &= 0, &(3.319)\\ 2\cos 2\pi\lambda &= 2, &(3.320)\\ \cos 2\pi\lambda &= 1. &(3.321)\end{aligned}$$

This can only be achieved if we select

$$\lambda = n, \qquad n = 0, 1, 2, \ldots. \tag{3.322}$$

Thus,

$$B(\theta) = C_1 \sin n\theta + C_2 \cos n\theta. \tag{3.323}$$

Then Eq. (3.312) reduces to

$$r_*{}^2 \frac{d^2 A}{dr_*{}^2} + r_* \frac{dA}{dr_*} - n^2 A = 0. \tag{3.324}$$

This is a second order ordinary differential equation with variable coefficients. It is known as Euler's[12] equation. If $n = 0$, it reduces to

$$r_* \frac{d^2 A}{dr_*^2} + \frac{dA}{dr_*} = 0. \tag{3.325}$$

With $A' = dA/dr_*$, this becomes

$$\begin{aligned} \frac{dA'}{dr_*} &= -\frac{A'}{r_*}, &(3.326)\\ \frac{dA'}{A'} &= -\frac{dr_*}{r_*}, &(3.327)\\ \ln A' &= C - \ln r_*, &(3.328)\\ A' &= \frac{\hat{C}}{r_*}. &(3.329)\end{aligned}$$

Here $\hat{C} = e^C$. Continue then to find

$$\begin{aligned} \frac{dA}{dr_*} &= \frac{\hat{C}}{r_*}, &(3.330)\\ A(r_*) &= \tilde{C} + \hat{C} \ln r_*, \qquad n = 0. &(3.331)\end{aligned}$$

For $n \neq 0$, we can find solutions by assuming solutions of the form $A(r_*) = r_*{}^b$. Substituting, we find

$$\begin{aligned} r_*{}^2 b(b-1) r_*{}^{b-2} + r_* b r_*{}^{b-1} - n^2 r_*{}^b &= 0, &(3.332)\\ b(b-1) + b - n^2 &= 0, &(3.333)\\ b^2 - n^2 &= 0, &(3.334)\\ b &= \pm n, \qquad n = 1, 2, \ldots &(3.335)\end{aligned}$$

[12] Leonhard Euler, 1707-1783, Swiss mathematician.

Thus

$$A(r_*) = C_3 {r_*}^n + C_4 {r_*}^{-n}. \tag{3.336}$$

Combining, we find

$$T_*(r_*,\theta) = \begin{cases} \left(\tilde{C} + \hat{C}\ln r_*\right)(C_2) & n = 0, \\ \left(C_3 {r_*}^n + C_4 {r_*}^{-n}\right)(C_1 \sin n\theta + C_2 \cos n\theta), & n = 1, 2, \dots \end{cases} \tag{3.337}$$

Now, we seek a bounded T_* at $r_* = 0$. To achieve this, we will insist that both $\hat{C} = C_4 = 0$, so that

$$T_*(r_*,\theta) = \begin{cases} \hat{C}_0 & n = 0, \\ {r_*}^n \left(\hat{C}_1 \sin n\theta + \hat{C}_2 \cos n\theta\right), & n = 1, 2, \dots \end{cases} \tag{3.338}$$

Here we have taken $\hat{C}_0 = \tilde{C}C_2$, $\hat{C}_1 = C_3 C_1$, and $\hat{C}_2 = C_3 C_2$. We can in fact form linear combinations of the various modes; doing this, and segregating the $n = 0$ term, defining the terms C_0, C_n and B_n for convenience, and rearranging, we can say

$$T_*(r_*,\theta) = C_0 + \sum_{n=1}^{\infty} C_n r_*^n \cos n\theta + \sum_{n=1}^{\infty} B_n r_*^n \sin n\theta. \tag{3.339}$$

Now when $r_* = 1$, we have

$$T(1,\theta) = f(\theta) = -1 + 2H(\pi - \theta) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta. \tag{3.340}$$

The following orthogonality properties are easily verified for nonnegative integers n and m:

$$\int_0^{2\pi} \cos n\theta \cos m\theta \; d\theta = \begin{cases} 2\pi & n = m = 0, \\ \pi & n = m \neq 0, \\ 0 & n \neq m, \end{cases} \tag{3.341}$$

$$\int_0^{2\pi} \sin n\theta \sin m\theta \; d\theta = \begin{cases} 0 & n = m = 0, \\ \pi & n = m \neq 0, \\ 0 & n \neq m. \end{cases} \tag{3.342}$$

Let us see how to find B_n. Let us operate on Eq. (3.340) first by multiplying by $\sin m\theta$:

$$f(\theta)\sin m\theta = C_0 \sin m\theta + \sum_{n=1}^{\infty} C_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta, \tag{3.343}$$

$$\begin{aligned} \int_0^{2\pi} f(\theta)\sin m\theta \; d\theta &= C_0 \underbrace{\int_0^{2\pi} \sin m\theta \; d\theta}_{=0} + \sum_{n=1}^{\infty} C_n \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta \; d\theta}_{=0} \\ &\quad + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{2\pi} \sin n\theta \sin m\theta \; d\theta}_{=\pi\delta_{mn}}, \end{aligned} \tag{3.344}$$

$$= \sum_{n=1}^{\infty} B_n \pi \delta_{mn}, \tag{3.345}$$

$$= \pi B_m, \tag{3.346}$$

$$\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \; d\theta = B_n. \tag{3.347}$$

Then, one could used the same procedure to find C_0 and C_n. The general trigonometric Fourier coefficients for $f(\theta)$ are easily shown to be

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta, \tag{3.348}$$

$$C_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \qquad n = 1, 2, \dots \tag{3.349}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta, \qquad n = 1, 2, \dots \tag{3.350}$$

We find then for $f(\theta) = -1 + 2H(\pi - \theta)$ from Eq. (3.307) that

$$C_0 = 0, \tag{3.351}$$

$$C_n = \{0, 0, 0, 0, \dots\}, \tag{3.352}$$

$$B_n = \left\{\frac{4}{\pi}, 0, \frac{4}{3\pi}, 0, \frac{4}{5\pi}, \dots\right\}. \tag{3.353}$$

Only odd powers of n have value in B_n. Recognizing this we can thus write the solution compactly as

$$T_*(r_*, \theta) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{{r_*}^{2n-1}}{2n-1}\sin((2n-1)\theta) = \frac{4r_*\sin\theta}{\pi} + \frac{4{r_*}^3\sin 3\theta}{3\pi} + \frac{4{r_*}^5\sin 5\theta}{5\pi} + \dots. \tag{3.354}$$

With $x_* = x/a$ and $y_* = y/a$, surface and contour plots of T_* composed from 25 nonzero terms is shown in Fig. 3.19. Also shown is the heat flux vector field $\mathbf{q} = -\nabla T_*$. We note there is no particular

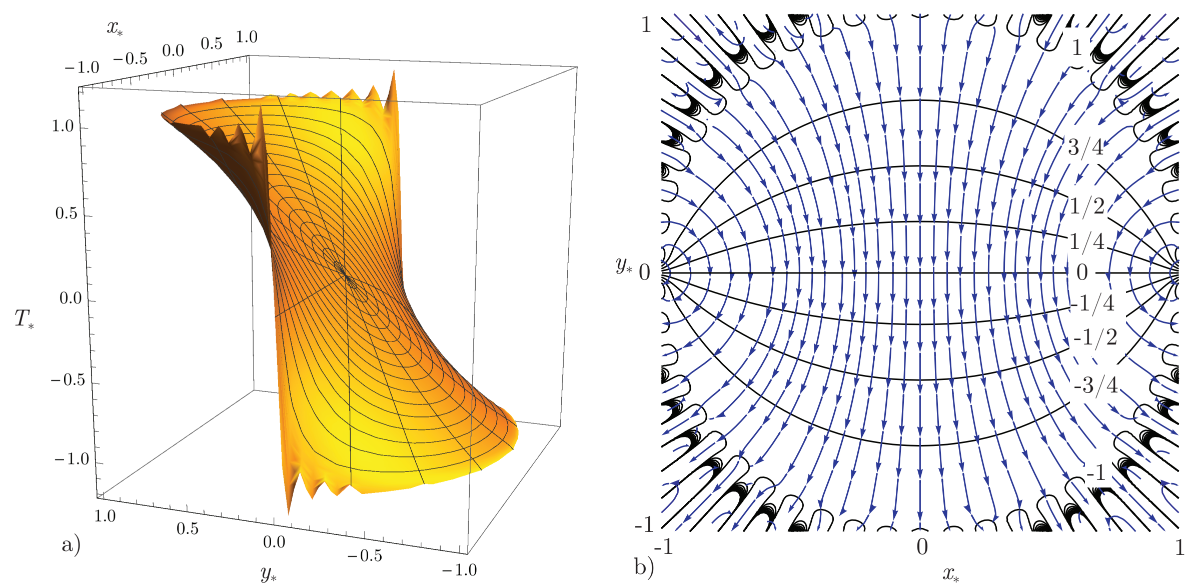


Figure 3.19: Plots of T_* which satisfy $\nabla^2 T_* = 0$ with $T_* = -1$ on the lower circular boundary and $T_* = 1$ on the upper circular boundary: a) surface plot, b) contour plot along with heat flux vector field.

difficulty in T_* at the origin $r_* = 0$. However T_* is nonunique at $(r_*, \theta) = (1, 0)$ and $(1, \pi)$, the locations of the jumps in T_*.

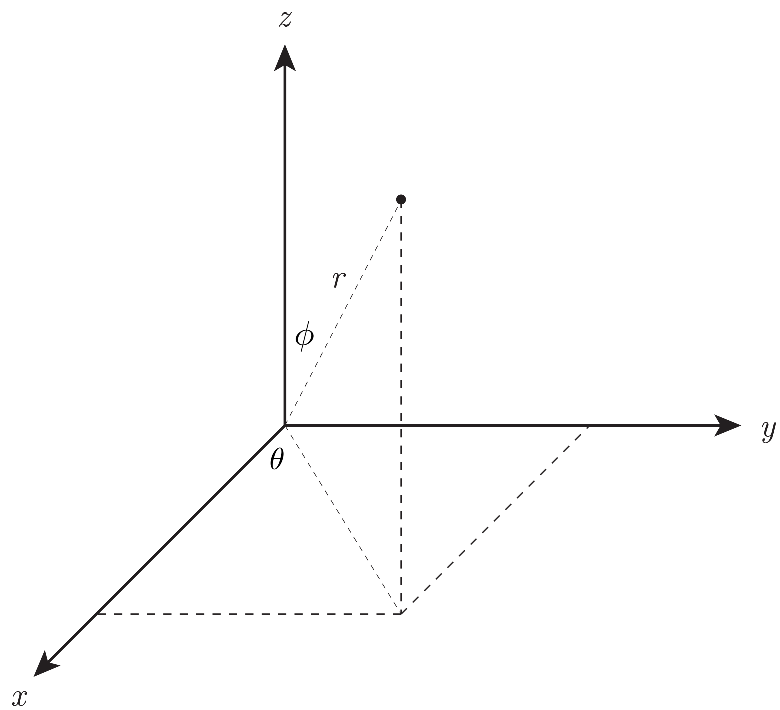


Figure 3.20: Spherical coordinate geometry.

3.3.2 Spherical

One can transform from the Cartesian system with (x, y, z) as coordinates to the spherical system with (r, ϕ, θ) as coordinates via

$$\begin{aligned} x &= r\cos\theta\sin\phi, && (3.355)\\ y &= r\sin\theta\sin\phi, && (3.356)\\ z &= r\cos\phi. && (3.357) \end{aligned}$$

A sketch of the geometry is shown in Fig. 3.20. We will consider the domain $r \in [0, \infty)$, $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$. Then, with the exception of the z-axis, $(x, y, z) = (0, 0, z)$ every (x, y, z) will map to a unique (r, ϕ, θ).

The Jacobian of the transformation is

$$\begin{aligned} \mathbf{J} &= \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)}, && (3.358)\\ &= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}, && (3.359) \end{aligned}$$

$$= \begin{pmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\phi\sin\theta & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}. \tag{3.360}$$

We have $J = |\mathbf{J}| = r^2\sin\phi$, so the transformation is singular and thus nonunique when either $r = 0$, $\phi = 0$, or $\phi = \pi$. It is orientation-preserving for $r > 0$, $\phi \in [0, \pi]$ and it is volume-preserving only for $r^2 \sin\phi = 1$; thus, in general it does not preserve volume.

The metric tensor $\mathbf{G}$ is

$$\mathbf{G} = \mathbf{J}^T\cdot\mathbf{J}, \tag{3.361}$$

$$= \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ r\cos\theta\cos\phi & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\theta\cos\phi & r\cos\theta\sin\phi & 0 \end{pmatrix}\begin{pmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\phi\sin\theta & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}, \tag{3.362}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\phi \end{pmatrix}. \tag{3.363}$$

Because $\mathbf{G}$ is diagonal, the new coordinates axes are also orthogonal.

The gradient operator in the Cartesian system is related to that of the spherical system via

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = (\mathbf{J}^T)^{-1}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \theta} \end{pmatrix}, \tag{3.364}$$

$$= \begin{pmatrix} \cos\theta\sin\phi & \frac{\cos\theta\cos\phi}{r} & -\frac{\csc\phi\sin\theta}{r} \\ \sin\theta\sin\phi & \frac{\cos\phi\sin\theta}{r} & \frac{\cos\theta\csc\phi}{r} \\ \cos\phi & -\frac{\sin\phi}{r} & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \theta} \end{pmatrix}, \tag{3.365}$$

$$= \begin{pmatrix} \cos\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\phi} - \frac{\csc\phi\sin\theta}{r}\frac{\partial}{\partial\theta} \\ \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi\sin\theta}{r}\frac{\partial}{\partial\phi} + \frac{\cos\theta\csc\phi}{r}\frac{\partial}{\partial\theta} \\ \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi} \end{pmatrix}. \tag{3.366}$$

We then have the Laplacian operator, $\nabla^2 = \nabla^T\cdot\nabla$, which is, following extensive reduction

$$\nabla^2 = \nabla^T\cdot\nabla = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right). \tag{3.367}$$

Example 3.12

Find the distribution of T within a sphere of radius a which satisfies $\nabla^2 T = 0$ with boundary conditions of $T = T_1$ on the upper half of the sphere and $T = T_2$ on the lower half.

In general, we could expect $T = T(r, \phi, \theta)$. However, we notice symmetry in the boundary conditions such that we are motivated to seek solutions $T = T(r, \phi)$; that is, we will neglect any variation in θ. Our governing equation and boundary conditions then reduce to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T}{\partial\phi}\right) = 0, \qquad T(a,\phi) = \begin{cases} T_1, & \phi \in \left[0, \frac{\pi}{2}\right], \\ T_2, & \phi \in \left(\frac{\pi}{2}, \pi\right]. \end{cases} \tag{3.368}$$

Similar to the related example in polar coordinates we select scaled variables $r_* = r/a$, and $T_* = 1 + 2(T - T_1)/(T_1 - T_2)$. The problem is then expressed as

$$\frac{\partial}{\partial r_*}\left({r_*}^2\frac{\partial T_*}{\partial r_*}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T_*}{\partial\phi}\right) = 0, \quad T_*(1,\phi) = -1 + 2H\left(\frac{\pi}{2} - \phi\right) = \begin{cases} 1, & \phi \in \left[0, \frac{\pi}{2}\right], \\ -1, & \phi \in \left(\frac{\pi}{2}, \pi\right]. \end{cases} \tag{3.369}$$

Let us separate variables and see if this leads to a solution. We can try

$$T_*(r_*, \phi) = A(r_*)B(\phi). \tag{3.370}$$

Substituting this assumption into Eq. (3.369) yields

$$B(\phi)\frac{d}{dr_*}\left({r_*}^2\frac{dA}{dr_*}\right) + \frac{A(r_*)}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dB}{d\phi}\right) = 0, \tag{3.371}$$

$$\frac{1}{A(r_*)}\frac{d}{dr_*}\left({r_*}^2\frac{dA}{dr_*}\right) = -\frac{1}{B(\phi)\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dB}{d\phi}\right) = \lambda. \tag{3.372}$$

This yields two ordinary differential equations:

$$\frac{d}{dr_*}\left({r_*}^2\frac{dA}{dr_*}\right) - \lambda A = 0, \tag{3.373}$$

$$\frac{d}{d\phi}\left(\sin\phi\frac{dB}{d\phi}\right) + \lambda\sin\phi\ B = 0. \tag{3.374}$$

Let us operate further on Eq. (3.374), performing some non-obvious transformations to render it into a standard form. First let us change the independent variable from ϕ to s via the transformation

$$s = \cos\phi. \tag{3.375}$$

With this transformation, we find from the chain rule that

$$\frac{d}{d\phi} = \frac{ds}{d\phi}\frac{d}{ds} = -\sin\phi\frac{d}{ds}. \tag{3.376}$$

Then Eq. (3.374) is rewritten as

$$-\sin\phi\frac{d}{ds}\left(-\sin^2\phi\frac{dB}{ds}\right) + \lambda\sin\phi\ B = 0. \tag{3.377}$$

Now we recognize that $\sin^2\phi = 1 - \cos^2\phi = 1 - s^2$, and we scale by $\sin\phi$, taking care that $\phi \in (0, \pi)$, so as to get

$$\frac{d}{ds}\left((1-s^2)\frac{dB}{ds}\right) + \lambda B = 0, \tag{3.378}$$

$$\underbrace{-\frac{d}{ds}\left((1-s^2)\frac{d}{ds}\right)}_{\mathbf{L}} B = \lambda B. \tag{3.379}$$

Here $\mathbf{L}$ is the well known positive definite Sturm-Liouville operator whose eigenvalues $\lambda = n(n+1)$, with $n = 0, 1, 2, \ldots$ and eigenfunctions are the Legendre[13] polynomials, $P_n(s)$:

$$P_n(s) = \frac{1}{2^n n!} \frac{d^n}{ds^n} \left(s^2 - 1\right)^n. \tag{3.380}$$

The first few eigenfunctions and eigenvalues are

$$\begin{aligned}
P_0(s) &= 1, \qquad \lambda = 0, && (3.381)\\
P_1(s) &= s, \qquad \lambda = 2, && (3.382)\\
P_2(s) &= \frac{1}{2}(-1 + 3s^2), \qquad \lambda = 6, && (3.383)\\
P_3(s) &= \frac{1}{2}s(-3 + 5s^2), \qquad \lambda = 12, && (3.384)\\
P_4(s) &= \frac{1}{8}(3 - 30s^2 + 35s^4), \qquad \lambda = 20, && (3.385)\\
&\vdots
\end{aligned}$$

It can be shown by direct substitution of the eigenvalues and their corresponding eigenfunctions that Eq. (3.379) is satisfied. Just as the Sturm-Liouville operator $-d^2/ds^2$ has two families of eigenfunctions (sin and cos), so does the Legendre Sturm-Liouville operator. The other set of complementary functions are known as $Q_n(s)$ and have logarithmic singularities at $s = \pm 1$. Because we desire a bounded solution, we will select the constant modulating $Q_n(s)$ to be zero, and thus consider it no further.

We thus take

$$B_n(\phi) = P_n(\cos\phi). \tag{3.386}$$

Thus,

$$\begin{aligned}
B_0(\phi) &= 1, \qquad \lambda = 0, && (3.387)\\
B_1(\phi) &= \cos\phi, \qquad \lambda = 2, && (3.388)\\
B_2(\phi) &= \frac{1}{2}(-1 + 3\cos^2\phi), \qquad \lambda = 6, && (3.389)\\
&\vdots
\end{aligned}$$

Now return to consider Eq. (3.373). With $\lambda = n(n+1)$, $n = 0, 1, 2, \ldots$ it is

$$\frac{d}{dr_*}\left({r_*}^2 \frac{dA}{dr_*}\right) - n(n+1)A = 0. \tag{3.390}$$

This is an Euler equation. We can assume solutions of the type $A(r_*) = C{r_*}^b$. With this assumption, Eq. (3.390) becomes

$$\begin{aligned}
\frac{d}{dr_*}\left({r_*}^2 \frac{d}{dr_*}\left(C{r_*}^b\right)\right) - n(n+1)C{r_*}^b &= 0, && (3.391)\\
\frac{d}{dr_*}\left(b{r_*}^{b+1}\right) - n(n+1){r_*}^b &= 0, && (3.392)\\
b(b+1){r_*}^b - n(n+1){r_*}^b &= 0, && (3.393)\\
b(b+1) - n(n+1) &= 0. && (3.394)
\end{aligned}$$

[13] Adrien-Marie Legendre, 1752-1833, French mathematician.

Solving this quadratic equation for b, we find two solutions: $b = n$ and $b = -(n+1)$, thus

$$A(r_*) = C_1 {r_*}^n + C_2 {r_*}^{-(n+1)}, \qquad n = 0, 1, 2, \ldots \tag{3.395}$$

To suppress unbounded T_* when $r_* = 0$, we set $C_2 = 0$ so that

$$A(r_*) = C_1 {r_*}^n, \qquad n = 0, 1, 2, \ldots \tag{3.396}$$

We can then combine our solutions for $B(\phi)$ and $A(r_*)$ in terms of arbitrary linear combinations to get

$$T_*(r_*, \phi) = \sum_{n=0}^{\infty} C_n {r_*}^n P_n(\cos\phi). \tag{3.397}$$

To determine the constants C_n, we can apply the boundary condition at $r_* = 1$ from Eq. (3.369):

$$T_*(1, \phi) = f(\phi) = \sum_{n=0}^{\infty} C_n P_n(\cos\phi). \tag{3.398}$$

This amounts to expressing $f(\phi)$ in terms of a Fourier-Legendre series, where the basis functions are the Legendre polynomials. To aid in this, let us again employ the transformation of Eq. (3.375), $s = \cos\phi$. Let us also define g such that

$$g(\cos\phi) = f(\phi). \tag{3.399}$$

For example, if $f(\phi) = \phi$, then $g(\cos\phi) = \arccos(\cos\phi)$; so g is the inverse cosine function. Then in terms of s, we seek the Fourier-Legendre expansion for

$$g(s) = \sum_{n=0}^{\infty} C_n P_n(s). \tag{3.400}$$

Now the Legendre polynomials are orthogonal on the domain $s \in [-1, 1]$ with it being easy to show that

$$\int_{-1}^{1} P_n(s) P_m(s)\, ds = \delta_{mn} \frac{2}{2n+1}. \tag{3.401}$$

Note that when $s = 1$, $\phi = 0$ and when $s = -1$, $\phi = \pi$, so the domain $s \in [-1, 1]$ sweeps through the entire sphere. Let us use the orthogonality property while operating on Eq. (3.400):

$$\begin{aligned}
g(s) P_m(s) &= \sum_{n=0}^{\infty} C_n P_n(s) P_m(s), && (3.402)\\
\int_{-1}^{1} g(s) P_m(s)\, ds &= \sum_{n=0}^{\infty} C_n \underbrace{\int_{-1}^{1} P_n(s) P_m(s)\, ds}_{=2\delta_{nm}/(2n+1)}, && (3.403)\\
\int_{-1}^{1} g(s) P_m(s)\, ds &= \sum_{n=0}^{\infty} C_n \frac{2\delta_{mn}}{2n+1}, && (3.404)\\
\int_{-1}^{1} g(s) P_m(s)\, ds &= \frac{2C_m}{2m+1}, && (3.405)\\
C_n &= \frac{2n+1}{2} \int_{-1}^{1} g(s) P_n(s)\, ds. && (3.406)
\end{aligned}$$

Now our two-parted domain transforms as follows. With $s = \cos\phi$, we see that $\phi \in [0, \pi/2]$ maps to $s \in [1, 0]$; moreover, $\phi \in (\pi/2, \pi]$ maps to $s \in (0, -1]$. So we can say that $g(s)$ is expressed as

$$g(s) = -1 + 2H(s) = \begin{cases} 1 & s \in [1, 0], \\ -1 & s \in (0, -1]. \end{cases} \tag{3.407}$$

With this $g(s)$, evaluation of Eq. (3.406) gives

$$C_n = \left\{0, \frac{3}{2}, 0, -\frac{7}{8}, 0, \frac{11}{16}, 0, -\frac{75}{128}, \dots\right\}, \qquad n = 0, 1, 2, \dots \tag{3.408}$$

Very detailed analysis reveals there is a general form for the C_n for arbitrary n, allowing one to write the solution compactly as

$$T_*(r_*, \phi) = \sum_{m=0}^{\infty} \frac{(-1)^m (4m+3)(2m)!}{2^{2m+1}(m+1)(m!)^2} {r_*}^{2m+1} P_{2m+1}(\cos\phi). \tag{3.409}$$

The first two nonzero terms of the series are

$$T_*(r_*, \phi) = \frac{3}{2} r_* P_1(\cos\phi) - \frac{7}{8} {r_*}^3 P_3(\cos\phi) + \dots, \tag{3.410}$$

$$= \frac{3}{2} r_* \cos\phi - \frac{7}{8} {r_*}^3 \frac{\cos\phi(-3 + 5\cos^2\phi)}{2} + \dots. \tag{3.411}$$

With $x_* = x/a$, $z_* = z/a$, and considering only the plane on which $y = 0$, surface and contour plots of T_* composed from 10 nonzero terms of Eq. (3.409) are shown in Fig. 3.21. Again, there is no particular

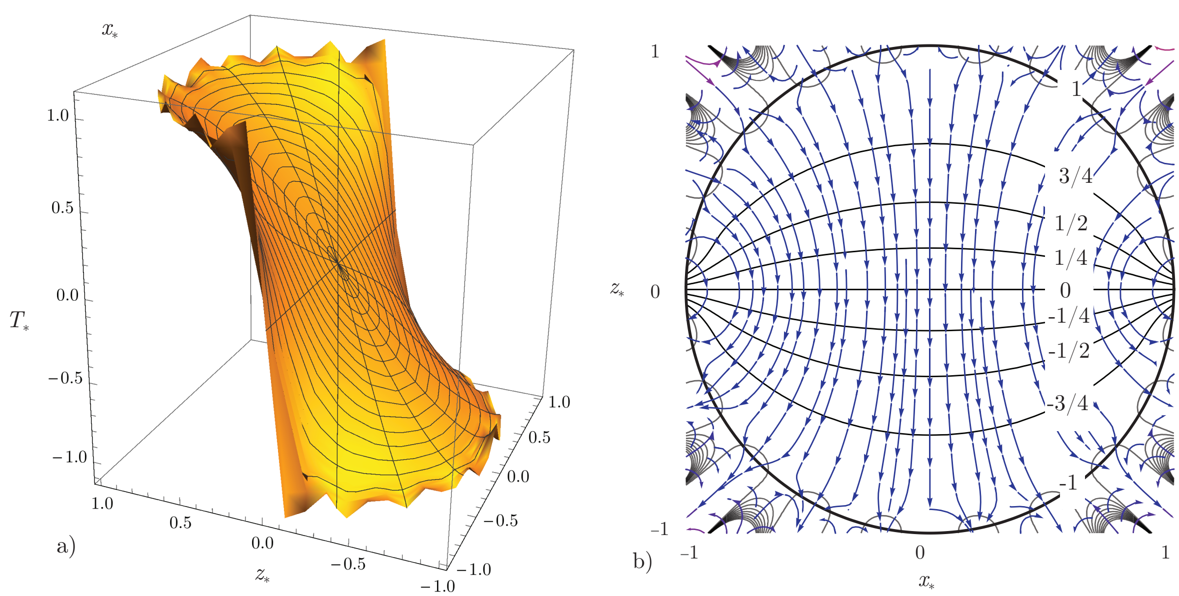


Figure 3.21: Plots from the plane $y_* = 0$ of T_* which satisfy $\nabla^2 T_* = 0$ with $T_* = 1$ on an upper hemispherical boundary and $T_* = -1$ on a lower hemispherical boundary: a) surface plot, b) contour plot and heat flux vector field.

difficulty in T_* at the origin $r_* = 0$. However T_* is nonunique at $(r_*, \phi) = (1, \pi/2)$, the locations of the jump in T_*. The plots of Fig. 3.21 are very similar to those of the cylindrical analog of Fig. 3.19; the small differences can be attributed to spherical versus cylindrical geometry.

3.4 Usage in a stability problem

Separation of variables is often an important component of problems of larger scope. Often when one wants to determine the stability of some solution of a nonlinear equation, one employs local linearization techniques so as to generate a linear problem which may be solved via separation of variables. As an example, let us consider an idealized problem motivated by combustion. The general problem is nonlinear, with a linear heat equation subjected to a nonlinear combustion source term. We shall determine a steady state solution from solving numerically a nonlinear problem, then determine its linear stability via separation of variables. We close by briefly examining its full nonlinear transient solution. Further physical and mathematical details are given by Powers.[14]

The domain is modeled to be a slab of infinite extent in the y and z directions and has length two in the x direction, with $x \in [-1, 1]$. The temperature at $x = \pm 1$ is held fixed at $T = 0$. The slab is initially unreacted. Exothermic conversion of material from reactants to products will generate an elevated temperature within the slab $T > 0$, for $x \in [-1, 1]$. If the thermal energy generated diffuses rapidly enough, the temperature within the slab will be $T \sim 0$, and the reaction rate will be low. If the energy generated by local reaction diffuses slowly, it will accumulate in the interior, accelerate the local reaction rate, and induce rapid energy release and high temperature.

We take our model equations to be

$$\frac{\partial T}{\partial t} = \frac{1}{\mathfrak{D}} \frac{\partial^2 T}{\partial x^2} + (1 - T) \exp\left(\frac{-\Theta}{1 + QT} \right), \tag{3.412}$$

$$T(-1, t) = 0, \qquad T(1, t) = 0, \qquad T(x, 0) = 0. \tag{3.413}$$

Here, the equations are dimensionless. Three dimensionless parameters appear, 1) Θ, the so-called dimensionless activation energy, motivated by so-called Arrhenius kinetics. In the Arrhenius kinetics model, reactions are suppressed at low temperature. At high temperature they are fully activated. The value of Θ plays a large role in determining at what temperature the transition from slow to fast occurs, 2) Q, the dimensionless heat release parameter which quantifies the exothermic nature of the reaction, and 3) $\mathfrak{D}$, the Damköhler[15] number, which gives the ratio of the time scales of energy diffusion to the time scales of chemical reaction. Note that the initial and boundary conditions are homogeneous. The only inhomogeneity lives in the exothermic reaction source term, which is nonlinear due to the $\exp(-1/T)$ term.

[14] Powers, J. M., 2024, *Lecture Notes on Fundamentals of Combustion*, University of Notre Dame; also see Powers, J. M., 2016, *Combustion Thermodynamics and Dynamics*, Cambridge University Press, New York.

[15] Gerhard Damköhler, 1908-1944, German chemist.

As an aside, let us consider the evolution of total energy within the domain. To do so we integrate Eq. (3.412) through the entire domain.

$$\int_{-1}^{1} \frac{\partial T}{\partial t}\, dx = \int_{-1}^{1} \frac{1}{\mathfrak{D}} \frac{\partial^2 T}{\partial x^2}\, dx + \int_{-1}^{1} (1-T) \exp\left(\frac{-\Theta}{1+QT}\right) dx, \tag{3.414}$$

$$\underbrace{\frac{d}{dt}\int_{-1}^{1} T\, dx}_{\text{thermal energy change}} = \underbrace{\frac{1}{\mathfrak{D}} \frac{\partial T}{\partial x}\bigg|_{x=1} - \frac{1}{\mathfrak{D}} \frac{\partial T}{\partial x}\bigg|_{x=-1}}_{\text{boundary heat flux}} + \underbrace{\int_{-1}^{1} (1-T) \exp\left(\frac{-\Theta}{1+QT}\right) dx}_{\text{internal conversion}}. \tag{3.415}$$

The total thermal energy in our domain changes due to two reasons: 1) diffusive energy flux at the isothermal boundaries, 2) internal conversion of chemical energy to thermal energy.

3.4.1 Spatially homogeneous solutions

For $\mathfrak{D} \to \infty$ diffusion becomes unimportant in Eq. (3.412), and we recover a balance between unsteady effects and reaction:

$$\frac{dT}{dt} = (1-T) \exp\left(\frac{-\Theta}{1+QT}\right), \qquad T(0) = 0. \tag{3.416}$$

Solutions $T(t)$ are independent of x and are considered spatially homogeneous. An asymptotic theory valid in the limit of large Θ predicts significant acceleration of reaction when

$$t \to \frac{e^{\Theta}}{Q\Theta}. \tag{3.417}$$

For $\Theta = 15$, $Q = 1$, we plot a numerical solution of $T(t)$ in Fig. 3.22. For these parameters,

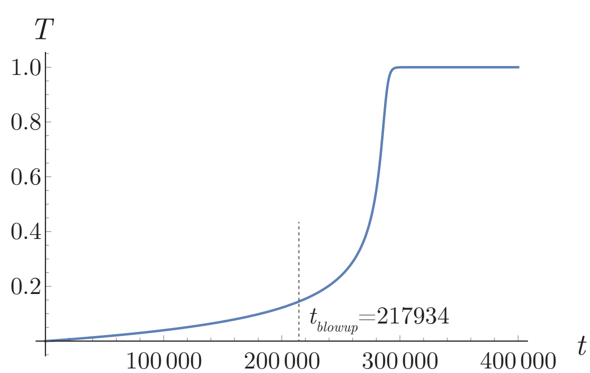


Figure 3.22: Plot of $T(t)$ for $\Theta = 15$, $Q = 1$, $\mathfrak{D} \to \infty$.

Eq. (3.417) estimates a blow-up phenomena at $t = 217934$. The results of Fig. 3.22 indicate our estimate is good. Physically, the exothermic heat release from initially slow reaction accumulates inducing a slow temperature increase. At a critical temperature, the extreme sensitivity of reaction rates induces a rapid rise of temperature to its final value of $T = 1$, where the material is completely reacted.

3.4.2 Steady solutions

Let us examine solutions to Eq. (3.412) in the steady state limit for which $\partial/\partial t = 0$:

$$0 = \frac{1}{\mathfrak{D}}\frac{d^2T}{dx^2} + (1-T)\exp\left(\frac{-\Theta}{1+QT}\right), \tag{3.418}$$

$$0 = T(-1) = T(1). \tag{3.419}$$

Rearrange to get

$$\frac{d^2T}{dx^2} = -\mathfrak{D}\,(1-T)\exp\left(\frac{-\Theta}{1+QT}\right), \tag{3.420}$$

$$= -\mathfrak{D}\frac{Q\Theta\exp(-\Theta)}{Q\Theta\exp(-\Theta)}\,(1-T)\exp\left(\frac{-\Theta}{1+QT}\right). \tag{3.421}$$

Now, defining for convenience

$$\delta = \mathfrak{D}Q\Theta\exp(-\Theta), \tag{3.422}$$

we get

$$\frac{d^2T}{dx^2} = -\delta\frac{\exp(\Theta)}{Q\Theta}\,(1-T)\exp\left(\frac{-\Theta}{1+QT}\right), \tag{3.423}$$

$$T(-1) = T(1) = 0. \tag{3.424}$$

Equations (3.423-3.424) can be solved by a numerical trial and error method for which we take $T(-1) = 0$ and guess $dT/dx|_{x=-1}$. We keep guessing until we have satisfied the boundary condition of $T(1) = 0$.

When we do this with $\delta = 0.4$, $\Theta = 15$, $Q = 1$ (so $\mathfrak{D} = \delta e^{\Theta}/Q/\Theta = 87173.8$), we find *three* steady solutions. For each we find a maximum temperature, T^m, at $x = 0$. One is at low temperature with $T^m = 0.016$. We find a second intermediate temperature solution with $T^m = 0.417$. And we find a high temperature solution with $T^m = 0.987$. Plots of $T(x)$ for high, low, and intermediate temperature solutions are given in Fig 3.23.

We can use a one-term collocation approximation to estimate the relationship between δ and T^m. Let us estimate that

$$T_a(x) = c_1(1-x^2). \tag{3.425}$$

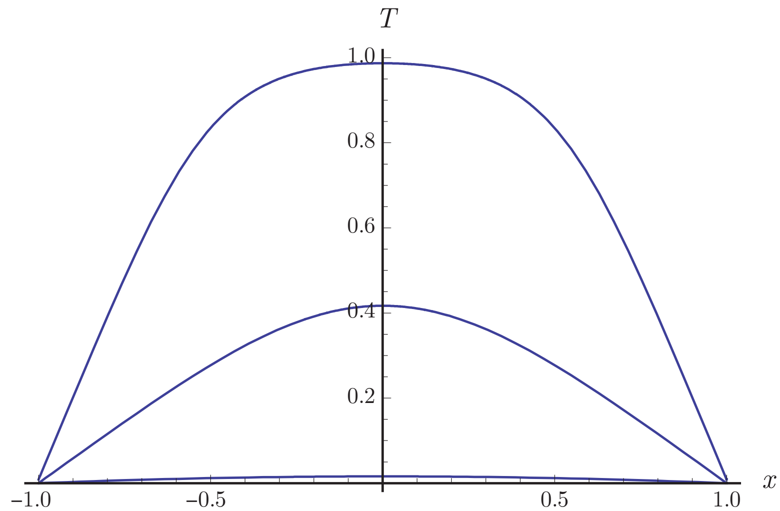


Figure 3.23: Plots of high, low, and intermediate temperature distributions $T(x)$ for $\delta = 0.4$, $Q = 1$, $\Theta = 15$.

This estimate certainly satisfies the boundary conditions. Substituting our choice into Eq. (3.423), we get a residual of

$$r(x) = -2c_1 + \frac{\delta}{Q\Theta} \exp\left(\Theta - \frac{\Theta}{1 + c_1 Q(1 - x^2)}\right)(1 - c_1(1 - x^2)). \tag{3.426}$$

We choose a one term collocation method with $\psi_1(x) = \delta_D(x)$. Then, setting $\int_{-1}^{1} \psi_1(x) r(x)dx = 0$ gives

$$r(0) = -2c_1 + \frac{\delta}{Q\Theta} \exp\left(\Theta - \frac{\Theta}{1 + c_1 Q}\right)(1 - c_1) = 0. \tag{3.427}$$

We solve for δ and get

$$\delta = \frac{2c_1}{1 - c_1} \frac{Q\Theta}{e^{\Theta}} \exp\left(\frac{\Theta}{1 + c_1 Q}\right). \tag{3.428}$$

The maximum temperature of the approximation is given by $T_a^m = c_1$ and occurs at $x = 0$. A plot of T_a^m versus δ is given in Fig 3.24. For $\delta < \delta_{c1} \sim 0.2$, one low temperature solution exists. For $\delta_{c1} < \delta < \delta_{c2} \sim 0.84$, three solutions exist. For $\delta > \delta_{c2}$, one high temperature solution exists.

3.4.3 Unsteady solutions

Let us now study the effects of time-dependency on our problem. Let us begin with Eq. (3.412).

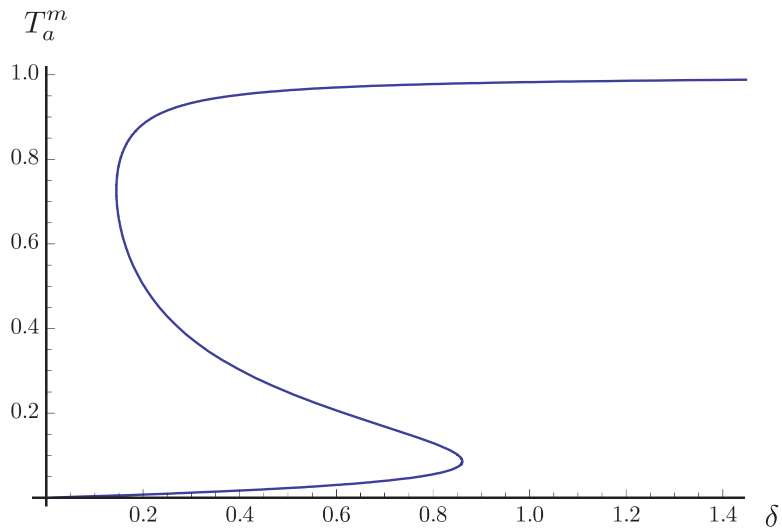


Figure 3.24: Plots of T_a^m versus δ, with $Q = 1$, $\Theta = 15$ from a one-term collocation approximate solution.

3.4.3.1 Linear stability

We will first consider small deviations from the steady solutions found earlier and see if those deviations grow or decay with time. This will allow us to make a definitive statement about the linear stability of those steady solutions.

3.4.3.1.1 Formulation First, recall that we have independently determined three exact numerical steady solutions to the time-independent version of Eq. (3.412). Let us call any of these $T_e(x)$. Note that by construction $T_e(x)$ satisfies the boundary conditions on T.

Let us subject a steady solution to a small perturbation and consider that to be our initial condition for an unsteady calculation. Take then

$$T(x, 0) = T_e(x) + \epsilon A(x), \qquad A(-1) = A(1) = 0, \tag{3.429}$$

$$A(x) = \mathcal{O}(1), \qquad 0 < \epsilon \ll 1. \tag{3.430}$$

Here, $A(x)$ is some function which satisfies the same boundary conditions as $T(x, t)$.

Now, let us assume that

$$T(x, t) = T_e(x) + \epsilon T'(x, t). \tag{3.431}$$

with

$$T'(x, 0) = A(x). \tag{3.432}$$

Here, T' is an $\mathcal{O}(1)$ quantity. We then substitute our Eq. (3.431) into Eq. (3.412) to get

$$\frac{\partial}{\partial t}\left(T_e(x) + \epsilon T'(x, t)\right) = \frac{1}{\mathfrak{D}} \frac{\partial^2}{\partial x^2}\left(T_e(x) + \epsilon T'(x, t)\right)$$

$$+\left(1-T_e(x)-\epsilon T'(x,t)\right)\exp\left(\frac{-\Theta}{1+QT_e(x)+Q\epsilon T'(x,t)}\right). \tag{3.433}$$

From here on we will understand that T_e is $T_e(x)$ and T' is $T'(x,t)$. Now, consider the exponential term:

$$\begin{aligned}
\exp\left(\frac{-\Theta}{1+QT_e+Q\epsilon T'}\right) &= \exp\left(\frac{-\Theta}{1+QT_e}\frac{1}{1+\frac{Q}{1+QT_e}\epsilon T'}\right), && (3.434)\\
&\sim \exp\left(\frac{-\Theta}{1+QT_e}\left(1-\frac{Q}{1+QT_e}\epsilon T'\right)\right), && (3.435)\\
&\sim \exp\left(\frac{-\Theta}{1+QT_e}\right)\exp\left(\frac{\epsilon\Theta Q}{(1+QT_e)^2}T'\right), && (3.436)\\
&\sim \exp\left(\frac{-\Theta}{1+QT_e}\right)\left(1+\frac{\epsilon\Theta Q}{(1+QT_e)^2}T'\right). && (3.437)
\end{aligned}$$

So, our Eq. (3.433) can be rewritten as

$$\begin{aligned}
\frac{\partial}{\partial t}\left(T_e+\epsilon T'\right) &= \frac{1}{\mathfrak{D}}\frac{\partial^2}{\partial x^2}\left(T_e+\epsilon T'\right)\\
&\quad +\left(1-T_e-\epsilon T'\right)\exp\left(\frac{-\Theta}{1+QT_e}\right)\left(1+\frac{\epsilon\Theta Q}{(1+QT_e)^2}T'\right),\\
&= \frac{1}{\mathfrak{D}}\frac{\partial^2}{\partial x^2}\left(T_e+\epsilon T'\right)\\
&\quad +\exp\left(\frac{-\Theta}{1+QT_e}\right)\left(1-T_e-\epsilon T'\right)\left(1+\frac{\epsilon\Theta Q}{(1+QT_e)^2}T'\right),\\
&= \frac{1}{\mathfrak{D}}\frac{\partial^2}{\partial x^2}\left(T_e+\epsilon T'\right)\\
&\quad +\exp\left(\frac{-\Theta}{1+QT_e}\right)\left((1-T_e)+\epsilon T'\left(-1+\frac{(1-T_e)\Theta Q}{(1+QT_e)^2}\right)+\mathcal{O}(\epsilon^2)\right),\\
&= \frac{\epsilon}{\mathfrak{D}}\frac{\partial^2 T'}{\partial x^2}+\underbrace{\frac{1}{\mathfrak{D}}\frac{\partial^2 T_e}{\partial x^2}+\exp\left(\frac{-\Theta}{1+QT_e}\right)(1-T_e)}_{=0}\\
&\quad +\exp\left(\frac{-\Theta}{1+QT_e}\right)\left(\epsilon T'\left(-1+\frac{(1-T_e)\Theta Q}{(1+QT_e)^2}\right)+\mathcal{O}(\epsilon^2)\right).
\end{aligned} \tag{3.438}$$

Now, we recognize the bracketed term as zero because $T_e(x)$ is constructed to satisfy the steady state equation. We also recognize that $\partial T_e(x)/\partial t = 0$. So, our equation reduces to, neglecting $\mathcal{O}(\epsilon^2)$ terms, and canceling ϵ

$$\frac{\partial T'}{\partial t} = \frac{1}{\mathfrak{D}}\frac{\partial^2 T'}{\partial x^2}+\exp\left(\frac{-\Theta}{1+QT_e}\right)\left(-1+\frac{(1-T_e)\Theta Q}{(1+QT_e)^2}\right)T'. \tag{3.439}$$

Equation (3.439) is a *linear* partial differential equation for $T'(x,t)$. It is of the form

$$\frac{\partial T'}{\partial t} = \frac{1}{\mathfrak{D}}\frac{\partial^2 T'}{\partial x^2} + B(x)T', \tag{3.440}$$

with

$$B(x) \equiv \exp\left(\frac{-\Theta}{1+QT_e(x)}\right)\left(-1+\frac{(1-T_e(x))\Theta Q}{(1+QT_e(x))^2}\right). \tag{3.441}$$

3.4.3.1.2 Separation of variables Let us use the standard technique of separation of variables to solve Eq. (3.440). We first assume that

$$T'(x,t) = H(x)K(t). \tag{3.442}$$

So, Eq. (3.440) becomes

$$H(x)\frac{dK(t)}{dt} = \frac{1}{\mathfrak{D}}K(t)\frac{d^2H(x)}{dx^2} + B(x)H(x)K(t), \tag{3.443}$$

$$-\frac{1}{K(t)}\frac{dK(t)}{dt} = -\frac{1}{\mathfrak{D}}\frac{1}{H(x)}\frac{d^2H(x)}{dx^2} - B(x) = \lambda. \tag{3.444}$$

Because the left side is a function of t and the right side is a function of x, the only way the two can be equal is if they are both the same constant. We will call that constant λ.

Now, Eq. (3.444) really contains two equations, the first of which is

$$\frac{dK(t)}{dt} + \lambda K(t) = 0. \tag{3.445}$$

This has solution

$$K(t) = C\exp(-\lambda t), \tag{3.446}$$

where C is some arbitrary constant. Clearly if $\lambda > 0$, this solution is stable, with time constant of relaxation $\tau = 1/\lambda$.

The second differential equation contained within Eq. (3.444) is

$$\underbrace{-\left(\frac{1}{\mathfrak{D}}\frac{d^2}{dx^2} + B(x)\right)}_{\mathcal{L}}H(x) = \lambda H(x). \tag{3.447}$$

This is of the classical eigenvalue form for a linear operator $\mathcal{L}$; that is $\mathcal{L}(H(x)) = \lambda H(x)$. We also must have

$$H(-1) = H(1) = 0, \tag{3.448}$$

to satisfy the spatially homogeneous boundary conditions on $T'(x,t)$.

This eigenvalue problem is difficult to solve because of the complicated nature of $B(x)$. Let us see how the solution would proceed in the limiting case of B as a constant. We will generalize later.

If B is a constant, we have

$$\frac{d^2H}{dx^2} + (B+\lambda)\mathfrak{D}H = 0, \qquad H(-1) = H(1) = 0. \tag{3.449}$$

The following mapping simplifies the problem somewhat:

$$y = \frac{x+1}{2}. \tag{3.450}$$

This takes our domain of $x \in [-1, 1]$ to $y \in [0, 1]$. By the chain rule

$$\frac{dH}{dx} = \frac{dH}{dy}\frac{dy}{dx} = \frac{1}{2}\frac{dH}{dy}.$$

So

$$\frac{d^2H}{dx^2} = \frac{1}{4}\frac{d^2H}{dy^2}.$$

So, our eigenvalue problem transforms to

$$\frac{d^2H}{dy^2} + 4\mathfrak{D}(B+\lambda)H = 0, \qquad H(0) = H(1) = 0. \tag{3.451}$$

This has solution

$$H(y) = C_1 \cos\left((\sqrt{4\mathfrak{D}(B+\lambda)}\ )y\right) + C_2 \sin\left((\sqrt{4\mathfrak{D}(B+\lambda)})\ y\right). \tag{3.452}$$

At $y = 0$ we have then

$$H(0) = 0 = C_1(1) + C_2(0), \tag{3.453}$$

so $C_1 = 0$. Thus,

$$H(y) = C_2 \sin\left((\sqrt{4\mathfrak{D}(B+\lambda)})\ y\right). \tag{3.454}$$

At $y = 1$, we have the other boundary condition:

$$H(1) = 0 = C_2 \sin\left((\sqrt{4\mathfrak{D}(B+\lambda)})\right). \tag{3.455}$$

Because $C_2 \neq 0$ to avoid a trivial solution, we must require that

$$\sin\left((\sqrt{4\mathfrak{D}(B+\lambda)})\right) = 0. \tag{3.456}$$

For this to occur, the argument of the sin function must be an integer multiple of π:

$$\sqrt{4\mathfrak{D}(B+\lambda)} = n\pi, \qquad n = 1, 2, 3, \ldots \tag{3.457}$$

Thus,

$$\lambda = \frac{n^2\pi^2}{4\mathfrak{D}} - B. \tag{3.458}$$

We need $\lambda > 0$ for stability. For large n and $\mathfrak{D} > 0$, we have stability. Depending on the value of B, low n, which corresponds to low frequency modes, could be unstable.

3.4.3.1.3 Numerical eigenvalue solution Let us return to the full problem where $B = B(x)$. Let us solve the eigenvalue problem via the method of finite differences. Let us take our domain $x \in [-1, 1]$ and discretize into N points with

$$\Delta x = \frac{2}{N-1}, \qquad x_i = (i-1)\Delta x - 1. \tag{3.459}$$

Note that when $i = 1$, $x_i = -1$, and when $i = N$, $x_i = 1$. Let us define $B(x_i) = B_i$ and $H(x_i) = H_i$.

We can rewrite Eq. (3.447) as

$$\frac{d^2 H(x)}{dx^2} + \mathfrak{D}(B(x) + \lambda)H(x) = 0, \qquad H(-1) = H(1) = 0. \tag{3.460}$$

Now, let us apply an appropriate equation at each node. At $i = 1$, we must satisfy the boundary condition so

$$H_1 = 0. \tag{3.461}$$

At $i = 2$, we discretize Eq. (3.460) with a second order central difference to obtain

$$\frac{H_1 - 2H_2 + H_3}{\Delta x^2} + \mathfrak{D}(B_2 + \lambda)H_2 = 0. \tag{3.462}$$

We get a similar equation at a general interior node i:

$$\frac{H_{i-1} - 2H_i + H_{i+1}}{\Delta x^2} + \mathfrak{D}(B_i + \lambda)H_i = 0. \tag{3.463}$$

At the $i = N - 1$ node, we have

$$\frac{H_{N-2} - 2H_{N-1} + H_N}{\Delta x^2} + \mathfrak{D}(B_{N-1} + \lambda)H_{N-1} = 0. \tag{3.464}$$

At the $i = N$ node, we have the boundary condition

$$H_N = 0. \tag{3.465}$$

These represent a linear tridiagonal system of equations of the form

$$\underbrace{\begin{pmatrix} \left(\frac{2}{\mathfrak{D}\Delta x^2} - B_2\right) & -\frac{1}{\mathfrak{D}\Delta x^2} & 0 & 0 & \dots & 0 \\ -\frac{1}{\mathfrak{D}\Delta x^2} & \left(\frac{2}{\mathfrak{D}\Delta x^2} - B_3\right) & -\frac{1}{\mathfrak{D}\Delta x^2} & 0 & \dots & 0 \\ 0 & -\frac{1}{\mathfrak{D}\Delta x^2} & \ddots & \ddots & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \dots \\ \vdots & \vdots & \dots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & \dots & \ddots & \ddots \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} H_2 \\ H_3 \\ \vdots \\ \vdots \\ \vdots \\ H_{N-1} \end{pmatrix}}_{\mathbf{h}} = \lambda \underbrace{\begin{pmatrix} H_2 \\ H_3 \\ \vdots \\ \vdots \\ \vdots \\ H_{N-1} \end{pmatrix}}_{\mathbf{h}} \tag{3.466}$$

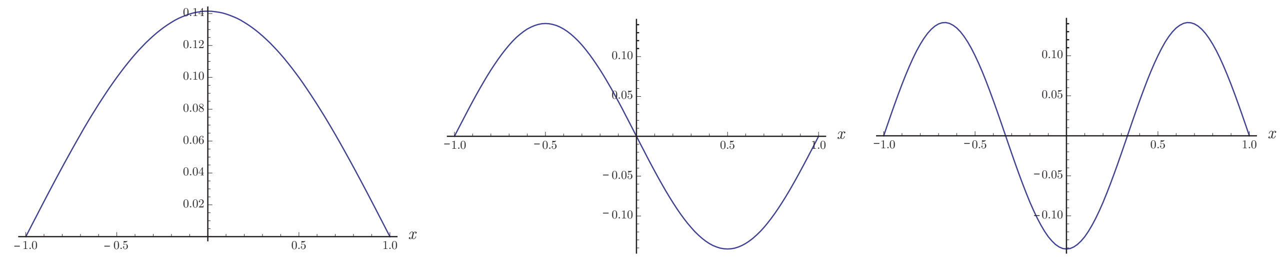


Figure 3.25: Plots of first, second, and third harmonic modes of eigenfunctions versus x, with $\delta = 0.4$, $Q = 1$, $\Theta = 15$, low temperature steady solution $T_e(x)$.

This is of the classical linear algebraic eigenvalue form $\mathbf{L} \cdot \mathbf{h} = \lambda \mathbf{h}$. All one need do is discretize and find the eigenvalues of the matrix $\mathbf{L}$. These will be good approximations to the eigenvalues of the differential operator $\mathcal{L}$. The eigenvectors of $\mathbf{L}$ will be good approximations of the eigenfunctions of $\mathcal{L}$. To get a better approximation, one need only reduce Δx.

Note because the matrix $\mathbf{L}$ is symmetric, the eigenvalues are guaranteed real, and the eigenvectors are guaranteed orthogonal. This is actually a consequence of the original problem being in Sturm-Liouville form, which is guaranteed to be self-adjoint with real eigenvalues and orthogonal eigenfunctions.

Low temperature transients For our case of $\delta = 0.4$, $Q = 1$, $\Theta = 15$ (so $\mathfrak{D} = 87173.8$), we can calculate the stability of the low temperature solution. Choosing $N = 101$ points to discretize the domain, we find a set of eigenvalues. They are all positive, so the solution is stable. The first few are

$$\lambda = 0.0000232705, 0.000108289, 0.000249682, 0.000447414, \ldots. \tag{3.467}$$

The first few eigenvalues can be approximated by inert theory with $B(x) = 0$, see Eq. (3.458):

$$\lambda \sim \frac{n^2\pi^2}{4\mathfrak{D}} = 0.0000283044, 0.000113218, 0.00025474, 0.00045287, \ldots. \tag{3.468}$$

The first eigenvalue is associated with the longest time scale $\tau = 1/0.0000232705 = 42972.9$ and a low frequency mode, whose shape is given by the associated eigenvector, plotted in Fig. 3.25. This represents the fundamental mode, also known as the first harmonic mode. Shown also in Fig. 3.25 are the second and third harmonic modes. Note that harmonic modes are not closely related to harmonic functions that were defined in Sec. 2.2.3 as solutions to Laplace's equation.

Intermediate temperature transients For the intermediate temperature solution with $T^m = 0.417$, we find the first few eigenvalues to be

$$\lambda = -0.0000383311, 0.0000668221, 0.000209943, \ldots \tag{3.469}$$

Except for the first, all the eigenvalues are positive. The first eigenvalue of $\lambda = -0.0000383311$ is associated with an *unstable* fundamental mode. This mode is also known as the first harmonic mode. We plot the first three harmonic modes in Fig. 3.26.

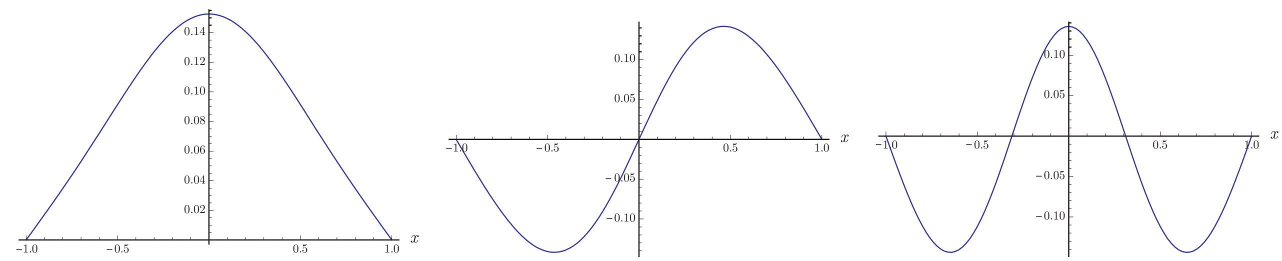


Figure 3.26: Plot of first, second, and third harmonic modes of eigenfunctions versus x, with $\delta = 0.4$, $Q = 1$, $\Theta = 15$, intermediate temperature steady solution $T_e(x)$.

High temperature transients For the high temperature solution with $T^m = 0.987$, we find the first few eigenvalues to be

$$\lambda = 0.000146419, 0.00014954, 0.000517724, \ldots \tag{3.470}$$

All the eigenvalues are positive, so all modes are stable. We plot the first three modes in Fig. 3.27.

3.4.3.2 Full transient solution

We can get a full transient solution to Eq. (3.412) with numerical methods. We omit details of such numerical methods, which can be found in standard texts.

3.4.3.2.1 Low temperature solution For our case of $\delta = 0.4$, $Q = 1$, $\Theta = 15$ (so $\mathfrak{D} = 87173.8$), we show a plot of the full transient solution in Fig. 3.28. Also seen in

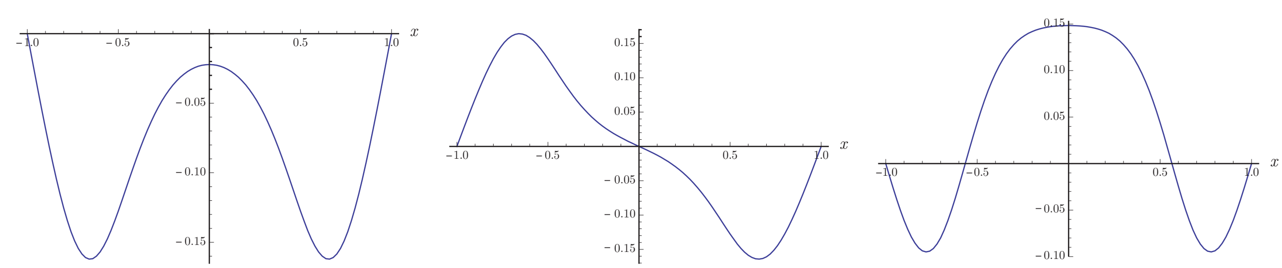


Figure 3.27: Plot of first, second, and third harmonic modes of eigenfunctions versus x, with $\delta = 0.4$, $Q = 1$, $\Theta = 15$, high temperature steady solution $T_e(x)$.

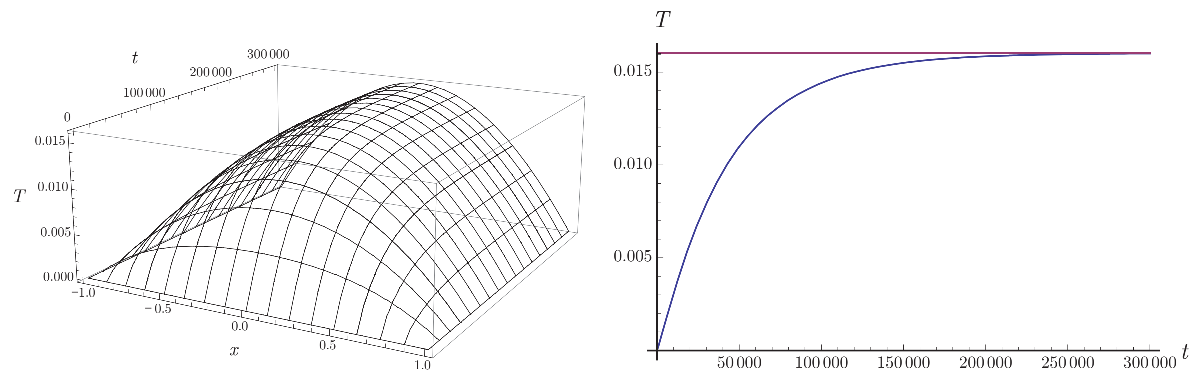


Figure 3.28: Plot of $T(x,t)$ and plot of $T(0,t)$ along with the long time exact low temperature centerline solution, $T_e(0)$, with $\delta = 0.4$, $Q = 1$, $\Theta = 15$.

Fig. 3.28 is that the centerline temperature $T(0,t)$ relaxes to the long time value predicted by the low temperature steady solution:

$$\lim_{t\to\infty} T(0,t) = 0.016. \tag{3.471}$$

3.4.3.2.2 High temperature solution We next select a value of $\delta = 1.2 > \delta_c$. This should induce transition to a high temperature solution. We maintain $\Theta = 15$, $Q = 1$. We get $\mathfrak{D} = \delta e^{\Theta}/\Theta/Q = 261521$. The full transient solution is shown in Fig. 3.29. Also shown

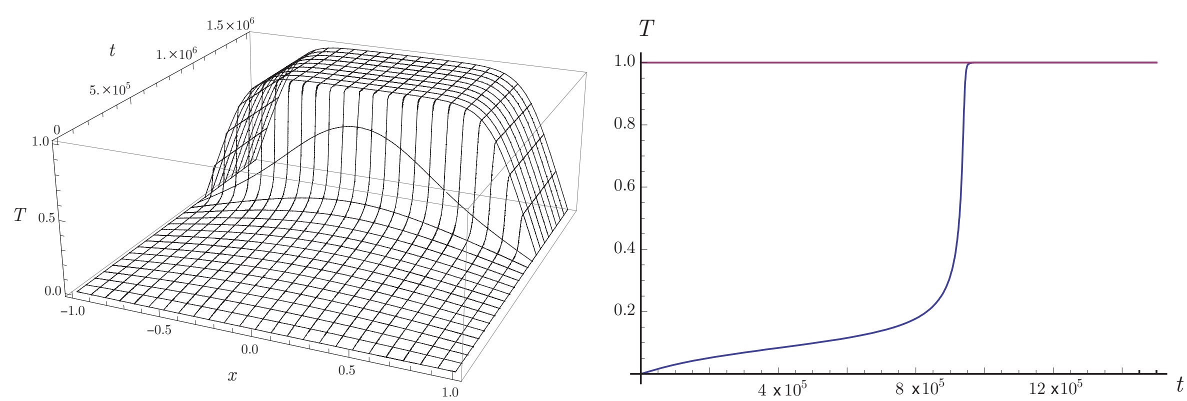


Figure 3.29: Plot of $T(x,t)$ and plot of $T(0,t)$ along with the long time exact low temperature centerline solution, $T_e(0)$, with $\delta = 1.2$, $Q = 1$, $\Theta = 15$.

in Fig. 3.29 is the centerline temperature $T(0,t)$. We see it relaxes to the long time value predicted by the high temperature steady solution:

$$\lim_{t\to\infty} T(0,t) = 0.9999185. \tag{3.472}$$

It is clearly seen that there is a rapid acceleration of the reaction for $t \sim 10^6$. This compares with the prediction of the induction time from the infinite Damköhler number, $\mathfrak{D} \to \infty$, thermal explosion theory of explosion to occur when

$$t \to \frac{e^{\Theta}}{Q\Theta} = \frac{e^{15}}{(1)(15)} = 2.17934 \times 10^5. \tag{3.473}$$

The estimate under-predicts the value by a factor of five. This is likely due to 1) cooling of the domain due to the low temperature boundaries at $x = \pm 1$, and 2) effects of finite activation energy.

3.5 Nonlinear separation of variables

We adopt here some presentation and an example first given by Powers and Sen.[16] We close this chapter by extending the notion of separation of variables to nonlinear systems. This will allow us to illustrate some important principles:

- Solution of a partial differential equation can always be cast in terms of solving an infinite set of ordinary differential equations.
- Approximate solution of a partial differential equation can be cast in terms of solving a finite set of ordinary differential equations.
- A linear partial differential equation induces an uncoupled system of linear ordinary differential equations.
- A nonlinear partial differential equation induces a coupled set of nonlinear ordinary differential equations.

There are many viable methods to represent a partial differential equation as system of ordinary differential equations. Among them are methods in which one or more dependent and independent variables are discretized; important examples are the finite difference and finite element methods, which will not be considered here. Another key method involves projecting the dependent variable onto a set of basis functions and truncating this infinite series. We will illustrate such a process here with an example involving a projection incorporating the method of weighted residuals.

Example 3.13

Convert the nonlinear partial differential equation, initial and boundary conditions

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left((1+\epsilon T)\frac{\partial T}{\partial x}\right), \qquad T(x,0) = x - x^2, \quad T(0,t) = T(1,t) = 0, \tag{3.474}$$

[16] J. M. Powers and M. Sen, *Mathematical Methods in Engineering*, Cambridge University Press, New York, 2015. See Section 9.10.

to a system of ordinary differential equations using a Galerkin[17] projection method and find a two-term approximation.

Equation (3.474) is an extension of the heat equation, Eq. (1.81), when one modifies Fourier's law, Eq. (1.77), to allow for a variation of thermal conductivity k with temperature T. Omitting details, it can be shown to describe the time-evolution of a spatial temperature field in a one-dimensional geometry with material properties which have weak temperature-dependency when $0 < \epsilon \ll 1$. The boundary conditions are homogeneous, and the initial condition is symmetric about $x = 1/2$. We can think of T as temperature, x as distance, and t as time, all of which have been suitably scaled. For $\epsilon = 0$, the material properties are constant, and the equation is linear; otherwise, the material properties are temperature-dependent, and the equation is nonlinear due to the product $T\partial T/\partial x$. We can use the product rule to rewrite Eq. (3.474) as

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \epsilon T \frac{\partial^2 T}{\partial x^2} + \epsilon \left(\frac{\partial T}{\partial x} \right)^2. \tag{3.475}$$

Now let us assume that $T(x,t)$ can be approximated in an N-term series by

$$T(x,t) = \sum_{n=1}^{N} \alpha_n(t) \varphi_n(x). \tag{3.476}$$

This amounts to a separation of variables, which we note does not require that our system be linear. We presume the exact solution is approached as $N \to \infty$. We can consider $\alpha_n(t)$ to be a set of N time-dependent amplitudes which modulate each spatial basis function, $\varphi_n(x)$. For convenience, we will insist that the spatial basis functions satisfy the spatial boundary conditions $\varphi_n(0) = \varphi_n(1) = 0$ as well as an orthonormality condition for $x \in [0,1]$:

$$\langle \varphi_n, \varphi_m \rangle = \delta_{nm}. \tag{3.477}$$

At the initial state, we have

$$T(x,0) = x - x^2 = \sum_{n=1}^{N} \alpha_n(0) \varphi_n(x). \tag{3.478}$$

The terms $\alpha_n(0)$ are simply the constants in the Fourier series expansion of $x - x^2$:

$$\alpha_n(0) = \langle \varphi_n, (x - x^2) \rangle. \tag{3.479}$$

The partial differential equation expands as

$$\underbrace{\sum_{n=1}^{N} \frac{d\alpha_n}{dt} \varphi_n(x)}_{\partial T/\partial t} = \underbrace{\sum_{n=1}^{N} \alpha_n(t) \frac{d^2\varphi_n}{dx^2}}_{\partial^2 T/\partial x^2} + \epsilon \underbrace{\left(\sum_{n=1}^{N} \alpha_n(t) \varphi_n(x) \right)}_{T} \underbrace{\left(\sum_{n=1}^{N} \alpha_n(t) \frac{d^2\varphi_n}{dx^2} \right)}_{\partial^2 T/\partial x^2} + \epsilon \underbrace{\left(\sum_{n=1}^{N} \alpha_n(t) \frac{d\varphi_n}{dx} \right)^2}_{(\partial T/\partial x)^2}, \tag{3.480}$$

We change one of the dummy indices in each of the nonlinear terms from n to m and rearrange to find

$$\sum_{n=1}^{N} \frac{d\alpha_n}{dt} \varphi_n(x) = \sum_{n=1}^{N} \alpha_n(t) \frac{d^2\varphi_n}{dx^2} + \epsilon \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n(t) \alpha_m(t) \left(\varphi_n(x) \frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx} \frac{d\varphi_m}{dx} \right). \tag{3.481}$$

[17]Boris Grigoryevich Galerkin, 1871-1945, Soviet engineer and mathematician.

Next, for the Galerkin procedure, one selects the weighting functions $\psi_l(x)$ to be the basis functions $\varphi_l(x)$ and takes the inner product of the equation with the weighting functions, yielding

$$\left\langle \varphi_l(x), \sum_{n=1}^{N} \frac{d\alpha_n}{dt}\varphi_n(x) \right\rangle = \left\langle \varphi_l(x), \sum_{n=1}^{N} \alpha_n(t)\frac{d^2\varphi_n}{dx^2} + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n(t)\alpha_m(t)\left(\varphi_n(x)\frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx}\frac{d\varphi_m}{dx}\right)\right\rangle. \tag{3.482}$$

$$\sum_{n=1}^{N} \frac{d\alpha_n}{dt}\underbrace{\langle \varphi_l(x), \varphi_n(x)\rangle}_{\delta_{ln}} = \left\langle \varphi_l(x), \sum_{n=1}^{N} \alpha_n(t)\frac{d^2\varphi_n}{dx^2} + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n(t)\alpha_m(t)\left(\varphi_n(x)\frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx}\frac{d\varphi_m}{dx}\right)\right\rangle. \tag{3.483}$$

Because of the orthonormality of the basis functions, the left side has obvious simplifications, yielding

$$\frac{d\alpha_l}{dt} = \left\langle \varphi_l(x), \sum_{n=1}^{N} \alpha_n(t)\frac{d^2\varphi_n}{dx^2} + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n(t)\alpha_m(t)\left(\varphi_n(x)\frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx}\frac{d\varphi_m}{dx}\right)\right\rangle. \tag{3.484}$$

The right side can also be simplified via a complicated set of integration by parts and application of boundary conditions. If we further select $\varphi_n(x)$ to be an eigenfunction of d^2/dx^2, the first term on the right side will simplify considerably, though this choice is not required. Let us here take that choice, thus requiring

$$\frac{d^2}{dx^2}\varphi_n(x) = \lambda_n\varphi_n(x). \tag{3.485}$$

Then we expand Eq. (3.484) as follows

$$\begin{aligned}
\frac{d\alpha_l}{dt} &= \sum_{n=1}^{N} \alpha_n(t)\left\langle \varphi_l(x), \frac{d^2\varphi_n}{dx^2}\right\rangle + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n(t)\alpha_m(t)\left\langle \varphi_l(x), \left(\varphi_n(x)\frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx}\frac{d\varphi_m}{dx}\right)\right\rangle, && (3.486)\\
&= \sum_{n=1}^{N} \alpha_n(t)\underbrace{\langle \varphi_l(x), \lambda_n\varphi_n\rangle}_{\lambda_n\delta_{ln}} + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n(t)\alpha_m(t)\left\langle \varphi_l(x), \left(\varphi_n(x)\frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx}\frac{d\varphi_m}{dx}\right)\right\rangle,\\
&= \sum_{n=1}^{N} \lambda_n\alpha_n(t)\delta_{ln} + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n(t)\alpha_m(t)\left\langle \varphi_l(x), \left(\varphi_n(x)\frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx}\frac{d\varphi_m}{dx}\right)\right\rangle, && (3.487)\\
&= \lambda_l\alpha_l(t) + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n(t)\alpha_m(t)\underbrace{\left\langle \varphi_l(x), \left(\varphi_n(x)\frac{d^2\varphi_m}{dx^2} + \frac{d\varphi_n}{dx}\frac{d\varphi_m}{dx}\right)\right\rangle}_{C_{lnm}}, && (3.488)\\
&= \lambda_l\alpha_l(t) + \epsilon \sum_{n=1}^{N}\sum_{m=1}^{N} C_{lnm}\alpha_n(t)\alpha_m(t). && (3.489)
\end{aligned}$$

Here C_{lmn} is set of constants obtained after forming the various integrals of the basis functions and their derivatives. Note that for the limit in which nonlinear effects are negligible, $\epsilon \to 0$, we get a set of N *uncoupled* linear ordinary differential equations for the time-dependent amplitudes. *Thus for the linear limit, the time-evolution of each mode is independent of the other modes.* In contrast, *for $\epsilon \neq 0$, the system of N ordinary differential equations for amplitude time-evolution is fully coupled and nonlinear.*

In any case, this all serves to remove the explicit dependency on x, thus yielding a system of N ordinary differential equations of the classical form of a nonlinear dynamical system:

$$\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{f}(\boldsymbol{\alpha}). \tag{3.490}$$

where $\boldsymbol{\alpha}$ is a vector of length N, and $\mathbf{f}$ is in general a nonlinear function of $\boldsymbol{\alpha}$. We summarize some important ideas for this equations of this type. For further background, one can consult Powers and Sen.[18]

- The system, Eq. (3.490), is in equilibrium when

$$\mathbf{f}(\boldsymbol{\alpha}) = \mathbf{0}. \tag{3.491}$$

  This constitutes a system of nonlinear algebraic equations. Because the system is nonlinear, existence and uniqueness of equilibria is not guaranteed. Thus, we could expect to find no roots, one root, or multiple roots, depending on $\mathbf{f}(\boldsymbol{\alpha})$. Equilibrium point are also known as *critical points* or *fixed points*. We distinguish an equilibrium point from a general point by an overline, thus, taking equilibrium points to be $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}}$, and requiring that

$$\mathbf{f}(\overline{\boldsymbol{\alpha}}) = \mathbf{0}. \tag{3.492}$$

- Stability of each equilibrium can be ascertained by a local linear analysis in the neighborhood of each equilibrium. Local Taylor[19] series analysis of Eq. (3.490) in such a neighborhood allows it to be rewritten as

$$\frac{d\boldsymbol{\alpha}}{dt} = \underbrace{\mathbf{f}(\overline{\boldsymbol{\alpha}})}_{\mathbf{0}} + \left.\frac{\partial \mathbf{f}}{\partial \boldsymbol{\alpha}}\right|_{\boldsymbol{\alpha}=\overline{\boldsymbol{\alpha}}} \cdot (\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}) + \ldots. \tag{3.493}$$

  Take the constant Jacobian of $\mathbf{f}$ evaluated at $\overline{\boldsymbol{\alpha}}$ as $\mathbf{J}$:

$$\mathbf{J} = \left.\frac{\partial \mathbf{f}}{\partial \boldsymbol{\alpha}}\right|_{\boldsymbol{\alpha}=\overline{\boldsymbol{\alpha}}} \tag{3.494}$$

  Use this and the fact that $\overline{\boldsymbol{\alpha}}$ is a constant to rewrite Eq. (3.493) as

$$\frac{d}{dt}\left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}\right) = \mathbf{J} \cdot \left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}\right). \tag{3.495}$$

- With $\mathbf{c}$ as an arbitrary constant, this linear system has an exact solution in terms of the matrix exponential:

$$\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}} = \mathbf{c} \cdot e^{\mathbf{J}t}. \tag{3.496}$$

  With $\mathbf{S}$ as the matrix whose columns are populated by the eigenvectors of $\mathbf{J}$ and $\boldsymbol{\Lambda}$ as the diagonal matrix whose diagonal is populated by the eigenvalues of $\mathbf{J}$, taking care to ensure the order is such that the correct eigenvalues correspond to the correct eigenvectors, the solution can be recast as

$$\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}} = \mathbf{c} \cdot \mathbf{S} \cdot e^{\boldsymbol{\Lambda}t} \cdot \mathbf{S}^{-1}. \tag{3.497}$$

[18] J. M. Powers and M. Sen, *Mathematical Methods in Engineering*, Cambridge University Press, New York, 2015. See Sections 9.3-9.6.

[19] Brook Taylor, 1685-1731, English mathematician and artist, Cambridge-educated, published on capillary action, magnetism, and thermometers, adjudicated the dispute between Newton and Leibniz over priority in developing calculus, contributed to the method of finite differences, invented integration by parts, name ascribed to Taylor series of which variants were earlier discovered by Gregory, Newton, Leibniz, Johann Bernoulli, and de Moivre.

Obviously, the eigenvalues of $\mathbf{J}$ determine that stability of each equilibrium. For stability, the real parts of each eigenvalue cannot be positive. A *source* has all real parts positive. A *sink* has all real parts negative. A *center* has all eigenvalues purely imaginary. A *saddle* has some real parts positive and some negative.

Returning to our problem, we select our orthonormal basis functions as the eigenfunctions of d^2/dx^2 that also satisfy the appropriate boundary conditions,

$$\varphi_n(x) = \sqrt{2}\sin((2n-1)\pi x), \qquad n = 1, \dots, N. \tag{3.498}$$

Because of the symmetry of our system about $x = 1/2$, it can be shown that only odd multiples of πx are present in the trigonometric sin approximation. Had we chosen an initial condition without such symmetry, we would have required both even and odd powers. We then apply the necessary Fourier expansion to find $\alpha_n(0)$, perform a detailed analysis of all of the necessary inner products, select $N = 2$, and arrive at the following nonlinear system of ordinary differential equations for the evolution of the time-dependent amplitudes:

$$\frac{d\alpha_1}{dt} = -\pi^2\alpha_1 + \sqrt{2}\pi\epsilon\left(-\frac{4}{3}\alpha_1^2 + \frac{8}{15}\alpha_2\alpha_1 - \frac{36}{35}\alpha_2^2\right), \qquad \alpha_1(0) = \frac{4\sqrt{2}}{\pi^3}, \tag{3.499}$$

$$\frac{d\alpha_2}{dt} = -9\pi^2\alpha_2 + \sqrt{2}\pi\epsilon\left(\frac{12}{5}\alpha_1^2 - \frac{648}{35}\alpha_2\alpha_1 - 4\alpha_2^2\right), \qquad \alpha_2(0) = \frac{4\sqrt{2}}{27\pi^3}. \tag{3.500}$$

The system is in equilibrium at all points (α_1, α_2) where

$$f_1(\alpha_1, \alpha_2) = -\pi^2\alpha_1 + \sqrt{2}\pi\epsilon\left(-\frac{4}{3}\alpha_1^2 + \frac{8}{15}\alpha_2\alpha_1 - \frac{36}{35}\alpha_2^2\right) = 0, \tag{3.501}$$

$$f_2(\alpha_1, \alpha_2) = -9\pi^2\alpha_2 + \sqrt{2}\pi\epsilon\left(\frac{12}{5}\alpha_1^2 - \frac{648}{35}\alpha_2\alpha_1 - 4\alpha_2^2\right) = 0. \tag{3.502}$$

The general form of the Jacobian matrix $\mathbf{J}$ is

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f_1}{\partial \alpha_1} & \frac{\partial f_1}{\partial \alpha_2} \\ \frac{\partial f_2}{\partial \alpha_1} & \frac{\partial f_2}{\partial \alpha_2} \end{pmatrix} = \begin{pmatrix} -\pi^2 - \sqrt{2}\pi\epsilon\left(\frac{8}{3}\alpha_1 - \frac{8}{15}\alpha_2\right) & \sqrt{2}\pi\epsilon\left(\frac{8}{15}\alpha_1 - \frac{72}{35}\alpha_2\right) \\ \sqrt{2}\pi\epsilon\left(\frac{24}{5}\alpha_1 - \frac{648}{35}\alpha_2\right) & -9\pi^2 - \sqrt{2}\pi\epsilon\left(\frac{648}{35}\alpha_1 + 8\alpha_2\right) \end{pmatrix}. \tag{3.503}$$

For $\epsilon = 1/5$, we find four sets of equilibria, (α_1, α_2), and their eigenvalues, (λ_1, λ_2), associated with local values of their Jacobian matrix, $\mathbf{J}$, all given as follows

$$(\alpha_1, \alpha_2) = (0, 0), \qquad (\lambda_1, \lambda_2) = (-\pi^2, -9\pi^2), \qquad \text{sink}, \tag{3.504}$$

$$(\alpha_1, \alpha_2) = (-4.53, -6.04), \qquad (\lambda_1, \lambda_2) = (-17.4, 44.2), \qquad \text{saddle}, \tag{3.505}$$

$$(\alpha_1, \alpha_2) = (-8.78, -2.54), \qquad (\lambda_1, \lambda_2) = (9.70, 73.7), \qquad \text{source}, \tag{3.506}$$

$$(\alpha_1, \alpha_2) = (-5.15, 3.46), \qquad (\lambda_1, \lambda_2) = (-43.3, 18.6), \qquad \text{saddle}. \tag{3.507}$$

When $\epsilon = 0$, and because we selected our basis functions to be the eigenfunctions of d^2/dx^2, we see the system is linear and uncoupled with exact solution

$$\alpha_1(t) = \frac{4\sqrt{2}}{\pi^3}e^{-\pi^2 t}, \tag{3.508}$$

$$\alpha_2(t) = \frac{4\sqrt{2}}{27\pi^3}e^{-9\pi^2 t}. \tag{3.509}$$

Thus, for $\epsilon = 0$ the two-term approximation is

$$T(x,t) \approx \frac{8}{\pi^3}e^{-\pi^2 t}\sin(\pi x) + \frac{8}{27\pi^3}e^{-9\pi^2 t}\sin(3\pi x). \tag{3.510}$$

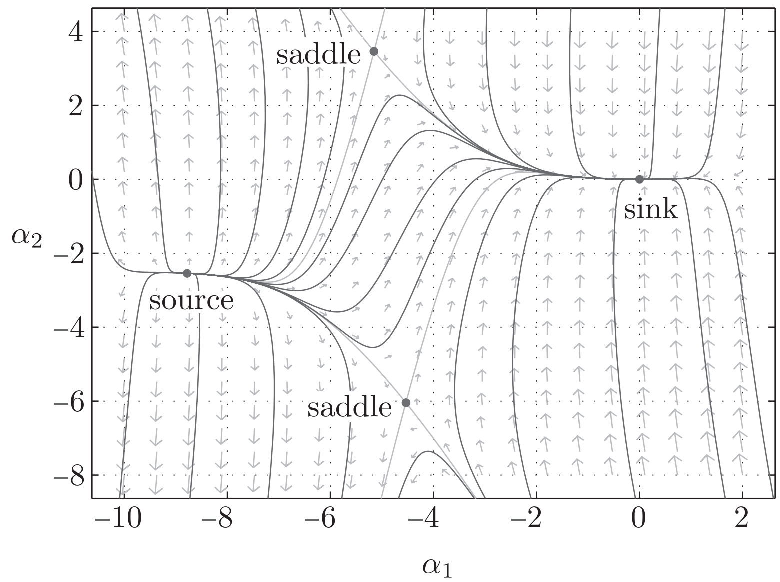


Figure 3.30: Phase plane dynamics of $N = 2$ amplitudes of spatial modes of solution to a weakly nonlinear heat equation.

For $\epsilon \neq 0$, numerical solution is required. We do so for $\epsilon = 1/5$ and plot the phase plane dynamics in Fig. 3.30 for arbitrary initial conditions. Many initial conditions lead one to the finite sink at $(0, 0)$. It is likely that the dynamics are also influenced by equilibria at infinity, not shown here. One can show that the solutions in the neighborhood of the sink are the most relevant to the underlying physical problem.

We plot results of $\alpha_1(t)$, $\alpha_2(t)$ for our initial conditions in Fig. 3.31. We see the first mode has significantly more amplitude than the second mode. Both modes are decaying rapidly to the sink at $(0, 0)$. The $N = 2$ solution with full time and space dependency is

$$T(x, t) \approx \alpha_1(t)\sqrt{2}\sin(\pi x) + \alpha_2(t)\sqrt{2}\sin(3\pi x), \tag{3.511}$$

and is plotted in Fig. 3.32.

Problems

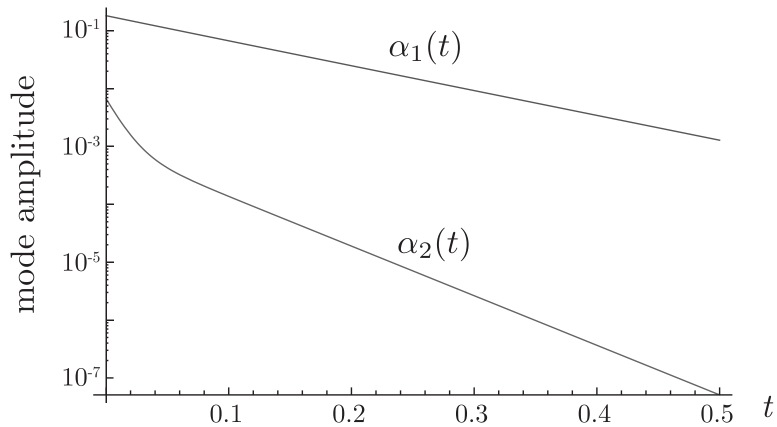


Figure 3.31: Evolution of $N = 2$ amplitudes of spatial modes of solution to a weakly nonlinear heat equation.

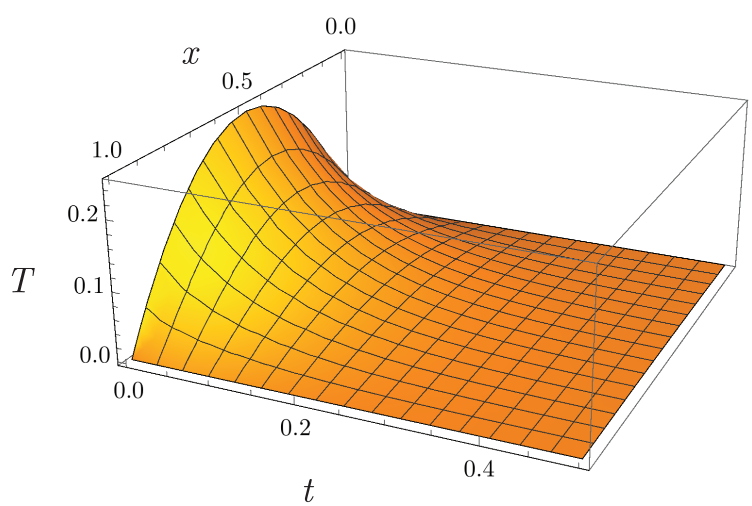


Figure 3.32: $T(x,t)$ from $N = 2$ term Galerkin projection for a weakly nonlinear heat equation.

Chapter 4

One-dimensional waves

see Mei, Chapters 1, 3.

Here we consider further aspects of one-dimensional wave propagation. We build on notions explored in Sec. 1.1. We will not focus on those one-dimensional waves which propagate in two modes, left and right, such as studied in Secs. 2.2.1, 3.2.

4.1 One-dimensional conservation laws

As described by LeVeque[1] the proper way to arrive at differential equations arising from physical conservation principles is to use a more primitive form of the conservation laws, expressed in terms of integrals of conservative form quantities balanced by fluxes and source terms of those quantities. From such primitive forms, we shall often deduce continuum differential equations; in certain cases, we will admit discontinuous solutions.

4.1.1 Multiple conserved variables

Consider the scenario of Fig. 4.1. In both Fig. 4.1a,b, we have a volume bounded in the x direction by x_1 and x_2. If $\mathbf{q}$ is a set of variables representing some quantity which is conserved, and $\mathbf{f}(\mathbf{q})$ is the flux of $\mathbf{q}$ (e.g. for mass conservation, density ρ is a conserved variable and ρu is the mass flux), and $\mathbf{s}(\mathbf{q})$ is an internal source term, then the primitive form of the conservation law can be written as

$$\frac{d}{dt}\int_{x_1}^{x_2} \mathbf{q}(x,t)\ dx = \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)) + \int_{x_1}^{x_2} \mathbf{s}(\mathbf{q}(x,t))\ dx. \tag{4.1}$$

Here, we have considered flow into and out of a one-dimensional box for $x \in [x_1, x_2]$. In Fig. 4.1a, the state variables $\mathbf{q}$ are allowed to have discontinuous jumps, while in Fig. 4.1b, the state variables $\mathbf{q}$ are continuous. For problems with embedded discontinuous jumps, the

[1] R. J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhäuser, Basel, 1992.

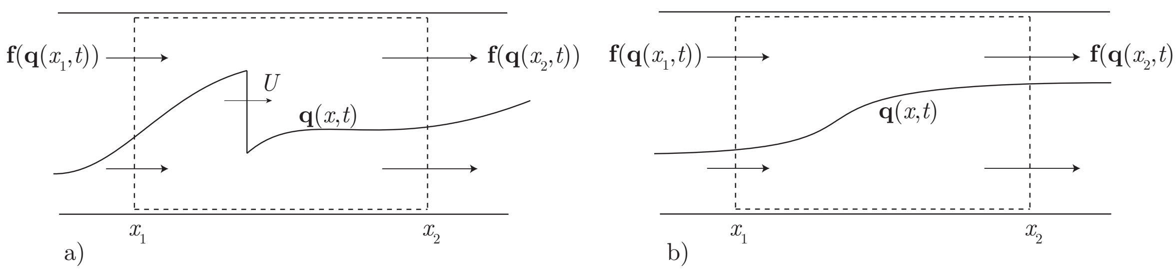


Figure 4.1: Schematic of general flux $\mathbf{f}$ into and out of finite volume in which general variable $\mathbf{q}$ evolves.

mean value theorem does not work because a local value of $\mathbf{q}$ depends on how one lets x_1 approach x_2. Because of this, we cannot use the limiting process employed in Sec. 1.1 to arrive at a partial differential equation. One cannot replace the mean value of $\mathbf{q}$ by its local value, and one cannot cast the conservation law in terms of a partial differential equation when there are embedded discontinuous jumps. If we assume there is a discontinuity in the region $x \in [x_1, x_2]$ propagating at speed U, we can find the Cauchy[2] principal value of the integral by splitting it into the form

$$\begin{aligned} \frac{d}{dt}\int_{x_1}^{x_1+Ut^-} \mathbf{q}(x,t)\ dx + \frac{d}{dt}\int_{x_1+Ut^+}^{x_2} \mathbf{q}(x,t)\ dx \\ = \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)) + \int_{x_1}^{x_2} \mathbf{s}(\mathbf{q}(x,t))\ dx. \end{aligned} \tag{4.2}$$

Here, $x_1 + Ut^-$ lies just before the discontinuity and $x_1 + Ut^+$ lies just past the discontinuity. Using Leibniz's rule, we get

$$\begin{aligned} \mathbf{q}(x_1+Ut^-,t)U - 0 + \int_{x_1}^{x_1+Ut^-} \frac{\partial \mathbf{q}}{\partial t}\ dx + 0 - \mathbf{q}(x_1+Ut^+,t)U + \int_{x_1+Ut^+}^{x_2} \frac{\partial \mathbf{q}}{\partial t}\ dx \\ = \mathbf{f}(\mathbf{q}(x_1,t)) - \mathbf{f}(\mathbf{q}(x_2,t)) + \int_{x_1}^{x_2} \mathbf{s}(\mathbf{q}(x,t))\ dx. \end{aligned} \tag{4.3}$$

Now, if we assume that $x_2 - x_1 \to 0$ and that on either side of the discontinuity the volume of integration is sufficiently small so that the time and space variation of $\mathbf{q}$ is negligibly small, we get

$$\mathbf{q}(x_1)U - \mathbf{q}(x_2)U = \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)), \tag{4.4}$$

$$U\left(\mathbf{q}(x_1) - \mathbf{q}(x_2)\right) = \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)). \tag{4.5}$$

[2]Augustin-Louis Cauchy, 1789-1857, French mechanician.

Note that the contribution of the source term $\mathbf{s}$ is negligible as $x_2 - x_1 \to 0$. Defining next the notation for a jump as

$$[\![\mathbf{q}(x)]\!] \equiv \mathbf{q}(x_2) - \mathbf{q}(x_1), \tag{4.6}$$

the jump conditions are rewritten as

$$U\,[\![\mathbf{q}(x)]\!] = [\![\mathbf{f}(\mathbf{q}(x))]\!]\,. \tag{4.7}$$

If $U = 0$, as is the case when we transform to the frame where the wave is at rest, we simply recover

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{q}(x_1)) - \mathbf{f}(\mathbf{q}(x_2)), && (4.8)\\ \mathbf{f}(\mathbf{q}(x_1)) &= \mathbf{f}(\mathbf{q}(x_2)), && (4.9)\\ [\![\mathbf{f}(\mathbf{q}(x))]\!] &= \mathbf{0}. && (4.10) \end{aligned}$$

That is, the fluxes on either side of the discontinuity are equal. We also get a more general result for $U \neq 0$, which is the well-known

$$U = \frac{\mathbf{f}(\mathbf{q}(x_2)) - \mathbf{f}(\mathbf{q}(x_1))}{\mathbf{q}(x_2) - \mathbf{q}(x_1)} = \frac{[\![\mathbf{f}(\mathbf{q}(x))]\!]}{[\![\mathbf{q}(x)]\!]}. \tag{4.11}$$

In contrast, if there is no discontinuity, Eq. (4.1) reduces to a partial differential equation describing a continuum. We achieve this by rewriting Eq. (4.1) as

$$\left(\frac{d}{dt}\int_{x_1}^{x_2} \mathbf{q}(x,t)\ dx\right) + \left(\mathbf{f}(\mathbf{q}(x_2,t)) - \mathbf{f}(\mathbf{q}(x_1,t))\right) = \int_{x_1}^{x_2} \mathbf{s}(\mathbf{q}(x,t))\ dx. \tag{4.12}$$

Now, if we assume *continuity* of all fluxes and variables, we can use Taylor series expansion and Leibniz's rule to say

$$\left(\int_{x_1}^{x_2} \frac{\partial}{\partial t}\mathbf{q}(x,t)\ dx\right) + \left(\left(\mathbf{f}(\mathbf{q}(x_1,t)) + \frac{\partial \mathbf{f}}{\partial x}(x_2 - x_1) + \dots\right) - \mathbf{f}(\mathbf{q}(x_1,t))\right) = \int_{x_1}^{x_2} \mathbf{s}(\mathbf{q}(x,t))\ dx. \tag{4.13}$$

We let $x_2 \to x_1$ and get

$$\begin{aligned} \left(\int_{x_1}^{x_2} \frac{\partial}{\partial t}\mathbf{q}(x,t)\ dx\right) + \left(\frac{\partial \mathbf{f}}{\partial x}(x_2 - x_1)\right) &= \int_{x_1}^{x_2} \mathbf{s}(\mathbf{q}(x,t))\ dx, && (4.14)\\ \left(\int_{x_1}^{x_2} \frac{\partial}{\partial t}\mathbf{q}(x,t)\ dx\right) + \int_{x_1}^{x_2} \frac{\partial \mathbf{f}}{\partial x}\ dx &= \int_{x_1}^{x_2} \mathbf{s}(\mathbf{q}(x,t))\ dx. && (4.15)\\ & && (4.16) \end{aligned}$$

Combining all terms under a single integral, we get

$$\int_{x_1}^{x_2} \left(\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} - \mathbf{s}\right) dx = \mathbf{0}. \tag{4.17}$$

Now, this integral must be zero for an arbitrary x_1 and x_2, so the integrand itself must be zero, and we get our partial differential equation:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} - \mathbf{s} = \mathbf{0}, \tag{4.18}$$

$$\frac{\partial}{\partial t}\mathbf{q}(x,t) + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{q}(x,t)) = \mathbf{s}(\mathbf{q}(x,t)), \tag{4.19}$$

which applies away from jumps.

4.1.2 Single conserved variable

Let us consider a simple and important form in which there is a single conserved variable and no source term. For such a case, we study Eq. (4.19) with $\mathbf{q} = u$, $\mathbf{f}(\mathbf{q}) = f(u)$, $\mathbf{s} = 0$. Then, we have the conservative form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0. \tag{4.20}$$

Assuming no discontinuities, Eq. (4.20) may be rewritten using the chain rule in characteristic form as

$$\frac{\partial u}{\partial t} + \frac{df}{du}\frac{\partial u}{\partial x} = 0. \tag{4.21}$$

Here the local speed of propagation of waves is df/du.

The function $f(u)$ may be *convex* or *non-convex*. A function is convex if its *epigraph*, the set of points on or above the graph of the function, form a convex set. It is easy to show that a function is convex iff its second derivative is non-negative over its whole domain. Plots of examples of convex ($f(u) = 1/2 + u^2$) and non-convex ($f(u) = 3/2 - u^2$) functions are shown in Fig. 4.2. Note that the example convex function has $d^2f/du^2 = 2 > 0$ and the example non-convex function has $d^2f/du^2 = -2 < 0$.

Example 4.1

Find the jump equations for the simple wave propagation of Sec. 1.1.

We start with Eq. (1.10), replacing $x_1 + \Delta x$ by x_2 and otherwise using the notation of Sec. 1.1:

$$\frac{dm}{dt} = -\left(\rho|_{x_2} Aa - \rho|_{x_1} Aa\right), \tag{4.22}$$

$$A\frac{d}{dt}\int_{x_1}^{x_2} \rho\, dx = -\left(\rho|_{x_2} Aa - \rho|_{x_1} Aa\right), \tag{4.23}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\, dx = -\left(\rho|_{x_2} a - \rho|_{x_1} a\right). \tag{4.24}$$

Here our vector $\mathbf{q}$ has one entry $\mathbf{q} = (\rho)$. And our flux vector $\mathbf{f}$ also has one entry $\mathbf{f} = (\rho a)$. And there is no source of mass, so the vector $\mathbf{s} = \mathbf{0} = (0)$.

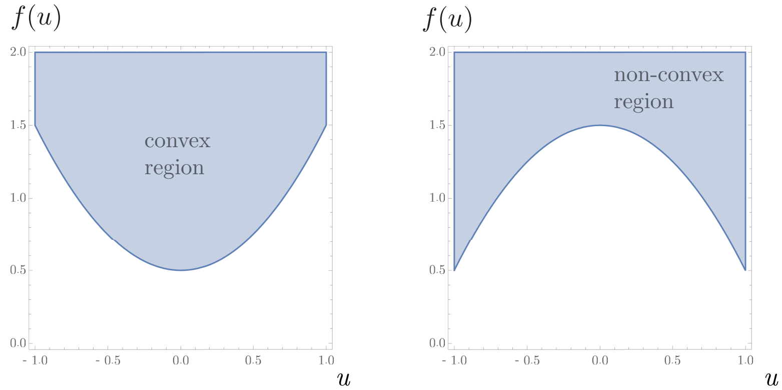


Figure 4.2: Convex function, $f = 1/2 + u^2$, and non-convex function, $f = 3/2 - u^2$.

Equation (4.11) tells us that discontinuous jumps propagate at speed

$$U = \frac{[\![\mathbf{f}(\mathbf{q}(x))]\!]}{[\![\mathbf{q}(x)]\!]} = \frac{[\![\rho a]\!]}{[\![\rho]\!]} = \frac{\rho_2 a - \rho_1 a}{\rho_2 - \rho_1} = a. \tag{4.25}$$

And for steady jumps for which $U = 0$, we simply have no jump in ρ: $\rho_2 = \rho_1$. And for situations where there is no jumps, we recover from Eq. (4.19) the continuous partial differential equation

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0. \tag{4.26}$$

We note as an aside that here $u = \rho$ and $f(u) = f(\rho) = a\rho$. Thus $d^2f/d\rho^2 = 0$. Because it is non-negative, the flux function is convex.

Example 4.2

If the conserved variable is $u(x,t)$ and the flux of u is given by $f(u) = u^2/2$, find appropriate jump equations and the appropriate partial differential equation for continuous values of u. Evaluate the possibles jumps admitted through in a steady wave for which u in the far field as $x \to -\infty$ takes on the value u_1.

Equation (4.11) tells us that discontinuous jumps propagate at speed

$$\begin{aligned}
U &= \frac{[\![\mathbf{f}(\mathbf{q}(x))]\!]}{[\![\mathbf{q}(x)]\!]}, && (4.27)\\
&= \frac{\left[\!\left[\frac{u^2}{2}\right]\!\right]}{[\![u]\!]}, && (4.28)\\
&= \frac{\frac{u_2^2}{2} - \frac{u_1^2}{2}}{u_2 - u_1}, && (4.29)\\
&= \frac{1}{2}\frac{(u_2 - u_1)(u_2 + u_1)}{u_2 - u_1}, && (4.30)\\
&= \frac{u_2 + u_1}{2}. && (4.31)
\end{aligned}$$

The jump propagates at the average value of u over the jump.

Then Eq. (4.19) gives us, if u is continuous,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0. \tag{4.32}$$

This can be expanded by the rules of calculus to get the so-called *inviscid Bateman*[3]*-Burgers'*[4] *equation*[5], usually known as the *inviscid Burgers' equation*:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{4.33}$$

If the wave is steady, $\partial/\partial t = 0$, and Eq. (4.32) reduces to the ordinary differential equation

$$\frac{d}{dx}\left(\frac{u^2}{2}\right) = 0, \qquad u(x \to -\infty) = u_1. \tag{4.34}$$

Integrating, we obtain

$$\frac{u^2}{2} = C. \tag{4.35}$$

To satisfy the far-field boundary condition, we need $C = u_1^2/2$, giving

$$\frac{u^2}{2} = \frac{u_1^2}{2}, \tag{4.36}$$

$$u = \pm u_1. \tag{4.37}$$

Note that

- Only one of the solutions matches the boundary condition in the far field, but
- There is nothing preventing the existence of a stationary discontinuity with $U = 0$ sitting at *any finite* x where the solution jumps from $u = u_1$ to $u = -u_1$. Such a solution will satisfy the governing differential equations and boundary condition. Additionally, it will satisfy the jump equations at the discontinuity.
- The flux function here $f(u) = u^2/2$ is convex because $d^2f/du^2 = 1 > 0$.

For $u_1 = 1$, we give a plot of $u(x)$ with a discontinuity located at $x = 1$ in Fig. 4.3. Obviously $U = 0$ because via Eq. (4.31), $U = (u_1 + u_2)/2 = (u_1 - u_1)/2 = 0$.

If we had $u_1 = 0$ and $u_2 = 2$, a solution would exist with a discontinuity linking the two states. However, the discontinuity would be propagating at $U = (0 + 2)/2 = 1$. If we had transformed to the frame where the wave were stationary, $\hat{u} = u - U = u - 1$, we would have $\hat{u}_1 = -1$ and $\hat{u}_2 = 1$. For general u_1 and u_2, we could transform via $\hat{u} = u - U = u - (u_1 + u_2)/2$. Then $\hat{u}_1 = (u_1 - u_2)/2$ and $\hat{u}_2 = (u_2 - u_1)/2 = -\hat{u}_1$.

[3] Harry Bateman, 1882-1946, English mathematician.

[4] Johannes Martinus Burgers, 1895-1981, Dutch physicist.

[5] The viscous version of the model equation, $\partial u/\partial t + u\, \partial u/\partial x = \nu\, \partial^2 u/\partial x^2$, is widely known as Burgers' equation and is often cited as originating from J. M. Burgers, 1948, A mathematical model illustrating the theory of turbulence, *Advances in Applied Mathematics*, 1: 171-199. However, the viscous version was given earlier by H. Bateman, 1915, Some recent researches in the motion of fluids, *Monthly Weather Review*, 43(4): 163-170.

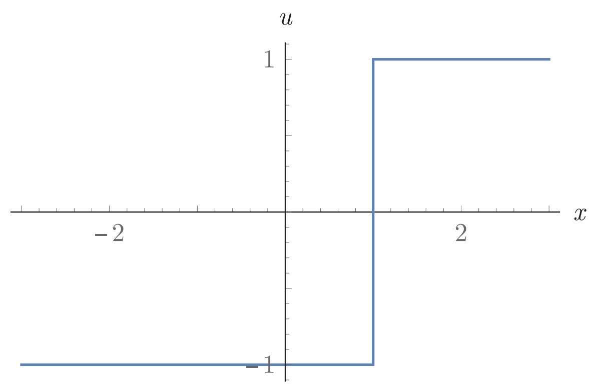


Figure 4.3: Solution to $\partial u/\partial t + \partial/\partial x(u^2/2)$ with $u(x \to -\infty) = -1$. A stationary discontinuity (thus $U = 0$) is arbitrarily located at $x = 1$.

Remarkably, if we incorrectly take as our starting point a continuous partial differential equation such as Eq. (4.33), it is possible to be led to an incorrect jump equation as illustrated by the following example.

Example 4.3

Show by multiplying Eq. (4.33) by u that one can be led to infer a jump condition which is inconsistent with Eq. (4.31).

We perform the multiplication to get

$$u\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} = 0. \tag{4.38}$$

The ordinary rules of calculus suggest then that we can say

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) = 0. \tag{4.39}$$

So our jump condition might be expected to be

$$\begin{aligned} U &= \frac{\left[\!\left[\frac{u^3}{3}\right]\!\right]}{\left[\!\left[\frac{u^2}{2}\right]\!\right]}, & (4.40)\\ &= \frac{2}{3}\frac{u_2^3 - u_1^3}{u_2^2 - u_1^2}, & (4.41)\\ &= \frac{2}{3}\frac{(u_2 - u_1)(u_1^2 + u_1u_2 + u_2^2)}{(u_2 - u_1)(u_2 + u_1)}, & (4.42)\\ &= \frac{2}{3}\frac{u_1^2 + u_1u_2 + u_2^2}{u_2 + u_1}, & (4.43)\\ &= \frac{u_2 + u_1}{2}\left(1 + \frac{1}{3}\left(\frac{u_2 - u_1}{u_2 + u_1}\right)^2\right). & (4.44) \end{aligned}$$

Had we done the same analysis with the continuous equivalent $\partial u/\partial t + \partial/\partial x(u^2/2) = 0$, we would have arrived at a different result: $U = (u_2 + u_1)/2$, as seen in Eq. (4.31). Clearly we cannot perform ad hoc operations on continuous equations and expect to infer a consistent expression for the propagation speed of a discontinuity. It is thus essential to infer propagation speeds of discontinuities from the more fundamental integral form of the conservation equations such as that of Eq. (4.1).

4.2 Inviscid Burgers' equation

Let us analyze the inviscid Burgers' equation, Eq. (4.33), in the context of coordinate transformations that have the general form

$$x = x(\xi, \tau), \tag{4.45}$$
$$t = t(\xi, \tau). \tag{4.46}$$

We assume the transformation to be unique and invertible. The Jacobian matrix of the transformation is

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \tau} \\ \frac{\partial t}{\partial \xi} & \frac{\partial t}{\partial \tau} \end{pmatrix}. \tag{4.47}$$

And we have

$$J = \det \mathbf{J} = \frac{\partial x}{\partial \xi}\frac{\partial t}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial t}{\partial \xi}. \tag{4.48}$$

Now

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} = (\mathbf{J}^T)^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \tau} \end{pmatrix} = \frac{1}{J}\begin{pmatrix} \frac{\partial t}{\partial \tau} & -\frac{\partial t}{\partial \xi} \\ -\frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \xi} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \tau} \end{pmatrix} = \frac{1}{J}\begin{pmatrix} \frac{\partial t}{\partial \tau}\frac{\partial}{\partial \xi} - \frac{\partial t}{\partial \xi}\frac{\partial}{\partial \tau} \\ -\frac{\partial x}{\partial \tau}\frac{\partial}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \tau} \end{pmatrix}. \tag{4.49}$$

With these transformation rules, Eq (4.33) is rewritten as

$$\underbrace{\frac{1}{J}\left(-\frac{\partial x}{\partial \tau}\frac{\partial u}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau}\right)}_{\partial u/\partial t} + u \underbrace{\frac{1}{J}\left(\frac{\partial t}{\partial \tau}\frac{\partial u}{\partial \xi} - \frac{\partial t}{\partial \xi}\frac{\partial u}{\partial \tau}\right)}_{\partial u/\partial x} = 0. \tag{4.50}$$

Now by assumption, $J \neq 0$, so we can multiply by J to get

$$-\frac{\partial x}{\partial \tau}\frac{\partial u}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} + u\frac{\partial t}{\partial \tau}\frac{\partial u}{\partial \xi} - u\frac{\partial t}{\partial \xi}\frac{\partial u}{\partial \tau} = 0. \tag{4.51}$$

Let us now restrict our transformation to satisfy the following requirements:

$$\frac{\partial x}{\partial \tau} = u\frac{\partial t}{\partial \tau}, \tag{4.52}$$
$$t(\xi, \tau) = \tau. \tag{4.53}$$

The first says that if we insist that ξ is held fixed, that the ratio of the change in x to the change in t will be u; this is equivalent to the more standard statement that on a characteristic line we have $dx/dt = u$. The second is a convenience simply equating τ to t. Applying the second restriction to the first, we can also say

$$\frac{\partial x}{\partial \tau} = u. \tag{4.54}$$

With these restrictions, our inviscid Burgers' equation becomes

$$\begin{aligned}
-\underbrace{\frac{\partial x}{\partial \tau}}_{u}\frac{\partial u}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} + u\underbrace{\frac{\partial t}{\partial \tau}}_{1}\frac{\partial u}{\partial \xi} - u\underbrace{\frac{\partial t}{\partial \xi}}_{0}\frac{\partial u}{\partial \tau} &= 0, && (4.55)\\
\cancel{-u\frac{\partial u}{\partial \xi}} + \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} + \cancel{u\frac{\partial u}{\partial \xi}} &= 0, && (4.56)\\
\frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \tau} &= 0. && (4.57)
\end{aligned}$$

Let us further require that $\partial x/\partial \xi \neq 0$. Then we have

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= 0, && (4.58)\\
u &= f(\xi). && (4.59)
\end{aligned}$$

Here f is an arbitrary function. Substitute this into Eq. (4.52) to get

$$\frac{\partial x}{\partial \tau} = f(\xi)\frac{\partial t}{\partial \tau}. \tag{4.60}$$

We can integrate Eq. (4.60) to get

$$x = f(\xi)t + g(\xi). \tag{4.61}$$

Here $g(\xi)$ is an arbitrary function. Note the coordinate transformation can be chosen for our convenience. To this end, remove t in favor of τ and set $g(\xi) = \xi$ so that x maps to ξ when $t = \tau = 0$ giving

$$x(\xi, \tau) = f(\xi)\tau + \xi. \tag{4.62}$$

We can then state the solution to the inviscid Burgers' equation, Eq. (4.33), parametrically as

$$\begin{aligned}
u(\xi, \tau) &= f(\xi), && (4.63)\\
x(\xi, \tau) &= f(\xi)\tau + \xi, && (4.64)\\
t(\xi, \tau) &= \tau. && (4.65)
\end{aligned}$$

For this transformation, we have from Eq. (4.47) that

$$\mathbf{J} = \begin{pmatrix} 1 + \frac{df}{d\xi}\tau & f(\xi) \\ 0 & 1 \end{pmatrix}. \tag{4.66}$$

Thus

$$J = \det \mathbf{J} = 1 + \frac{df}{d\xi}\tau. \tag{4.67}$$

We have a singularity in the coordinate transformation whenever $J = 0$, implying a difficulty when

$$\tau = -\frac{1}{\frac{df}{d\xi}}. \tag{4.68}$$

Example 4.4

Solve the inviscid Burgers' equation, $\partial u/\partial t + u\ \partial u/\partial x = 0$, Eq. (4.33), if

$$u(x, 0) = 1 + \sin \pi x, \qquad x \in [0, 1] \tag{4.69}$$

Let us not be concerned with that portion of u which at $t = 0$ has $x < 0$ or $x > 1$. The analysis is easily modified to address this.

We know the solution is given in general by Eqs. (4.63-4.65). At $t = 0$, we have $\tau = 0$, and thus $x = \xi$. And we have

$$f(\xi) = 1 + \sin \pi\xi. \tag{4.70}$$

Thus we can say by inspection that the solution is

$$u(\xi, \tau) = 1 + \sin \pi\xi, \tag{4.71}$$
$$x(\xi, \tau) = (1 + \sin \pi\xi)\,\tau + \xi, \tag{4.72}$$
$$t(\xi, \tau) = \tau. \tag{4.73}$$

Results are plotted in Fig. 4.4. One notes the following:

- The signal propagates to the right; this is a consequence of $u > 0$ in the domain we consider.
- Portions of the signal with higher u propagate faster.
- The signal distorts as t increases.
- The wave appears to "break" at $t = t_s$, where $1/4 \lesssim t_s \lesssim 1/2$. For $t > t_s$, it is possible to find multiple values of u at a given x and t. If u were a physical variable, we would not expect to see such multivaluedness in nature.
- Because of the convexity of the flux function, the right side of the wave form steepens, and the left side of the wave form becomes more shallow.

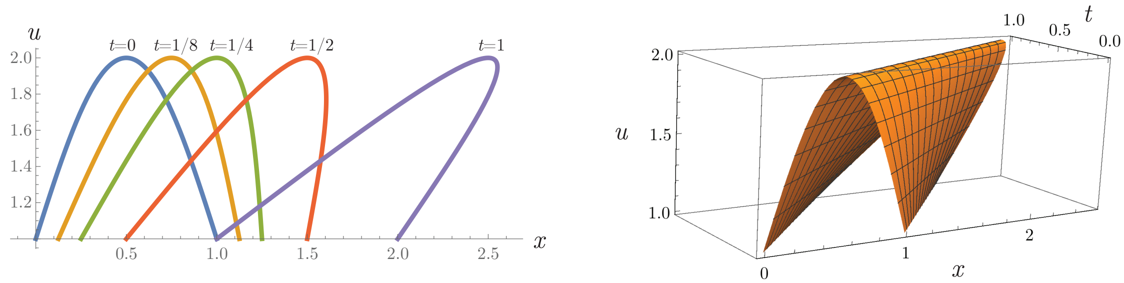


Figure 4.4: Solution to $\partial u/\partial t + u\partial u/\partial x$ with $u(x,0) = 1 + \sin \pi x$.

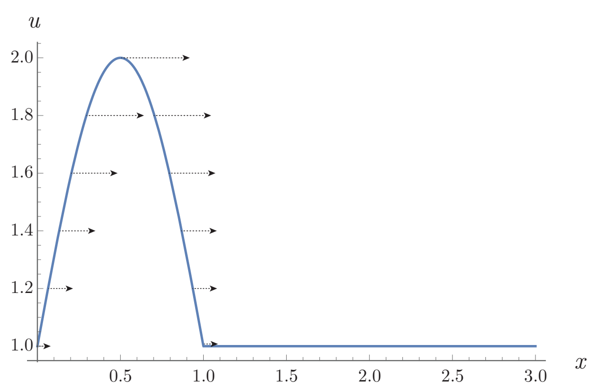


Figure 4.5: Sketch of response of u which satisfies the inviscid Burgers' equation $\partial u/\partial t + u\partial u/\partial x$ with $u(x,0) = 1 + \sin \pi x$.

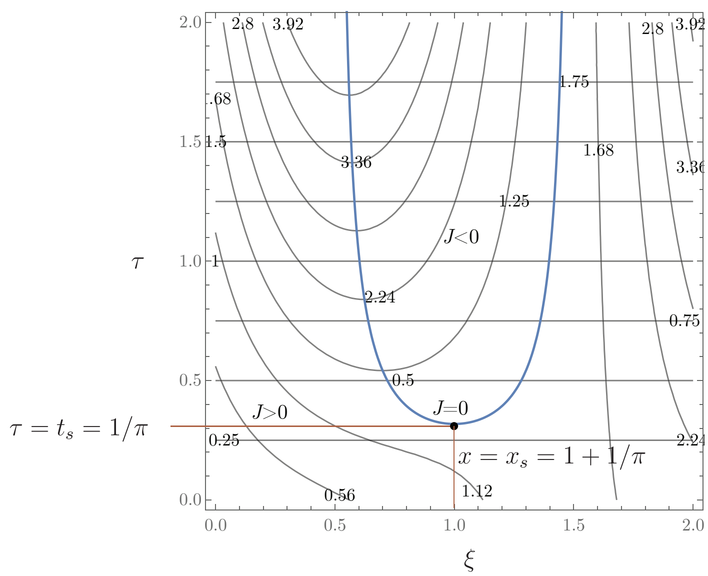


Figure 4.6: Curves where $J = 0$ and of constant x and t in the (ξ, τ) plane for our coordinate transformation.

The sketch of Fig. 4.5 shows how one can envision the portion of the initial sine wave with $x > 1/2$ steepening, while that portion with $x < 1/2$ flattens. We place arrows whose magnitude is proportional to the local value of u on the plot itself.

For our value of $f(\xi)$, we have from from Eq. (4.67) that

$$J = 1 + \pi\tau \cos \pi\xi. \tag{4.74}$$

Clearly, there exist values of (ξ, τ) for which $J = 0$. At such points, we can expect difficulties in our solution. In Fig. 4.6, we plot a portion of the locus of points for which $J = 0$ in the (ξ, τ) plane. We also see portions of this plane where the transformation is orientation-preserving, for which $J > 0$, and orientation-reversing, for which $J < 0$. Also shown in Fig. 4.6 are contours of constant x and t. Clearly when $J = 0$, the contours of constant x are parallel to those of constant t, and there are not enough linearly independent vectors to form a basis.

From Eq. (4.68), we can expect a singular coordinate transformation when

$$\tau = -\frac{1}{\frac{df}{d\xi}} = -\frac{1}{\pi \cos \pi\xi}. \tag{4.75}$$

We then substitute this into Eqs. (4.72, 4.73) to get a parametric curve for when the transformation is singular, $x_s(\xi), t_s(\xi)$:

$$x_s(\xi) = -\frac{1 + \sin \pi\xi}{\pi \cos \pi\xi} + \xi, \tag{4.76}$$

$$t_s(\xi) = -\frac{1}{\pi \cos \pi\xi}. \tag{4.77}$$

A portion of this curve for where the transformation is singular is shown in Fig. 4.7. Figure 4.7a plots

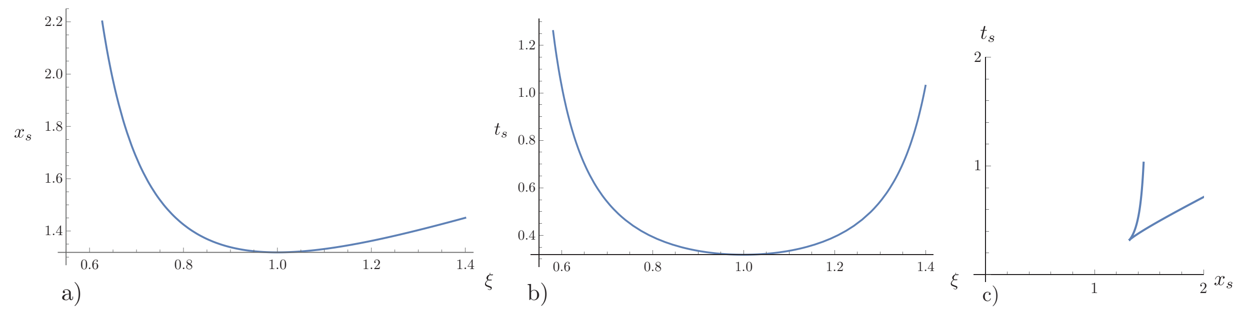


Figure 4.7: Plots indicating where the coordinate transformation of Eqs. (4.72,4.73) is singular: a) $x_s(\xi)$ from Eq. (4.76), b) $t_s(\xi)$ from Eq. (4.77), c) representation of the curve of singularity in (x,t) space.

$x_s(\xi)$ from Eq. (4.76). Figure 4.7b plots $t_s(\xi)$ from Eq. (4.77). We see a parametric plot of the same quantities in Fig. 4.7c. At early time the system is free of singularities. It is easily shown that both $x_s(\xi)$ and $t_s(\xi)$ have a local minimum at $\xi = 1$, at which point, we have

$$x_s(1) = 1 + \frac{1}{\pi}, \tag{4.78}$$

$$t_s(1) = \frac{1}{\pi}. \tag{4.79}$$

Examining Fig. 4.4, this appears to be the point at which the solution becomes multivalued. Examining Fig. 4.6, this is the point on the curve $J = 0$ that is a local minimum. So while x_s and t_s are well-behaved as functions of ξ for the domain considered, when the curves are projected into the (x,t) plane, there is a cusp at $(x,t) = (x_s(1), t_s(1)) = (1 + 1/\pi, 1/\pi)$.

4.3 Viscous Burgers' equation

Our predictions of $u(x,t)$ change dramatically when diffusion is introduced. Consider the viscous Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}. \tag{4.80}$$

4.3.1 Comparison to inviscid solution

When we simulate the same problem whose diffusion-free solution is plotted in Fig. 4.4 for which $u(x,0) = 1 + \sin \pi x$, we obtain the results plotted in Fig. 4.8 for four different values of $\nu = 1/1000$, $1/100$, $1/10$, and 1. While we will soon outline a method to obtain an exact solution to the viscous Burgers' equation, in practice, it is complicated. It is often easier to

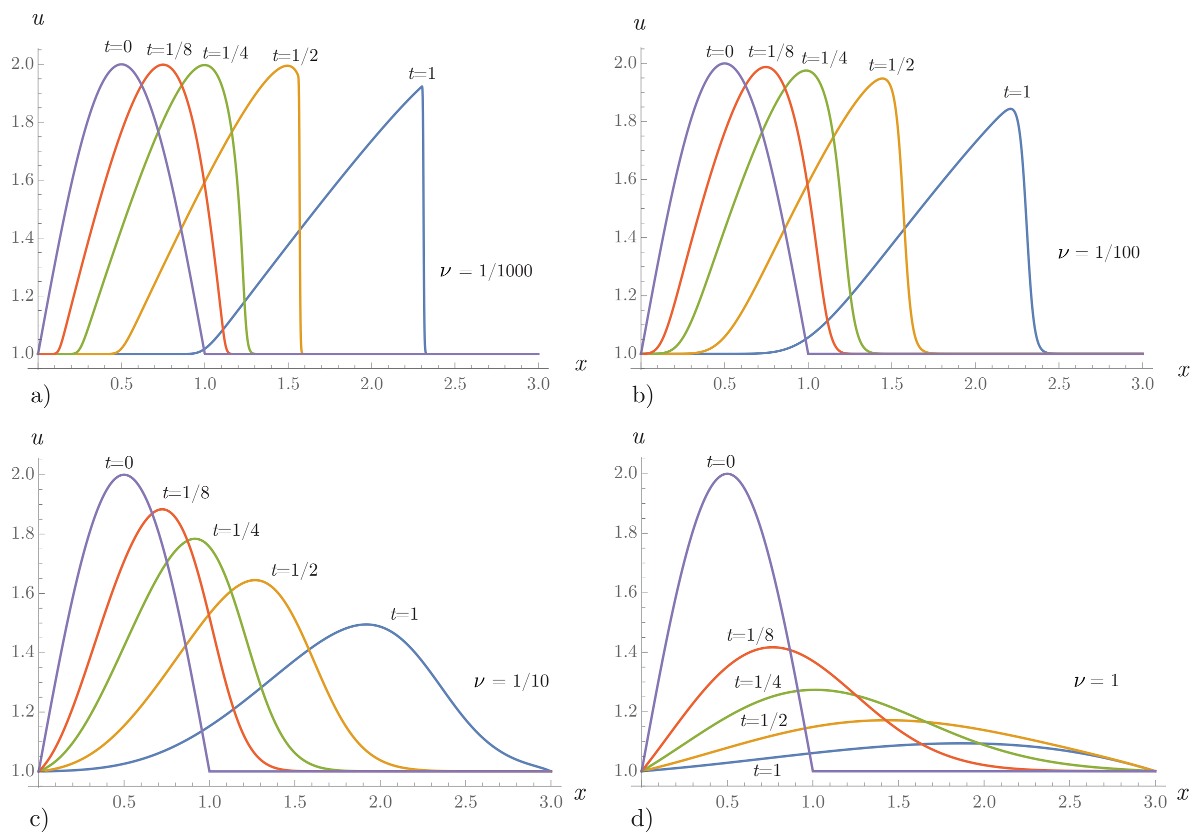


Figure 4.8: Solution to the viscous Burgers' equation $\partial u/\partial t + u\partial u/\partial x = \nu\partial^2 u/\partial x^2$ with $u(x,0) = 1 + \sin \pi x$ and various values of ν: a) 1/1000, b) 1/100, c) 1/10, d) 1.

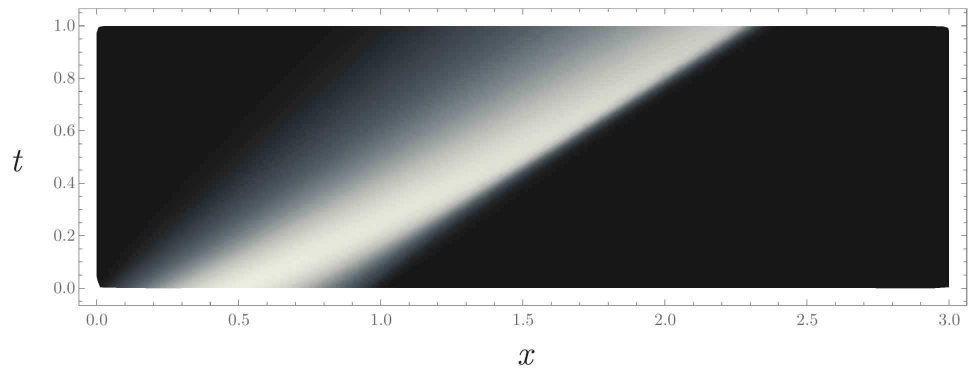


Figure 4.9: $x - t$ diagram for solution to the viscous Burgers' equation $\partial u/\partial t + u\partial u/\partial x = \nu \partial^2 u/\partial x^2$ with $u(x, 0) = 1 + \sin \pi x$, $\nu = 1/100$.

obtain results by numerical discretization, and that is what we did here. The scheme used was sufficiently resolved to capture the thin zones present when ν was small. For the case where $\nu = 1/100$, we plot the $x - t$ diagram, where the shading is proportional to the local value of u, in Fig. 4.9.

We note:

- We restricted our study to positive values of ν, which can be shown to be necessary for a stable solution as $t \to \infty$.
- If $\nu = 0$, our viscous Burgers' equation reduces to the inviscid Burgers' equation.
- As $\nu \to 0$, solutions to the viscous Burgers' equation seem to relax to a solution with an infinitely thin discontinuity; they do not relax to those solutions displayed in Fig. 4.4.
- For all values of ν, the solution $u(x, t)$ at a given time has a single value of u for a single value of x, in contrast to multi-valued solutions exhibited by the diffusion-free analog.
- As $\nu \to 0$, the peaks retain a larger magnitude. Thus one can conclude that enhancing ν smears peaks.
- At early time the solutions to the viscous Burgers' equation resemble those of the inviscid Burgers' equation.

Let us try to understand this behavior. Fundamentally, it will be seen that in many cases, nonlinearity, manifested in $u\ \partial u/\partial x$ can serve to steepen a waveform. If that steepening is unchecked by diffusion, either a formal discontinuity is admitted, or multi-valued solutions. Now diffusion acts most strongly when gradients are steep, that is when $\partial u/\partial x$ has large magnitude. As a wave steepens due to nonlinear effects, diffusion, which many have been initially unimportant, can reassert its importance and serve to suppress the growth due to the nonlinearity.

4.3.2 Steadily propagating waves

Let us examine solutions to Eq. (4.80) that can link a constant state where $u(-\infty,t)=u_1$ to a second constant state where $u(\infty,t)=u_2$. We shall see that this can be achieved by what is known as a steadily propagating wave solution. For such solutions a waveform is maintained as the wave translates with a given velocity.

We can employ both a coordinate transformation and a change of variables. Let us take as new coordinates

$$\xi = x - at, \tag{4.81}$$
$$\tau = t. \tag{4.82}$$

Here a is a constant which we will see fit to specify later. From this, we get in matrix form

$$\begin{pmatrix}\xi\\ \tau\end{pmatrix} = \underbrace{\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}}_{\mathbf{J}^{-1}}\begin{pmatrix}x\\ t\end{pmatrix}. \tag{4.83}$$

The inverse transformation is

$$\begin{pmatrix}x\\ t\end{pmatrix} = \underbrace{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}}_{\mathbf{J}}\begin{pmatrix}\xi\\ \tau\end{pmatrix}. \tag{4.84}$$

Here the Jacobian matrix is related, but not identical, to that defined in previous analysis, see Sec. 1.1; moreover $J=\det\mathbf{J}=1$. Similar to our analysis of Sec. 1.1, we get

$$\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}\end{pmatrix} = \mathbf{J}^{T^{-1}}\begin{pmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \tau}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -a & 1\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \tau}\end{pmatrix}. \tag{4.85}$$

Thus, we see

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \tag{4.86}$$
$$\frac{\partial}{\partial t} = -a\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau}. \tag{4.87}$$

We apply this coordinate transformation to Eq. (4.80) to get

$$\underbrace{-a\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau}}_{\frac{\partial u}{\partial t}} + \underbrace{u\frac{\partial u}{\partial \xi}}_{u\frac{\partial u}{\partial x}} = \nu\underbrace{\frac{\partial^2 u}{\partial \xi^2}}_{\frac{\partial^2 u}{\partial x^2}}, \tag{4.88}$$
$$\frac{\partial u}{\partial \tau} + (u-a)\frac{\partial u}{\partial \xi} = \nu\frac{\partial^2 u}{\partial \xi^2}. \tag{4.89}$$

This form suggests we will realize further simplification by defining

$$w = u - a. \tag{4.90}$$

In physics, this is known as a *Galilean*[6] *transformation*, with w being the relative velocity. Doing so, we get

$$\frac{\partial}{\partial \tau}(w+a) + w\frac{\partial}{\partial \xi}(w+a) = \nu \frac{\partial^2}{\partial \xi^2}(w+a), \tag{4.91}$$

$$\frac{\partial w}{\partial \tau} + w\frac{\partial w}{\partial \xi} = \nu \frac{\partial^2 w}{\partial \xi^2}. \tag{4.92}$$

Remarkably, Eq. (4.92) has precisely the same form as Eq. (4.80), with w standing in for u. Leaving aside for now any concerns about initial and boundary conditions, we say that our Galilean transformation, which transforms both the dependent and independent variables, has mapped Eq. (4.80) into itself. It is analogous to certain geometrical transformations. For example, if we rotate a square through an angle of $\pi/4$, it appears askew relative to its original orientation. But if we rotate a square through the angles $\pm n\pi/2$, where n an integer, one cannot detect a difference between the transformed square and the original square. The form of the square is thus *invariant* to rotation through angles of $\pm n\pi/2$. It has a particular type of symmetry. The geometric form circle is invariant under rotations through *any* angle. It has a different type of symmetry. Idealized snowflakes may thought to be invariant under rotations of $\pm n\pi/3$. Our Burgers' equation too has a symmetry in that its form is invariant under a Galilean transformation. Mathematical models that transform under a mapping into themselves are also known as *self-similar* and are one of the key features of what is known as *group theory*. A further discussion of similarity will be given in Ch. 6.

Our boundary conditions transform to

$$w(-\infty, \tau) = u_1 - a \equiv w_1, \tag{4.93}$$

$$w(\infty, \tau) = u_2 - a \equiv w_2. \tag{4.94}$$

We shall see there is an additional requirement for a for symmetry, to be determined.

We trivially note that if we seek solutions that are independent of ξ, Eq. (4.92) reduces to $dw/d\tau = 0$, which gives us $w = C$. The boundary conditions are only satisfied in the special case when $w_1 = w_2$, giving $w = w_1$. This is not particularly useful. We find nontrivial results when we seek solutions that are independent of τ; that is we seek $w = w(\xi)$. Then Eq. (4.92) reduces to

$$w\frac{dw}{d\xi} = \nu\frac{d^2 w}{d\xi^2}, \tag{4.95}$$

$$\frac{d}{d\xi}\left(\frac{w^2}{2}\right) = \nu\frac{d^2 w}{d\xi^2}, \tag{4.96}$$

$$\frac{w^2}{2} + C = \nu\frac{dw}{d\xi}. \tag{4.97}$$

[6] Galileo di Vincenzo Bonaiuti de' Galilei, 1564-1642, Florentine polymath.

Now as $\xi \to -\infty$, we expect $w \to w_1$ and $dw/d\xi \to 0$. Thus,

$$\frac{w_1^2}{2} + C = 0, \tag{4.98}$$

$$C = -\frac{w_1^2}{2}. \tag{4.99}$$

Thus,

$$\nu \frac{dw}{d\xi} = \frac{1}{2}(w^2 - w_1^2), \tag{4.100}$$

$$\frac{dw}{w_1^2 - w^2} = -\frac{d\xi}{2\nu}, \tag{4.101}$$

$$\frac{1}{w_1} \tanh^{-1}\left(\frac{w}{w_1}\right) = -\frac{\xi}{2\nu} + C, \tag{4.102}$$

$$w(\xi) = w_1 \tanh\left(-\frac{w_1}{2\nu}\xi + C w_1\right). \tag{4.103}$$

Examination of this solution reveals that $\lim_{\xi \to -\infty} w(\xi) = w_1$ and $\lim_{\xi \to \infty} w(\xi) = -w_1$ and $w(\xi) = 0$ when $\xi = 2\nu C$. Let us make the convenient assumption that $C = 0$ to place the somewhat arbitrary zero-crossing at $\xi = 0$. Other choices would simply translate the zero-crossing and not otherwise affect the solution. Now we see we have satisfied the boundary condition at $\xi \to -\infty$ but not at $\xi = +\infty$. We can satisfy both boundary conditions at $\pm\infty$ by making the correct choice of the as of yet unspecified wave speed a. We thus would like to choose a such that

$$-w_1 = w_2. \tag{4.104}$$

Using our definitions, Eqs. (4.93,4.94), we get

$$-(u_1 - a) = u_2 - a, \tag{4.105}$$

$$2a = u_1 + u_2, \tag{4.106}$$

$$a = \frac{u_1 + u_2}{2}. \tag{4.107}$$

Then

$$w_1 = u_1 - a = \frac{u_1 - u_2}{2}, \tag{4.108}$$

$$w_2 = u_2 - a = -\frac{u_1 - u_2}{2} = -w_1. \tag{4.109}$$

Our solution is then

$$w(\xi) = \frac{u_1 - u_2}{2} \tanh\left(-\frac{u_1 - u_2}{4\nu}\xi\right). \tag{4.110}$$

In terms of our untransformed variables, we have

$$u(x,t) = \frac{u_1+u_2}{2} + \frac{u_1-u_2}{2}\tanh\left(-\frac{u_1-u_2}{4\nu}\left(x - \frac{u_1+u_2}{2}t\right)\right). \tag{4.111}$$

It is easy to verify by direct calculation that Eq. (4.111) satisfies the viscous Burgers' equation, Eq. (4.80). Additionally, it satisfies both boundary conditions at $x = \pm\infty$. By inspection of the solution, the thickness ℓ of the zone were u adjusts from u_1 to u_2 is given by

$$\ell = \left|\frac{4\nu}{u_1-u_2}\right| \tag{4.112}$$

In the limit as $\nu \to 0$, we see $\ell \to 0$, and $u(x,t)$ suffers a jump from $u = u_1$ to $u = u_2$ at $x = (u_1+u_2)t/2$. This is fully consistent with our inviscid jump analysis given in Eq. (4.31). Because our independent analysis of the viscous Burgers' equation revealed that the propagation speed is $(u_1+u_2)/2$, we conclude that the appropriate form of the inviscid Burgers' equation is that of Eq. (4.32), and not one of the many others, such as Eq. (4.39). Results are plotted in Fig. 4.10 for three different values of $\nu = 1/1000$, $1/100$, and $1/10$ for $u_1 = 3/2$, $u_2 = 1/2$ and $t = 2$. Clearly, all solutions relax at $\pm\infty$ to the correct values of u_1 and u_2. The only effect of ν is the thickness of the zone where u relaxes from u_1 to u_2. Also the propagation speed $a = (u_1+u_2)/2 = 1$. Because the wave was centered at $x = 0$ at $t = 0$, we see at $t = 2$ its "center" has propagated to $x = 2$.

4.3.3 Cole-Hopf transformation

For more general conditions than those of a steadily propagating wave, the viscous Burgers' equation's analysis is simplified by a so-called Cole[7]-Hopf[8] transformation. Let us redefine u in terms of a new variable $\phi(x,t)$ via

$$u = -2\nu\frac{1}{\phi}\frac{\partial\phi}{\partial x}. \tag{4.113}$$

Then Eq. (4.80) becomes

$$\begin{aligned}
\frac{\partial}{\partial t}\left(-2\nu\frac{1}{\phi}\frac{\partial\phi}{\partial x}\right) - 2\nu\frac{1}{\phi}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left(-2\nu\frac{1}{\phi}\frac{\partial\phi}{\partial x}\right) &= \nu\frac{\partial^2}{\partial x^2}\left(-2\nu\frac{1}{\phi}\frac{\partial\phi}{\partial x}\right), &(4.114)\\
\frac{\partial}{\partial t}\left(\frac{1}{\phi}\frac{\partial\phi}{\partial x}\right) - 2\nu\frac{1}{\phi}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial\phi}{\partial x}\right) &= \nu\frac{\partial^2}{\partial x^2}\left(\frac{1}{\phi}\frac{\partial\phi}{\partial x}\right), &(4.115)\\
& &(4.116)
\end{aligned}$$

[7]Julian David Cole, 1925-1999, American mathematician.
[8]Eberhart Hopf, 1902-1983, Austrian-American mathematician and astronomer.

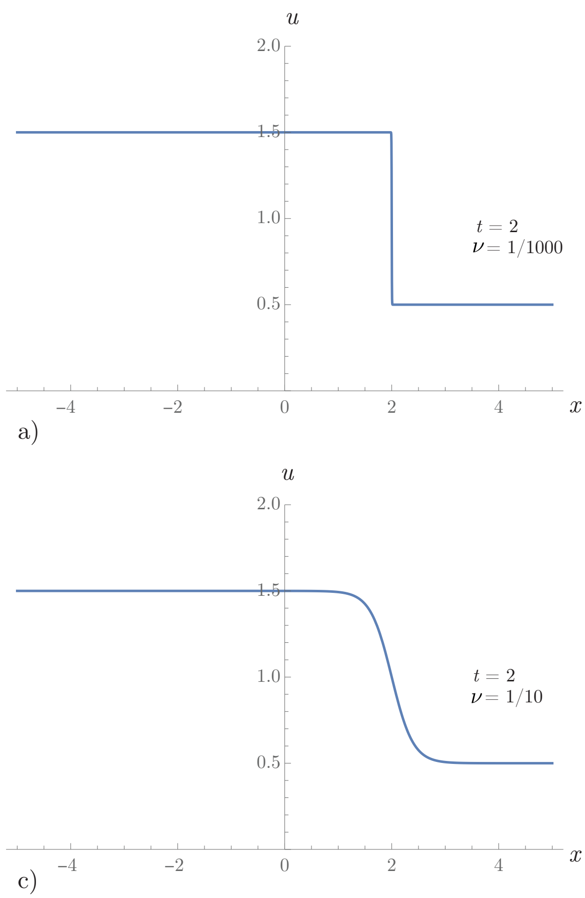


Figure 4.10: Propagating steady wave solution at $t = 2$ to the viscous Burgers' equation $\partial u/\partial t + u\ \partial u/\partial x = \nu\ \partial^2 u/\partial x^2$ with $u(-\infty, t) = u_1 = 3/2$, $u(\infty, t) = u_2 = 1/2$ and various values of ν: a) 1/1000, b) 1/100, c) 1/10 d) 1.

Detailed calculation verifies that this reduces to

$$\frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial \phi}{\partial t}\right) = \nu \frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial^2 \phi}{\partial x^2}\right). \tag{4.117}$$

Regrouping, we can say

$$\frac{\partial}{\partial x}\left(\frac{1}{\phi}\left(\frac{\partial \phi}{\partial t} - \nu\frac{\partial^2 \phi}{\partial x^2}\right)\right) = 0, \tag{4.118}$$

$$\frac{1}{\phi}\left(\frac{\partial \phi}{\partial t} - \nu\frac{\partial^2 \phi}{\partial x^2}\right) = f(t), \tag{4.119}$$

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial x^2} + \phi f(t). \tag{4.120}$$

It suffices to take $f(t) = 0$, leaving us to solve a heat equation:

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial x^2}. \tag{4.121}$$

If $u(x, 0) = g(x)$, then it can be shown that

$$u(x,t) = -2\nu\frac{\partial}{\partial x}\ln\left(\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-r)^2}{4\nu t} - \frac{1}{2\nu}\int_0^r g(s)\ ds\right)\ dr\right). \tag{4.122}$$

Example 4.5

Find $u(x,t)$ for solutions to the viscous Burgers' equation, Eq. (4.80) if

$$u(x,0) = U\frac{x}{L}. \tag{4.123}$$

Direct substitution into Eq. (4.122) gives

$$u(x,t) = -2\nu\frac{\partial}{\partial x}\ln\left(\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-r)^2}{4\nu t} - \frac{1}{2\nu}\int_0^r\left(\frac{Us}{L}\right)\ ds\right)\ dr\right), \tag{4.124}$$

$$= -2\nu\frac{\partial}{\partial x}\ln\left(\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-r)^2}{4\nu t} - \frac{1}{2\nu}\frac{Ur^2}{2L}\right)\ dr\right). \tag{4.125}$$

Symbolic computational software reveals the answer to be simply

$$u(x,t) = U\frac{\frac{x}{L}}{1+\frac{Ut}{L}}. \tag{4.126}$$

It is easily verified that both the initial condition as well as Eq. (4.80) are satisfied. Because the solution is linear in x, it does not depend on the coefficient ν. Thus, it is also a solution to the inviscid Burgers' equation. For $U = 1$, $L = 1$, we plot the solution in Fig. 4.11. Note for large t, more specifically for $Ut/L \gg 1$, our solution reduces to

$$\lim_{t\to\infty} u(x,t) = \frac{x}{t}. \tag{4.127}$$

It is easily seen that $u = x/t$ satisfies the viscous Burgers' equation by direct substitution.

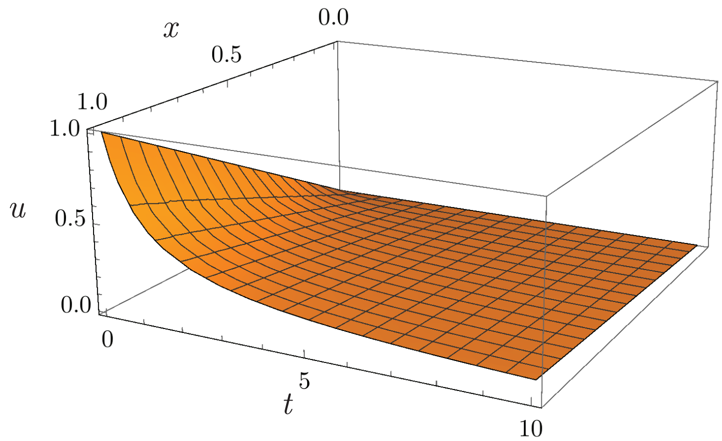


Figure 4.11: Solution to the viscous Burgers' equation $\partial u/\partial t + u\partial u/\partial x = \nu \partial^2 u/\partial x^2$ with $u(x,0) = Ux/L$ with $U = 1$, $L = 1$.

As an aside, we note that the previous example showed $u = x/t$ satisfied Burgers' equation. In fact, direct substitution verifies that

$$u(x,t) = \frac{x+C}{t}, \qquad C \in \mathbb{R}^1, \tag{4.128}$$

where C is any constant satisfies the Burgers' equation. This and a large family of exact solutions are described by Öziş and Aslan.[9] Exact solutions are obtained by first identifying appropriate transformations, then recasting the partial differential equation typically as a nonlinear ordinary differential equation that can be solved. For the limit when $\nu = 1$, one can verify that the solutions given next each satisfy Burgers' equation:

$$u(x,t) = \frac{1}{\sqrt{t}}\left(\frac{-2e^{-(x+t)^2/(4t)}}{C+\sqrt{\pi}\,\mathrm{erf}\left(\frac{x+t}{2\sqrt{t}}\right)}\right) - 1 \qquad C \in \mathbb{R}^1, \tag{4.129}$$

$$u(x,t) = \frac{1}{\sqrt{t}}\left(\frac{x+t}{\sqrt{t}} - \frac{2e^{(x+t)^2/(4t)}}{C+\sqrt{\pi}\,\mathrm{erfi}\left(\frac{x+t}{2\sqrt{t}}\right)}\right) - 1 \qquad C \in \mathbb{R}^1, \tag{4.130}$$

$$u(x,t) = 1 + t - \sqrt{2\left(x - t - \frac{t^2}{2} + C\right)}\,\frac{J_{-2/3}\left(\frac{\sqrt{2}}{3}\left(x - t - \frac{t^2}{2} + C\right)^{3/2}\right)}{J_{1/3}\left(\frac{\sqrt{2}}{3}\left(x - t - \frac{t^2}{2} + C\right)^{3/2}\right)}, \quad C \in \mathbb{R}^1. \tag{4.131}$$

[9] T. Öziş and İ. Aslan, 2017, Similarity solutions to Burgers' equation in terms of special functions of mathematical physics, *Acta Physica Polonica B*, 48(7): 1349-1369.

4.3.4 Method of manufactured solutions

Let us use the method of manufactured solutions, introduced in Sec. 3.2.7, to generate a source term for the viscous Burgers' equation, taking $\nu = 1$. We thus pose the viscous Burgers' equation with a source term as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + f(x,t). \tag{4.132}$$

Let us pose a simple, but non-trivial $u(x,t)$ and find the source term with which it is consistent. Such a solution is useful for verifying numerical methods intended to solve the Burgers' equation. One such simple solution is

$$u(x,t) = x^2 t. \tag{4.133}$$

We substitute this into Eq. (4.132) to get

$$\underbrace{x^2}_{\frac{\partial u}{\partial t}} + \underbrace{(x^2 t)}_{u}\underbrace{(2xt)}_{\frac{\partial u}{\partial x}} - \underbrace{2t}_{\frac{\partial^2 u}{\partial x^2}} = f(x,t). \tag{4.134}$$

So

$$f(x,t) = x^2 + 2x^3t^2 - 2t. \tag{4.135}$$

Thus, we can say the manufactured solution $u = x^2 t$ satisfies the partial differential equation, initial, and boundary conditions given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + x^2 + 2x^3t^2 - 2t, \quad u(x,0) = 0, \quad u(0,t) = 0, \quad u(1,t) = t. \tag{4.136}$$

A plot of the solution is given in Fig. 4.12.

4.4 Traffic flow model

One of the more straightforward and intuitive applications of the notions of this chapter comes in the study of ordinary traffic flow; see Fig. 4.13. Most students are familiar with suddenly and surprisingly coming to a halt in what was freely flowing traffic as a consequence of a red light or other constriction far upstream. One can imagine this as a discontinuity in vehicle density, and it propagates backwards from the site of the traffic blockage. Such a discontinuity is sometimes called a *shock wave*. In this scenario it propagates to the left. Most students are also familiar with the gradual decrease in vehicle density that accompanies a traffic light turning green. This decrease in density is known as a *rarefaction wave*. It is depicted as propagating to the left as well, though it could be moving to the left or the right.

Now let us develop a simple model for traffic flow. Let us take ρ as the vehicle density. For very light traffic density, we might imagine that a doubling of the vehicle density would

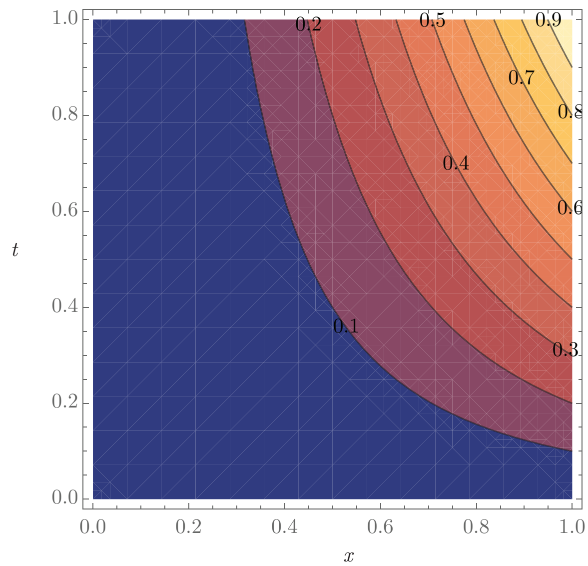


Figure 4.12: Plot of manufactured solution $u(x,t) = x^2t$ that satisfies viscous Burgers' equation with source term and associated initial and boundary conditions, Eq. (4.136).

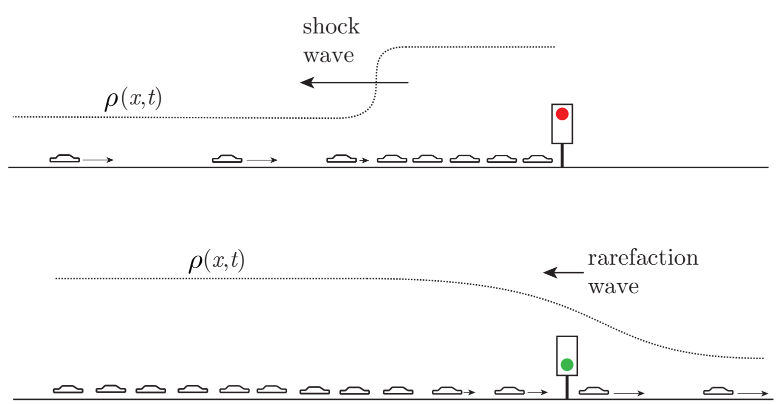


Figure 4.13: Sketch of vehicle traffic density response $\rho(x,t)$ to stop and go signals.

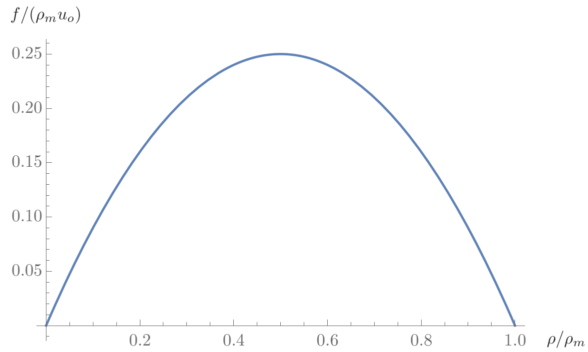


Figure 4.14: Plot of traffic flux f as a function of scaled traffic density ρ/ρ_m for simple quadratic flux model.

double the vehicle flux $f(\rho)$. Certainly when the road becomes too crowded, drivers slow down, so that one might imagine there to be a vehicle density where the flux attains its maximum value f_m. As we increase the density, the traffic begins to jam, and the flux goes down. At a maximum density, ρ_m, we can expect our road to resemble a parking lot: there will be no flux of vehicles, $f(\rho_m) = 0$. Let us take a simple quadratic model for the flux of vehicles:

$$f(\rho) = \rho u_0 \left(1 - \frac{\rho}{\rho_m}\right), \qquad \rho \in [0, \rho_m]. \tag{4.137}$$

Note that our flux function is slightly different than that of Mei's found on his p. 45; ours retains more analogs with the notation and precepts of fluid mechanics, and is easier to justify on dimensional grounds. We restrict density appropriately. Here u_0 is a constant, which we interpret as a characteristic velocity with $u_0 > 0$. A plot of $f/(\rho_m u_0)$ as a function of ρ/ρ_m is given in Fig. 4.14.

Clearly when $\rho = \rho_m$, the flux is zero: $f = 0$. Also when ρ is small, the flux linearly increases with increasing ρ. We have

$$\frac{df}{d\rho} = u_0 \left(1 - \frac{2\rho}{\rho_m}\right), \tag{4.138}$$

$$\frac{d^2 f}{d\rho^2} = -\frac{2u_0}{\rho_m}. \tag{4.139}$$

As the second derivative is strictly negative, any critical point must be a maximum. Importantly, the flux function f for this problem is non-convex. And when $df/d\rho = 0$, we must have

$$\rho = \frac{\rho_m}{2}. \tag{4.140}$$

Thus, the maximum flux is

$$f\left(\frac{\rho_m}{2}\right) = f_m = \frac{\rho_m u_0}{4}. \tag{4.141}$$

For conservation of vehicle density, we specialize Eq. (4.19) with $\mathbf{q} = \rho$, $\mathbf{f} = f$, and $\mathbf{s} = 0$, thus giving

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} f(\rho) = 0, \tag{4.142}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u_0 \left(1 - \frac{\rho}{\rho_m}\right)\right) = 0. \tag{4.143}$$

for the inviscid limit. Expanding the derivative, we find

$$\frac{\partial \rho}{\partial t} + u_0 \left(1 - 2\frac{\rho}{\rho_m}\right)\frac{\partial \rho}{\partial x} = 0. \tag{4.144}$$

Note that the characteristic curves are given by curves whose slope is

$$\frac{dx}{dt} = u_0 \left(1 - 2\frac{\rho}{\rho_m}\right), \tag{4.145}$$

The slope of the curve may be positive or negative and gives velocity of small disturbances; thus small disturbances may propagate to either the left or the right. The speed is dependent on the local value of ρ. Specializing Eq. (4.11) to find the speed of propagation of discontinuous jumps U, we get

$$U = \frac{[\![f(\rho)]\!]}{[\![\rho]\!]} = \frac{\left(\rho_2 u_0 \left(1 - \frac{\rho_2}{\rho_m}\right) - \rho_1 u_0 \left(1 - \frac{\rho_1}{\rho_m}\right)\right)}{\rho_2 - \rho_1}, \tag{4.146}$$

$$= u_0 \left(1 - \frac{\rho_1 + \rho_2}{\rho_m}\right). \tag{4.147}$$

We can postulate a viscous version of Eq. (4.143): Expanding the derivative, we find

$$\frac{\partial \rho}{\partial t} + u_0 \left(1 - 2\frac{\rho}{\rho_m}\right)\frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2}. \tag{4.148}$$

If we now define the transformed dependent variable as

$$w \equiv u_0 \left(1 - 2\frac{\rho}{\rho_m}\right), \tag{4.149}$$

we find Eq. (4.148) transforms to the Burgers' equation,

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \nu \frac{\partial^2 w}{\partial x^2}. \tag{4.150}$$

Note that from Eq. (4.145), w gives the speed of propagation of small disturbances. The inverse transformation gives

$$\rho = \frac{\rho_m}{2}\left(1 - \frac{w}{u_0}\right). \tag{4.151}$$

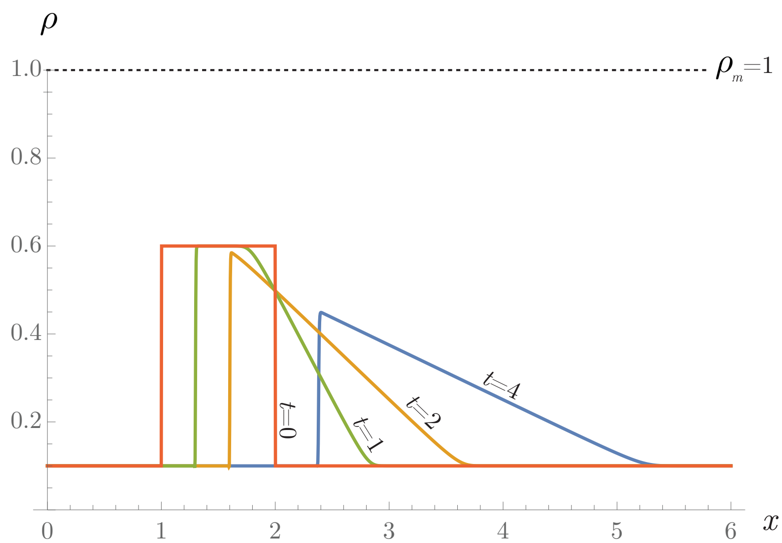


Figure 4.15: Plot of vehicle density ρ as a function of x at various t in a traffic snarl.

Example 4.6

Consider a traffic flow solution where $\rho_m = 1$, $u_0 = 1$, and $\nu = 1/1000$. At $t = 0$, the traffic density is low, except for a small region where it jumps to a significant fraction of the maximum density, then returns to the same low value. Specifically, we take

$$\rho(x,0) = \frac{1}{10} + \frac{1}{2}\left(H(x-1) - H(x-2)\right). \tag{4.152}$$

We can think of this as a small traffic snarl. Find the behavior of the traffic density for $t \in [0,4]$.

With $\rho_m = 1$, $u_0 = 1$, we have

$$f(\rho) = \rho(1-\rho), \tag{4.153}$$

$$f_m = \frac{1}{4}, \tag{4.154}$$

$$w = 1 - 2\rho, \tag{4.155}$$

$$\rho = \frac{1}{2}(1-w). \tag{4.156}$$

We solve the Burgers' equation numerically and perform the appropriate transformations to generate $\rho(x,t)$ A plot of the solution is given in Fig. 4.15. We clearly see a shock and rarefaction, both propagating to the right in the direction of increasing x. As vehicles approach from the right in the region where density is low, they suddenly encounter a steep jump. Vehicles on the downstream side of the snarl gradually decrease their density until they recover the freestream value of 1/10. In contrast to problems with a convex flux function, for this non-convex flux function, the head of the right-propagating wave is a rarefaction and the tail is a shock.

Specializing Eq. (4.147) for the parameters of this problem, we can expect jumps to propagate at speed

$$U = (1)\left(1 - \frac{\frac{1}{10} + \frac{6}{10}}{1}\right) = \frac{3}{10}. \tag{4.157}$$

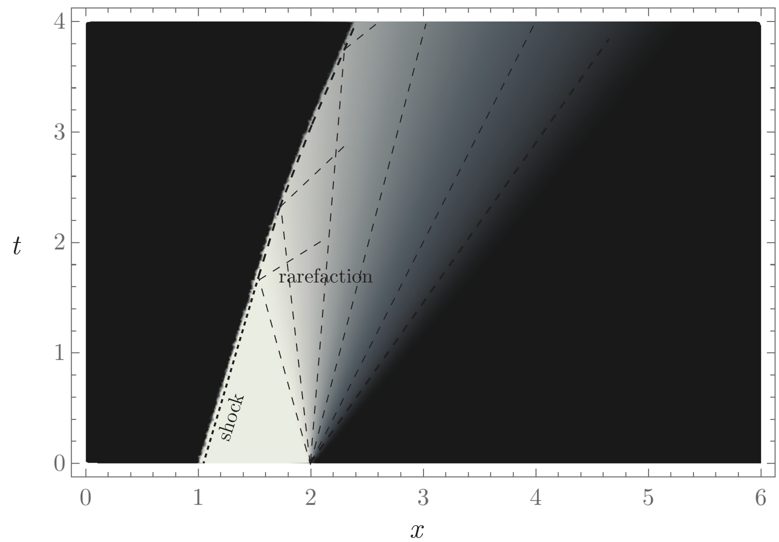


Figure 4.16: $x - t$ diagram of vehicle density ρ in a traffic snarl.

We see by examining Fig. 4.15 that the shock discontinuity at $x = 1$ at $t = 0$ has moved to $x = 1.3$ at $t = 1$, consistent with our theory of discontinuity propagation velocity.

Small disturbances propagate at $w = 1 - 2\rho$. A small disturbance in the low density region of the flow where $\rho = 1/10$ propagates at speed $w = 1 - 2/10 = 4/5$. A small disturbance in the high density region of the flow where $\rho = 3/5$ propagates at speed $w = 1 - 2(3/5) = -1/5$. There is a continuum of speeds for small disturbances that originate near the jump in ρ where $\rho \in [1/10, 3/5]$. These speeds range from $w = [4/5, -1/5]$.

The $x - t$ diagram of Fig. 4.16 summarizes many important concepts. Clearly at early time there is a nearly discontinuous shock propagating to the right at velocity $U = 3/10$. Simultaneously there is a continuous rarefaction, centered at $(x, t) = (2, 0)$. The tail of this rarefaction propagates in the negative x direction, with speed $-1/5$. Around $t = 1.6$, the infinitesimal tail of the rarefaction intersects with the shock, and modulates it. This modulation continues as more infinitesimal rarefaction waves intersect with the shock. Though it is difficult to discern from the plot, we have also sketched reflected waves after the rarefaction strikes the shock. The head of the rarefaction propagates to the right at speed $w = 4/5$.

4.5 Linear dispersive waves

Certainly the inviscid and viscous Burgers' equations we have studied have displayed the feature that their wave form distorts, sometimes dramatically, as time advances. Some of that is an inviscid effect, such as shown in Fig. 4.4; some is a viscous effect, such as shown in Fig. 4.8. The nonlinearity of the Burgers' equation makes closed form analysis difficult.

Let us study these distortions in the context of a simple linear model system,

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{evolution}} + \underbrace{a\frac{\partial u}{\partial x}}_{\text{advection}} = \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\text{diffusion}} + \underbrace{\beta\frac{\partial^3 u}{\partial x^3}}_{\text{dispersion}}. \tag{4.158}$$

The new term here is $\beta\ \partial^3 u/\partial x^3$. We will not provide a physical derivation, but note

- the term is known as a *dispersive* term as it will be seen to induce wave forms to disperse and thus lose their integrity,
- it can arise in a variety of physical scenarios such as in shallow water wave theory,
- it is a useful construct in evaluating higher order errors in various numerical approximations to partial differential equations.

To aid in our analysis, let us assume that we can separate variables so that

$$u(x,t) = A(t)e^{ikx}. \tag{4.159}$$

We have separated $u(x,t)$ into a time-dependent amplitude $A(t)$ and a single Fourier spatial mode with assumed wave number k. Here we use the term e^{ikx} as a convenience for analysis. It does introduce the imaginary number i; if real valued solutions are desired, they can always be achieved by suitably defining complex constants within $A(t)$. Also note that we are really assuming a spatially periodic solution in x with Euler's formula, see Sec. 8.3.1, giving

$$e^{ikx} = \cos kx + i\sin kx. \tag{4.160}$$

Recall from Eq. (3.75) the wave number is $k = 2\pi/\lambda$, where λ is the wavelength. Let us further imagine that we are in a doubly infinite domain, thus $x \in (-\infty, \infty)$. The consequence of this is that there is a *continuous spectrum* of k admitted as solutions in contrast to equations on a finite domain, where discrete spectra, such as displayed in Fig. 3.4, are the only types admitted.

Necessary derivatives of Eq. (4.159) are

$$\frac{\partial u}{\partial t} = \frac{dA}{dt}e^{ikx}, \tag{4.161}$$

$$\frac{\partial u}{\partial x} = ikAe^{ikx}, \tag{4.162}$$

$$\frac{\partial^2 u}{\partial x^2} = -k^2Ae^{ikx}, \tag{4.163}$$

$$\frac{\partial^3 u}{\partial x^3} = -ik^3Ae^{ikx}. \tag{4.164}$$

With these, we see that Eq. (4.158) reduces to

$$\begin{aligned}
\frac{dA}{dt}\cancel{e^{ikx}} + aikA\cancel{e^{ikx}} &= -\nu k^2 A\cancel{e^{ikx}} - \beta i k^3 A\cancel{e^{ikx}}, && (4.165)\\
\frac{dA}{dt} &= -\left(aik + \nu k^2 + \beta i k^3\right) A, && (4.166)\\
A(t) &= A_0 e^{-(\nu k^2 + i(ak+\beta k^3))t}, && (4.167)\\
&= A_0 e^{-\nu k^2 t} e^{-i(ak+\beta k^3)t}. && (4.168)
\end{aligned}$$

Here A_0 is the constant initial value of the amplitude of the Fourier mode. The term $e^{-\nu k^2 t}$ tells us that $A(t)$ has a decaying amplitude for $\nu > 0$ and for all k. Moreover the time scale of amplitude decay is $\tau = 1/(\nu k^2)$: rapid decay is induced by high wave number k and large diffusion coefficient ν. The term $e^{-i(ak+\beta k^3)t}$ is purely oscillatory and does not decay with time. We recombine to form $u(x,t)$ as

$$\begin{aligned}
u(x,t) &= A_0 e^{-\nu k^2 t} e^{-i(ak+\beta k^3)t} e^{ikx}, && (4.169)\\
&= A_0 e^{-\nu k^2 t} e^{ik(x-(a+\beta k^2)t)}. && (4.170)
\end{aligned}$$

Now considering the oscillatory part of $u(x,t)$, if $x - at - \beta k^2 t$ is fixed, a point on the propagating wave is fixed. Let us call that the *phase*, ϕ:

$$\phi = x - (a + \beta k^2)t. \qquad (4.171)$$

The phase has a velocity. If we hold ϕ fixed and differentiate with respect to time we get

$$\begin{aligned}
\underbrace{\frac{d}{dt}\phi}_{=0} &= \frac{dx}{dt} - (a + \beta k^2), && (4.172)\\
\frac{dx}{dt} = c &= a + \beta k^2. && (4.173)
\end{aligned}$$

We note, importantly,

- For $\beta \neq 0$, the phase speed of the Fourier mode depends on the wave number k. *Fourier modes with different k travel at different speeds.* This induces *dispersion* of an initial waveform.

- For $\beta > 0$, high frequency modes, that is those with large k, move rapidly, and are attenuated rapidly.

- For $\beta > 0$, low frequency modes move slowly and are attenuated slowly.

- If $\beta = 0$, all modes travel at the same speed a. Such waves are *non-dispersive.*

- The phase speed is independent of the diffusion coefficient ν.

So positive ν induces amplitude decay but not dispersion; nonzero β induces dispersion but no amplitude decay.

We can rewrite the oscillatory portion as

$$e^{ik(x-(a+\beta k^2)t)} = \exp\Big(i(kx-(ka+\beta k^3)t\Big), \quad (4.174)$$
$$= \exp\left(i(kx-\omega t)\right), \quad (4.175)$$

if we take

$$\omega = ka+\beta k^3. \quad (4.176)$$

In general, the relation

$$\omega = \omega(k), \quad (4.177)$$

is known as the *dispersion relation*. We refer to Whitham for details, where it is seen to be common to define the phase speed $c(k)$ as

$$c(k) = \frac{\omega(k)}{k}, \quad (4.178)$$

This is consistent with our Eq. (4.173) for which we have $c = a+\beta k^2$.

Leaving out details which are provided by Whitham, it is also common to define what is known as the *group velocity* $C(k)$ as

$$C(k) = \frac{d\omega}{dk}. \quad (4.179)$$

While individual Fourier modes propagate with individual phase speeds, it can be shown that the integrated energy of a signal in fact propagates with the group velocity. For our system, differentiating Eq. (4.176) shows the group velocity to be

$$C(k) = a+3\beta k^2. \quad (4.180)$$

Example 4.7

Consider a solution to

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3}. \quad (4.181)$$

Consider an initial condition of a top hat function:

$$u(x,0) = H(x-1) - H(x-2), \quad (4.182)$$

and four different parameter sets: i) $a=1$, $\nu=0$, $\beta=0$, ii) $a=1$, $\nu=1/100$, $\beta=0$, iii) $a=1$, $\nu=1/100$, $\beta=1/1000$, and iv) $a=1$, $\nu=1/1000$, $\beta=1/1000$.

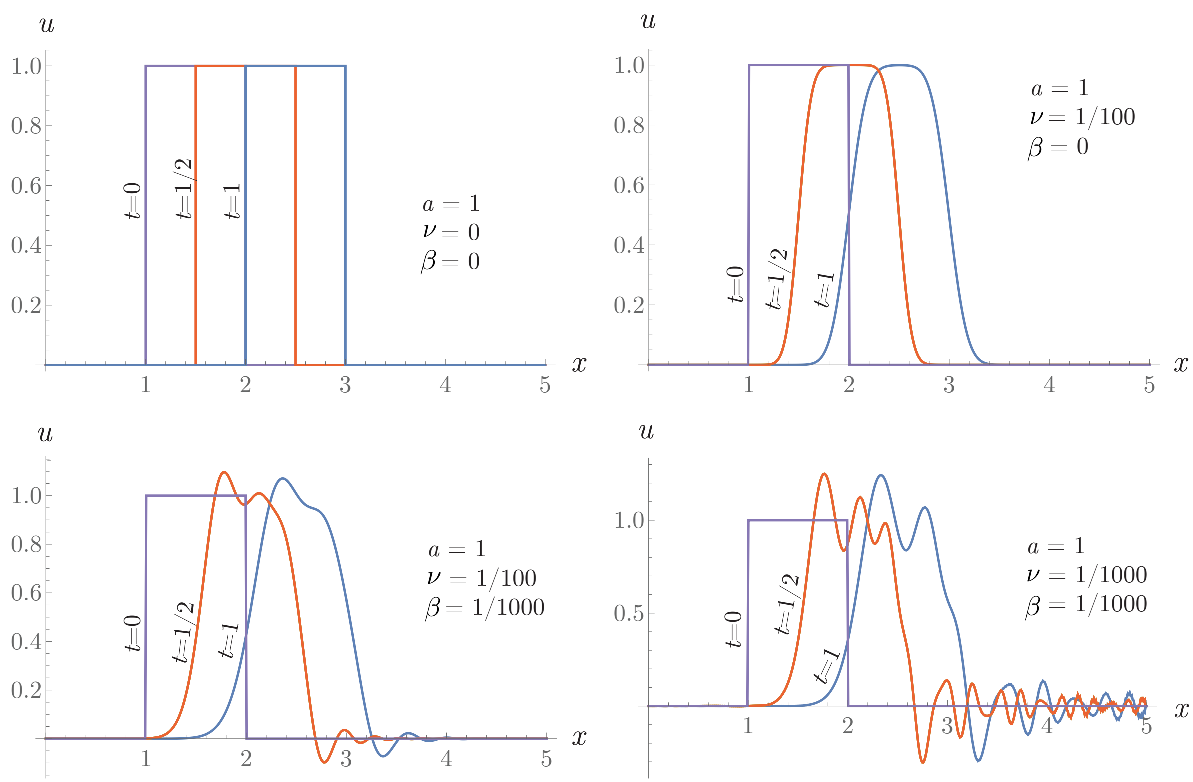


Figure 4.17: Solutions to Eq. (4.158) under conditions indicated.

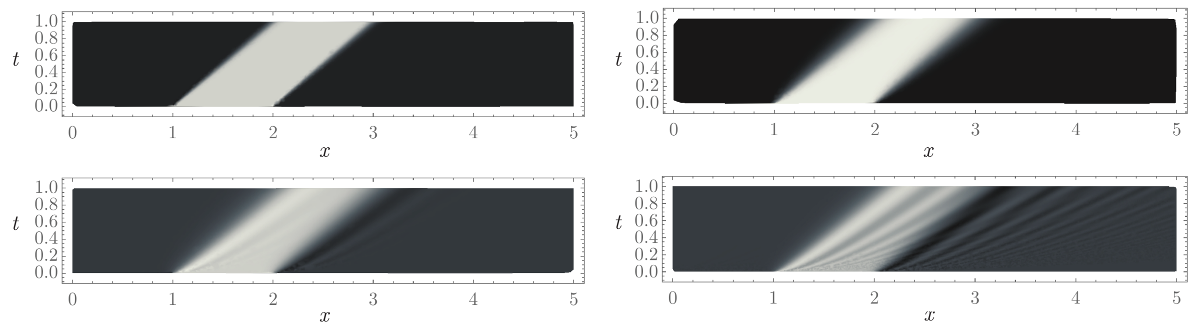


Figure 4.18: $x - t$ diagrams for solutions to Eq. (4.158) under conditions complementary to those of Fig. 4.17.

An individual Fourier mode, in terms of ordinary trigonometric functions, could be specialized from Eq. (4.170) to take the form

$$u(x,t) = A_0 e^{-\nu k^2 t} \sin\left(k(x - (a + \beta k^2)t)\right). \tag{4.183}$$

Omitting details of how to sum the various Fourier modes so as to match the initial conditions, we simply plot results in Fig. 4.17. For $a = 1$, $\nu = 0$, $\beta = 0$, the initial top hat advects to the right and the waveform is otherwise unchanged. For $a = 1$, $\nu = 1/100$, $\beta = 0$, the waveform advects to the right at the same rate, but the diffusion decays the amplitude of the high frequency modes near the sharp interface, smoothing the solution. The wave is advection and diffusing, but not dispersing. For $a = 1$, $\nu = 1/100$, $\beta = 1/1000$, we see some additional high frequency modes moving in front of the initial wave form. This is consistent with the notion that the phase speed for large k is large. This wave is advecting, diffusing, and dispersing. The dispersive effect is more apparent when diffusion is lowered so that $a = 1$, $\nu = 1/1000$, $\beta = 1/1000$.

We show complementary $x - t$ diagrams in Fig. 4.18.

4.6 Stokes' second problem

Let us consider what amounts to Stokes' second problem. It is a problem in one spatial dimension, so it is certainly "one-dimensional." As the governing equation is parabolic and not hyperbolic, it is not a traditional "wave." But in that it has a sinusoidally forced boundary condition, it does have wave-like features in that information from the boundary is propagated into the domain. The propagation mechanism is diffusion rather than advection. Stokes[10] addressed it in his original work which developed the Navier-Stokes equations in the mid-nineteenth century.[11] He addressed it in the context of momentum diffusion; here,

[10]George Gabriel Stokes, 1819-1903, Anglo-Irish mathematician and physicist.

[11]Stokes, G. G., 1851, "On the effect of the internal friction of fluids on the motion of pendulums," *Transactions of the Cambridge Philosophical Society*, 9(2): 8-106.

we shall study its analog in the context of energy diffusion. We shall consider Stokes' first problem later in Sec. 6.1.

Consider then the one-dimensional unsteady heat equation, Eq. (1.81) along with initial and boundary conditions as shown,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \tag{4.184}$$
$$T(x,0) = 0, \quad T(0,t) = T_0 \sin \Omega t, \quad T(\infty, t) < \infty. \tag{4.185}$$

We can imagine this problem physically as one in which a semi-infinite slab is subjected to an oscillatory temperature field at its boundary at $x = 0$. Such might be the case for the surface of the earth during a night-day cycle.

We may recall Euler's formula, Eq. (8.39), derived in Sec. 8.3.1:

$$e^{i\Omega t} = \cos \Omega t + i \sin \Omega t. \tag{4.186}$$

We also recall the real part is defined as

$$\Re \left(e^{i\Omega t} \right) = \cos \Omega t, \tag{4.187}$$

and the imaginary part is defined as

$$\Im \left(e^{i\Omega t} \right) = \sin \Omega t. \tag{4.188}$$

Let us define a related auxiliary problem, with T defined as a complex variable whose imaginary part is T: $\Im(\mathsf{T}) = T$. We then take our extended problem to be

$$\frac{\partial \mathsf{T}}{\partial t} = \alpha \frac{\partial^2 \mathsf{T}}{\partial x^2}, \tag{4.189}$$
$$\mathsf{T}(x,0) = 0, \quad \mathsf{T}(0,t) = T_0 e^{i\Omega t}, \quad |\mathsf{T}(\infty, t)| < \infty. \tag{4.190}$$

Next let us seek a solution which is valid at long time. That is to say, we will not require our solution to satisfy any initial condition but will require it to satisfy the partial differential equation and boundary conditions at $x = 0$ and $x \to \infty$. We will gain many useful insights even though we will not capture the initial condition, which, with extra effort, we could.

Let us separate variables in the following fashion. Assume that

$$\mathsf{T}(x,t) = f(x) e^{i\Omega t}, \tag{4.191}$$

where $f(x)$ is a function to be determined. Ultimately we will only be concerned with the imaginary portion of this solution, which is the portion we will need to match the boundary condition. With this assumption, we find formulæ for the various partial derivatives to be

$$\frac{\partial \mathsf{T}}{\partial t} = i\Omega f(x) e^{i\Omega t}, \tag{4.192}$$
$$\frac{\partial \mathsf{T}}{\partial x} = \frac{df}{dx} e^{i\Omega t}, \tag{4.193}$$
$$\frac{\partial^2 \mathsf{T}}{\partial x^2} = \frac{d^2 f}{dx^2} e^{i\Omega t}. \tag{4.194}$$

Then Eq. (4.189) becomes

$$i\Omega f \cancel{e^{i\Omega t}} = \alpha \frac{d^2 f}{dx^2} \cancel{e^{i\Omega t}}, \tag{4.195}$$

$$i\Omega f = \alpha \frac{d^2 f}{dx^2}. \tag{4.196}$$

Now assume that $f(x) = Ae^{ax}$, giving

$$i\Omega A e^{ax} = A a^2 \alpha e^{ax}, \tag{4.197}$$

$$i = a^2 \frac{\alpha}{\Omega}. \tag{4.198}$$

Now in a polar representation, we note that

$$i = e^{i\pi/2}. \tag{4.199}$$

More generally, we could say

$$i = e^{i(\pi/2 + 2n\pi)}, \quad n = 0, 1, 2, ... \tag{4.200}$$

Thus, Eq. (4.198) can be re-expressed as

$$e^{i(\pi/2+2n\pi)} = a^2 \frac{\alpha}{\Omega}, \tag{4.201}$$

$$\sqrt{\frac{\Omega}{\alpha}} e^{i(\pi/4+n\pi)} = a. \tag{4.202}$$

Using Euler's formula, Eq. (8.39), we could then say

$$a = \sqrt{\frac{\Omega}{\alpha}} \left(\cos\left(\frac{\pi}{4} + n\pi\right) + i \sin\left(\frac{\pi}{4} + n\pi\right) \right), \tag{4.203}$$

$$= \pm \sqrt{\frac{\Omega}{\alpha}} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right), \tag{4.204}$$

$$= \pm \sqrt{\frac{\Omega}{2\alpha}} (1 + i). \tag{4.205}$$

When n is even, we have the "plus" root; when odd, we have the "minus" root. For each root, we can have a solution; thus, we form linear combinations to get

$$f(x) = A_1 \exp\left(\sqrt{\frac{\Omega}{2\alpha}} (1+i) x \right) + A_2 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}} (1+i) x \right). \tag{4.206}$$

Now because we take $\Omega > 0$, $\alpha > 0$ and $x > 0$, we will need $A_1 = 0$ in order to keep $|\mathsf{T}|$ bounded as $x \to \infty$. So we have

$$f(x) = A_2 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}} (1+i) x \right). \tag{4.207}$$

Then recombining, we find that

$$\mathsf{T}(x,t) = A_2 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}}(1+i)x\right)\exp(i\Omega t). \tag{4.208}$$

Now at $x = 0$, we must have

$$T_0 \exp(i\Omega t) = A_2 \exp(i\Omega t). \tag{4.209}$$

We thus need to take $A_2 = T_0$ giving

$$\mathsf{T}(x,t) = T_0 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}}(1+i)x\right)\exp(i\Omega t). \tag{4.210}$$

We then find T by considering only the imaginary portion of T, giving

$$\begin{aligned}
T(x,t) &= \Im\left(T_0 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}}(1+i)x\right)\exp(i\Omega t)\right), && (4.211)\\
&= \Im\left(T_0 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}}(1+i)x + i\Omega t\right)\right), && (4.212)\\
&= \Im\left(T_0 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}}x + i\left(\Omega t - \sqrt{\frac{\Omega}{2\alpha}}x\right)\right)\right), && (4.213)\\
&= T_0 \exp\left(-\sqrt{\frac{\Omega}{2\alpha}}x\right)\sin\left(\Omega t - \sqrt{\frac{\Omega}{2\alpha}}x\right). && (4.214)
\end{aligned}$$

By inspection, the boundary condition is satisfied. Direct substitution reveals the solution also satisfies the heat equation. And clearly as $x \to \infty$, $T \to 0$.

Now the amplitude of this wave-like solution has decayed to roughly $T_0/100$ at a point where

$$\begin{aligned}
\sqrt{\frac{\Omega}{2\alpha}}x &= 4.5, && (4.215)\\
x &= 4.5\sqrt{\frac{2\alpha}{\Omega}}. && (4.216)
\end{aligned}$$

Thus the penetration depth of the wave into the domain is enhanced by high α and low Ω. And below this depth, the material is ambivalent to the disturbance at the boundary. With regards to the oscillatory portion of the solution, we see the angular frequency is Ω and the wave number is $k = \sqrt{\Omega/(2\alpha)}$.

The phase of the wave is given by

$$\phi = \Omega t - \sqrt{\frac{\Omega}{2\alpha}}x. \tag{4.217}$$

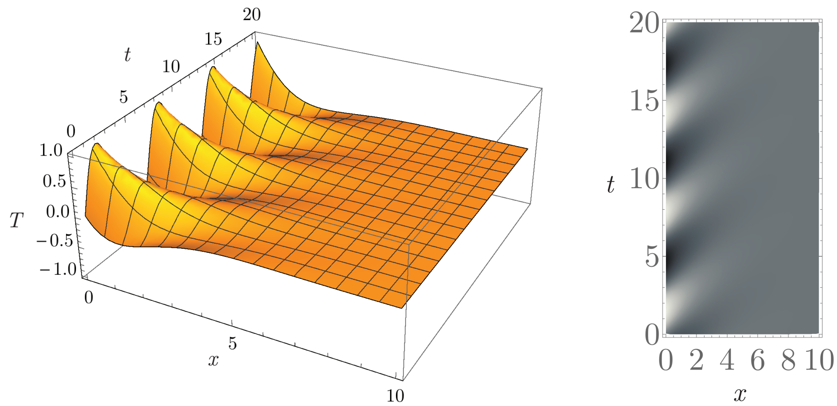


Figure 4.19: Solution to Stokes' second problem with $\alpha = 1$, $T_0 = 1$, $\Omega = 1$.

Let us get the phase speed. If the phase itself is constant, we differentiate to get

$$\frac{d\phi}{dt} = 0 = \Omega - \sqrt{\frac{\Omega}{2\alpha}}\frac{dx}{dt}, \tag{4.218}$$

$$\frac{dx}{dt} = \sqrt{2\alpha\Omega}. \tag{4.219}$$

For $\alpha = 1$, $T_0 = 1$, $\Omega = 1$, we plot $T(x,t)$ in Fig. 4.19. Clearly, for $x \approx 4.5\sqrt{(2)(1)/1} = 6.4$, the effect of the sinusoidal temperature variation at $x = 0$ has small effect.

Problems

Chapter 5

Two-dimensional waves

see Mei, Chapters 8, 10.

Here we consider aspects of two-dimensional wave propagation.

5.1 Helmholtz equation

Consider the multidimensional extension of the wave equation, Eq. (2.15):

$$\frac{\partial^2 \phi}{\partial t^2} = a^2 \nabla^2 \phi. \tag{5.1}$$

Here we have exchanged ϕ for y as we may wish to use y for a coordinate. With $\mathbf{x} = (x, y, z)^T$ representing three-dimensional spatial coordinates, we can separate variables as follows

$$\phi(\mathbf{x}, t) = u(t)v(\mathbf{x}). \tag{5.2}$$

With this assumption, Eq. (5.1) becomes

$$\begin{aligned} v\frac{d^2 u}{dt^2} &= a^2 u \nabla^2 v, & (5.3)\\ \frac{1}{a^2 u}\frac{d^2 u}{dt^2} &= \frac{1}{v}\nabla^2 v = -\frac{1}{\lambda^2}. & (5.4) \end{aligned}$$

Our choice of the constant to be $-1/\lambda^2$ is non-traditional, but will have an improved physical interpretation. This induces the equations

$$\begin{aligned} \nabla^2 v + \frac{1}{\lambda^2} v &= 0, & (5.5)\\ \frac{d^2 u}{dt^2} + \left(\frac{a}{\lambda}\right)^2 u &= 0. & (5.6) \end{aligned}$$

Equation (5.5) is known as a *Helmholtz*[1] equation. It is a linear elliptic partial differential equation. Because of its linearity, its solution can be decomposed into various eigenmodes. The appropriate eigenfunctions will depend on the particular geometry.

Equation (5.6) has solution

$$u(t) = C_1 \sin \frac{at}{\lambda} + C_2 \cos \frac{at}{\lambda}. \tag{5.7}$$

We note that if a has the units of velocity, and t time, then at has units of length, and so must λ.

5.2 Square domain

Let us consider Eq. (5.5) in a two-dimensional Cartesian geometry with $\mathbf{x} = (x, y)^T$, on the square $x \in [0, L]$, $y \in [0, L]$. Let us insist that $\phi(x, 0, t) = 0$, $\phi(x, L, t) = 0$, $\phi(L, y, t) = 0$. We will allow an inhomogeneous Dirichlet boundary condition at $\phi(0, y, t) = f(y, t)$. Equation (5.5) becomes

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{\lambda^2} v = 0. \tag{5.8}$$

Let us separate variables again by assuming

$$v(x, y) = w(x) z(y). \tag{5.9}$$

Substituting, we get

$$\begin{aligned} z \frac{d^2 w}{dx^2} + w \frac{d^2 z}{dy^2} + \frac{1}{\lambda^2} wz &= 0, && (5.10)\\ \frac{1}{w} \frac{d^2 w}{dx^2} + \frac{1}{z} \frac{d^2 z}{dy^2} + \frac{1}{\lambda^2} &= 0, && (5.11)\\ \frac{1}{z} \frac{d^2 z}{dy^2} = -\frac{1}{w} \frac{d^2 w}{dx^2} - \frac{1}{\lambda^2} &= -\frac{1}{\mu^2}. && (5.12) \end{aligned}$$

This induces

$$\begin{aligned} \frac{d^2 z}{dy^2} + \frac{1}{\mu^2} z &= 0, && (5.13)\\ \frac{d^2 w}{dx^2} + \left(\frac{1}{\lambda^2} - \frac{1}{\mu^2} \right) w &= 0. && (5.14) \end{aligned}$$

We find

$$z(y) = C_1 \sin \frac{y}{\mu} + C_2 \cos \frac{y}{\mu}. \tag{5.15}$$

[1]Herman Ludwig Ferdinand von Helmholtz, 1821-1894, German physician and physicist.

To satisfy the homogeneous Dirichlet boundary conditions, we must have $C_2 = 0$ and $1/\mu = n\pi/L$. Thus

$$z(y) = C_1 \sin \frac{n\pi y}{L}, \qquad n = 1, 2, \ldots. \tag{5.16}$$

And

$$\frac{d^2 w}{dx^2} + \left(\frac{1}{\lambda^2} - \frac{n^2\pi^2}{L^2}\right) w = 0. \tag{5.17}$$

For $\lambda < L/(n\pi)$, the solution is oscillatory; while for $\lambda > L/(n\pi)$, the solution will have an exponential character. In physics, the eigenfunctions for the oscillatory case are characterized as "bound states." In terms of the operator $-d^2/dx^2 - 1/\lambda^2 + n^2\pi^2/L^2$, we see that it is positive definite for $\lambda > L/(n\pi)$. Thus all its eigenvalues are positive. However for $\lambda < L/(n\pi)$, some of the eigenvalues may be negative, inducing the bound states. The solution is

$$w(x) = \begin{cases} C_3 \cosh\left(\sqrt{n^2\pi^2 - \frac{L^2}{\lambda^2}}\frac{x}{L}\right) + C_4 \sinh\left(\sqrt{n^2\pi^2 - \frac{L^2}{\lambda^2}}\frac{x}{L}\right), & \lambda > L/(n\pi), \\ C_3 \cos\left(\sqrt{\frac{L^2}{\lambda^2} - n^2\pi^2}\frac{x}{L}\right) + C_4 \sin\left(\sqrt{\frac{L^2}{\lambda^2} - n^2\pi^2}\frac{x}{L}\right) & \lambda < L/(n\pi). \end{cases} \tag{5.18}$$

One simple solution is

$$\phi(x,y,t) = \begin{cases} C \cos\left(\frac{at}{\lambda}\right) \sin\left(\frac{n\pi y}{L}\right) \cosh\left(\sqrt{n^2\pi^2 - \frac{L^2}{\lambda^2}}\frac{x}{L}\right)\left(1 - \frac{\tanh\left(\sqrt{n^2\pi^2 - \frac{L^2}{\lambda^2}}\frac{x}{L}\right)}{\tanh\left(\sqrt{n^2\pi^2 - \frac{L^2}{\lambda^2}}\right)}\right), \\ \qquad \lambda > L/(n\pi), \\ C \cos\left(\frac{at}{\lambda}\right) \sin\left(\frac{n\pi y}{L}\right) \cos\left(\sqrt{\frac{L^2}{\lambda^2} - n^2\pi^2}\frac{x}{L}\right)\left(1 - \frac{\tan\left(\sqrt{\frac{L^2}{\lambda^2} - n^2\pi^2}\frac{x}{L}\right)}{\tan\left(\sqrt{\frac{L^2}{\lambda^2} - n^2\pi^2}\right)}\right), \\ \qquad \lambda < L/(n\pi). \end{cases} \tag{5.19}$$

It satisfies the partial differential equation, the boundary conditions at $y = L$, and $x = L$. And it admits the inhomogeneous boundary condition at $x = 0$ of

$$\phi(0,y,t) = C\cos\left(\frac{at}{\lambda}\right)\sin\left(\frac{n\pi y}{L}\right), \tag{5.20}$$

$$= \frac{C}{2}\left(\sin\left(\frac{n\pi y}{L} - \frac{at}{\lambda}\right) + \sin\left(\frac{n\pi y}{L} + \frac{at}{\lambda}\right)\right), \tag{5.21}$$

$$= \frac{C}{2}\left(\sin\left(\frac{n\pi}{L}\left(y - \frac{at}{\lambda}\frac{L}{n\pi}\right)\right) + \sin\left(\frac{n\pi}{L}\left(y + \frac{at}{\lambda}\frac{L}{n\pi}\right)\right)\right). \tag{5.22}$$

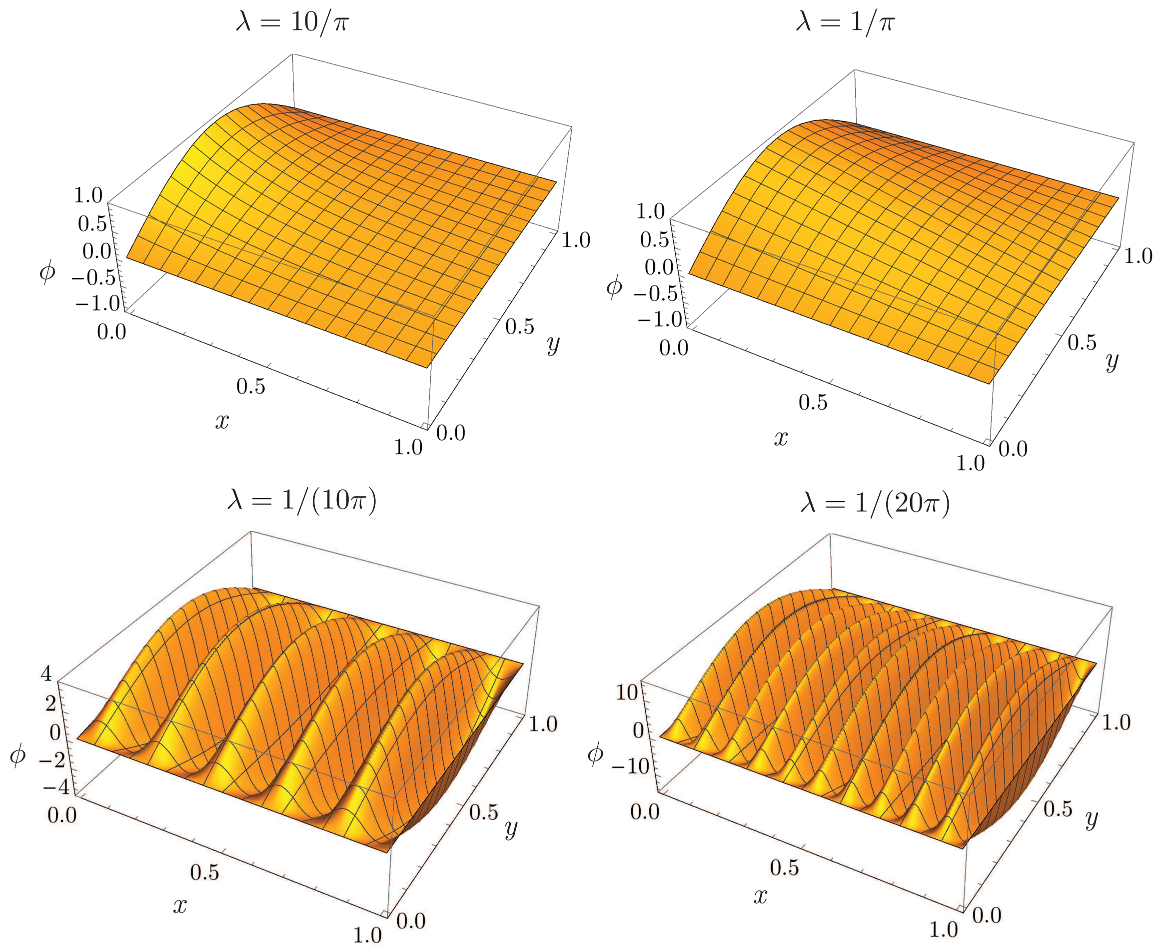


Figure 5.1: Plots of solution to the two-dimensional wave equation within a square domain for various values of λ with $n = 1$.

This boundary condition holds for all λ and n. We note the phase speed of the time-dependent boundary condition here is $aL/(\lambda n\pi)$. More general boundary conditions could be addressed with Fourier series expansions and use of the principle of superposition. A well-posed problem requires two initial conditions, and those can be deduced easily from our solution by examining $\phi(x, y, 0)$ and $\partial\phi/\partial t(x, y, 0)$. This analysis is not shown here.

We next consider some relevant plots for particular parameter values. In all plots we will take $C = 1$, $a = 1$, $L = 1$. We will study the effect of variable λ and variable n, which enter in the specification of the inhomogeneous boundary condition. We first fix $n = 1$ and present results at $t = 0$ for $\lambda = 10/\pi$, $1/\pi$, $1/(10\pi)$, and $1/(20\pi)$ in Fig. 5.1. We note that all figures display a matching of the various boundary conditions. We must envision the left boundary at $x = 0$ oscillating and propagating disturbances into the domain. With animation available with modern software, this can be visualized. We note for large λ that

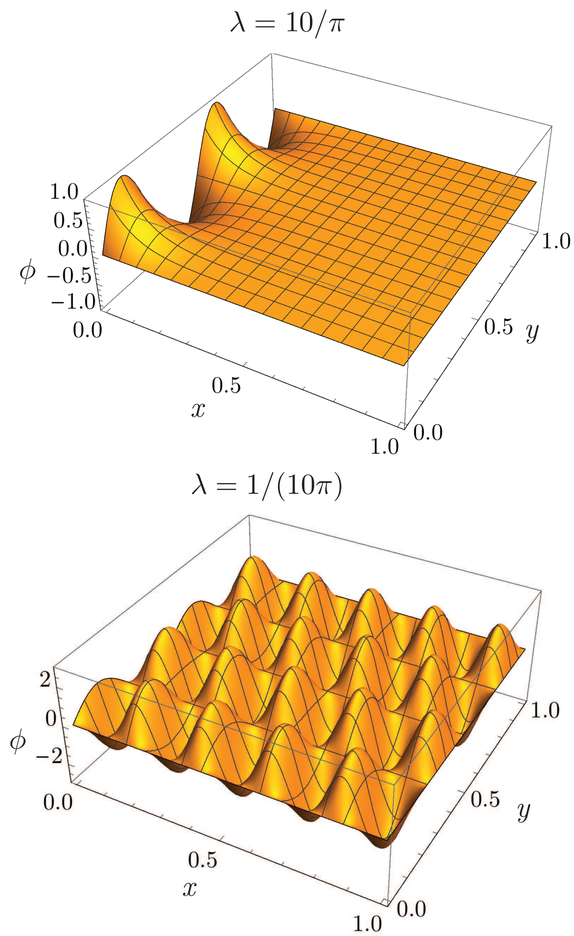


Figure 5.2: Plots of solution to the two-dimensional wave equation within a square domain for various values of λ with $n = 4$.

there is a simple decay of the boundary disturbance to zero at the other three boundaries. As λ increases, the essential behavior does not change until λ crosses below the threshold of $L/(n\pi) = 1/\pi$. Below this threshold, we find resonant structures oscillating within the domain. As λ decreases, we find the wavelength of those structures decreases, and the amplitude of the oscillations increases.

Let us next examine initial disturbances with higher wave number. We take $n = 4$ and present results at $t = 0$ for $\lambda = 10/\pi$, $1/(4\pi)$, $1/(10\pi)$, and $1/(20\pi)$ in Fig. 5.2. We see interesting phenomena here. First we note that large λ gives rise to a suppression of the signal penetration into the domain. For $\lambda \to \infty$, we see that the solution goes as $\cosh(n\pi x/L)$ and thus the penetration depth goes like $L/(n\pi)$. So high wave number disturbances are not felt in the interior. For a critical value of $\lambda = L/(n\pi) = 1/(4\pi)$ for us, the signal is felt through the entire domain, but it decays moderately to zero at the $x = 1$ boundary. For smaller λ, resonance patterns emerge and become the dominant structures.

5.3 Circular domain

Let us now consider Eq. (5.5) in a two-dimensional polar geometry with $\mathbf{x} = (r, \theta)^T$, within the domain bounded by $r \in [0, R]$, $\theta \in [0, 2\pi]$. Drawing upon Eq. (3.259), we specialize our earlier results for ∇^2 in cylindrical coordinates to the plane polar case and find Eq. (5.5) expands as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{\lambda^2}v = 0. \tag{5.23}$$

Let us separate variables once more:

$$v(r, \theta) = w(r)z(\theta). \tag{5.24}$$

Substituting, we get

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(wz)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(wz) + \frac{1}{\lambda^2}wz &= 0, && (5.25)\\
\frac{z}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) + \frac{w}{r^2}\frac{d^2 z}{d\theta^2} + \frac{1}{\lambda^2}wz &= 0, && (5.26)\\
\frac{r}{w}\frac{d}{dr}\left(r\frac{dw}{dr}\right) + \frac{1}{z}\frac{d^2 z}{d\theta^2} + \frac{r^2}{\lambda^2} &= 0, && (5.27)\\
\frac{r}{w}\frac{d}{dr}\left(r\frac{dw}{dr}\right) + \frac{r^2}{\lambda^2} &= -\frac{1}{z}\frac{d^2 z}{d\theta^2} = \alpha^2. && (5.28)
\end{aligned}$$

This induces two ordinary differential equations:

$$\begin{aligned}
\frac{d^2 z}{d\theta^2} + \alpha^2 z &= 0, && (5.29)\\
r\frac{d}{dr}\left(r\frac{dw}{dr}\right) + \left(\frac{r^2}{\lambda^2} - \alpha^2\right)w &= 0. && (5.30)
\end{aligned}$$

Solution to Eq. (5.29) is seen to be

$$z(\theta) = C_1 \sin\alpha\theta + C_2 \cos\alpha\theta. \tag{5.31}$$

Now we would like both ϕ and its derivatives to be periodic in θ. As done earlier in Sec. 3.3.1, we can achieve this by requiring $z(0) = z(2\pi)$ and $dz/d\theta(0) = dz/d\theta(2\pi)$. The two conditions are

$$\begin{aligned}
C_2 &= C_1 \sin 2\pi\alpha + C_2 \cos 2\pi\alpha, && (5.32)\\
\alpha C_1 &= \alpha C_1 \cos 2\pi\alpha - \alpha C_2 \sin 2\pi\alpha. && (5.33)
\end{aligned}$$

We write this as a linear system,

$$\begin{pmatrix} \sin 2\pi\alpha & \cos 2\pi\alpha - 1 \\ \cos 2\pi\alpha - 1 & -\sin 2\pi\alpha \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.34}$$

For a nontrivial solution, we insist the determinant of the coefficient matrix be zero, giving

$$\begin{aligned} -\sin^2 2\pi\alpha - (\cos 2\pi\alpha - 1)^2 &= 0, & (5.35)\\ -\sin^2 2\pi\alpha - (\cos^2 2\pi\alpha - 2\cos 2\pi\alpha + 1) &= 0, & (5.36)\\ \underbrace{-\sin^2 2\pi\alpha - \cos^2 2\pi\alpha}_{-1} + 2\cos 2\pi\alpha - 1 &= 0, & (5.37)\\ 2\cos 2\pi\alpha &= 2, & (5.38)\\ \cos 2\pi\alpha &= 1. & (5.39) \end{aligned}$$

For this, we require that

$$\alpha = n, \qquad n = 0, 1, 2, \ldots. \tag{5.40}$$

So, Eq. (5.31) reduces to

$$z(\theta) = C_1 \sin n\theta + C_2 \cos n\theta, \qquad n = 0, 1, 2, \ldots \tag{5.41}$$

With this, Eq. (5.30) becomes

$$r\frac{d}{dr}\left(r\frac{dw}{dr}\right) + \left(\frac{r^2}{\lambda^2} - n^2\right) w = 0, \tag{5.42}$$

that has solution

$$w(r) = C_3 J_n\left(\frac{r}{\lambda}\right) + C_4 Y_n\left(\frac{r}{\lambda}\right). \tag{5.43}$$

As $\lim_{r\to 0} Y_n(r/\lambda) \to -\infty$, we take $C_4 = 0$ to keep ϕ bounded.

We can compose a single mode of a solution as

$$\begin{aligned} \phi(r,\theta,t) &= u(t)w(r)z(\theta), & (5.44)\\ &= C\cos\left(\frac{at}{\lambda}\right) J_n\left(\frac{r}{\lambda}\right)\cos(n\theta). & (5.45) \end{aligned}$$

Of course, we could expand to include the sin component in both t and θ, and we could sum modes so as to match some specified initial condition. Realizing that is possible, let us simply study this simple solution, Eq. (5.45). Similar to the solution in the square domain of Sec. 5.2, let us restrict attention to $C = 1$, $R = 1$, and $a = 1$. So we have a special solution of

$$\phi(r,\theta,t) = \cos\left(\frac{t}{\lambda}\right) J_n\left(\frac{r}{\lambda}\right)\cos(n\theta). \tag{5.46}$$

At $t = 0$, this solution takes the form

$$\phi(r,\theta,0) = J_n\left(\frac{r}{\lambda}\right)\cos(n\theta). \tag{5.47}$$

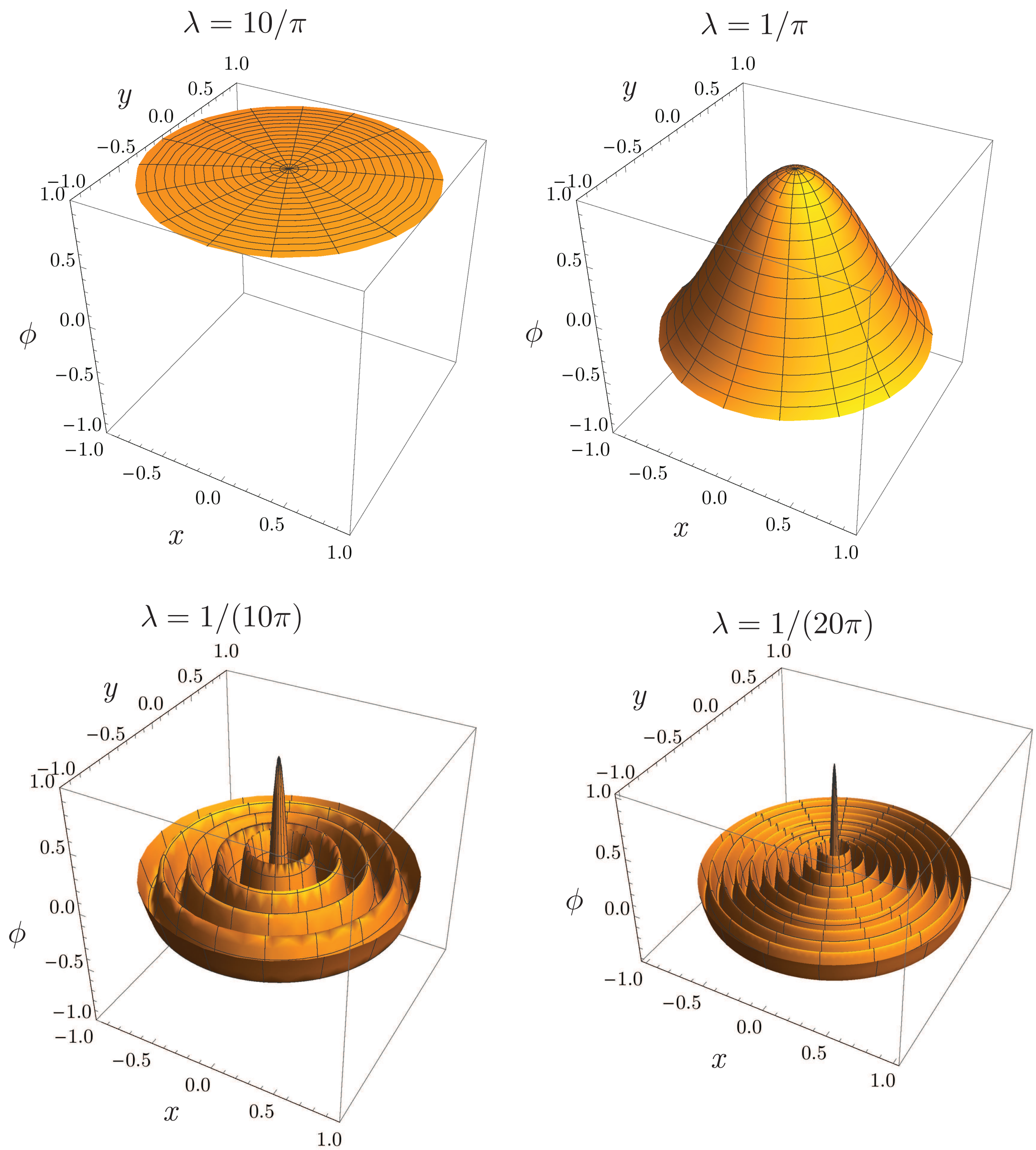


Figure 5.3: Plots at $t = 0$ of solution to the two-dimensional wave equation within a circular domain for various values of λ with $n = 0$.

We will study the effect of varying n and λ. We first fix $n = 0$ and present results at $t = 0$ for $\lambda = 10/\pi$, $1/\pi$, $1/(10\pi)$ and $1/(20\pi)$ in Fig. 5.3. For large λ, ϕ has little variation with space, and simply oscillates between ± 1 at a frequency dictated by λ. As λ decreases, more and more eigenmodes are bound within the domain. This is consistent with the results of Sec. 5.2. For solutions with $n = 0$, there is no variation with θ. In this section, we may imagine that ϕ at $r = 1$ is controlled; thus, the entire boundary of the circular domain may be nontrivial. This contrasts Sec. 5.2, where three of the boundaries were homogeneous and one was controlled.

While there appears to be a singularity at $r = 0$ for smaller values of λ, one can show that in fact the solution is finite for finite λ. In fact for $n = 0$, a Taylor series of ϕ taken in the limit of small r and small t gives

$$\phi \sim C\left(1 - \frac{r^2}{4\lambda^2} - \frac{a^2 t^2}{2\lambda^2} + \dots\right). \tag{5.48}$$

Certainly $\phi \sim C$ as $r \to 0$ and $t \to 0$. But we might expect some interesting behavior for small λ.

We next fix $n = 4$ and present results at $t = 0$ for $\lambda = 10/\pi$, $1/\pi$, $1/(10\pi)$ and $1/(20\pi)$ in Fig. 5.4. For large λ, the ϕ again has little variation with space. As λ decreases, more and more eigenmodes are bound within the domain. For solutions with $n = 4$, there is variation with θ.

Problems

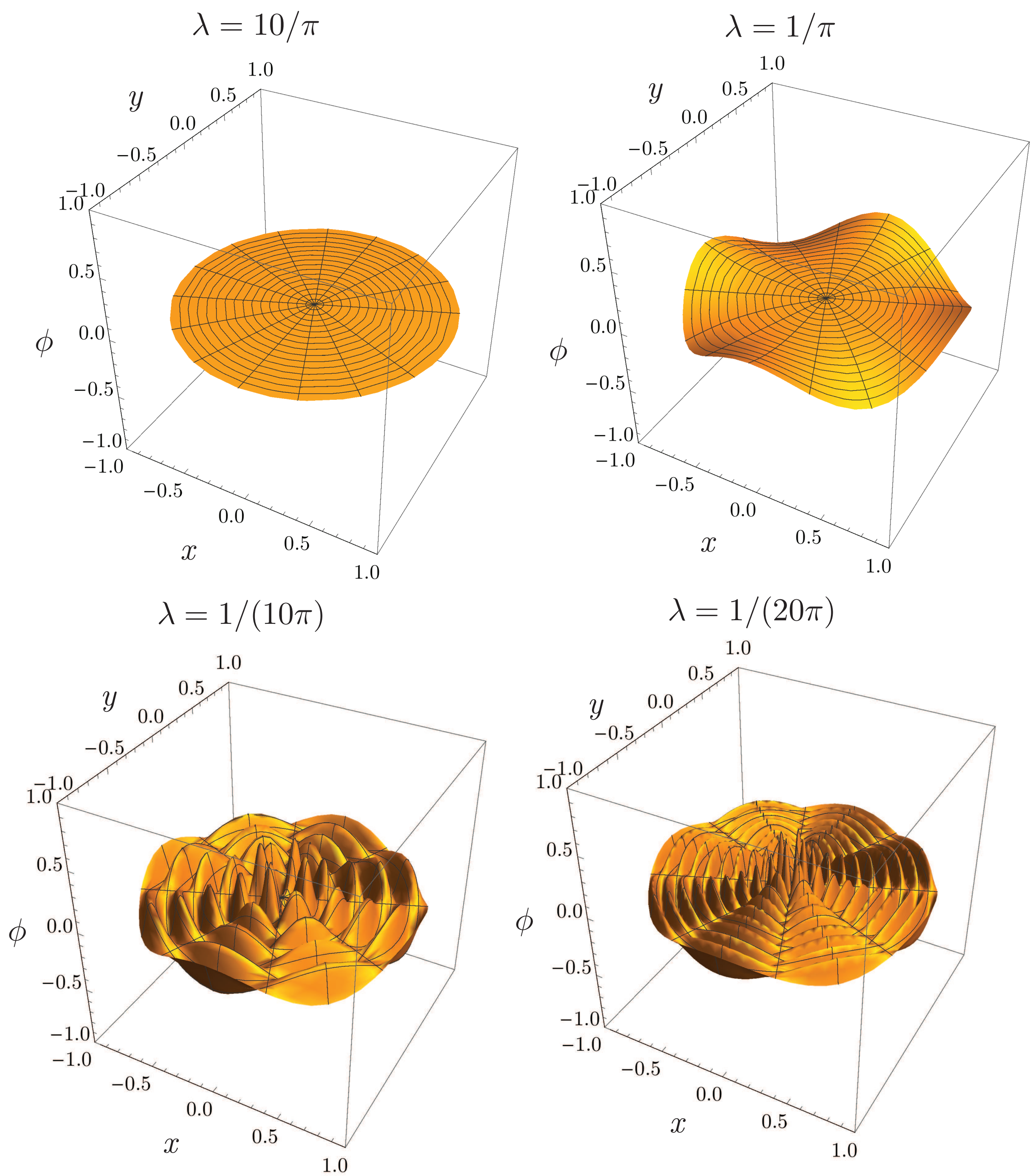


Figure 5.4: Plots of solution at $t = 0$ to the two-dimensional wave equation within a circular domain for various values of λ with $n = 4$.

Chapter 6

Self-similar solutions

see Cantwell.

Here we consider self-similar solutions. We will consider problems which can be addressed by what is known as a similarity transformation. The problems themselves will be fundamental ones which have variation in either time and one spatial coordinate, or with two spatial coordinates. Because two coordinates are involved, we must resort to solving partial differential equations. The similarity transformation actually reveals a hidden symmetry of the partial differential equations by defining a new independent variable, which is a grouping of the original independent variables, under which the partial differential equations transform into ordinary differential equations. We then solve the resulting ordinary differential equations by standard techniques.

6.1 Stokes' first problem

The first problem we will consider which uses a similarity transformation is known as Stokes' first problem. As with Stokes' second problem, Sec. 4.6, Stokes addressed it in his original work which developed the Navier-Stokes equations in the mid-nineteenth century.[1] The problem is described as follows, and is sketched in Figure 6.1. Consider a flat plate of infinite extent lying at rest for $t < 0$ on the $y = 0$ plane in $x - y - z$ space. In the volume described by $y > 0$ exists a fluid of semi-infinite extent which is at rest at time $t < 0$. At $t = 0$, the flat plate is suddenly accelerated to a constant velocity of U, entirely in the x direction. Because the no-slip condition is satisfied for the viscous flow, this induces the fluid at the plate surface to acquire an instantaneous velocity of $u(x = 0, t \geq 0) = U$. Because of diffusion of linear x momentum via tangential viscous shear forces, the fluid in the region above the plate begins to acquire a positive velocity in the x direction as well.

[1] Stokes, G. G., 1851, "On the effect of the internal friction of fluids on the motion of pendulums," *Transactions of the Cambridge Philosophical Society*, 9(2): 8-106.

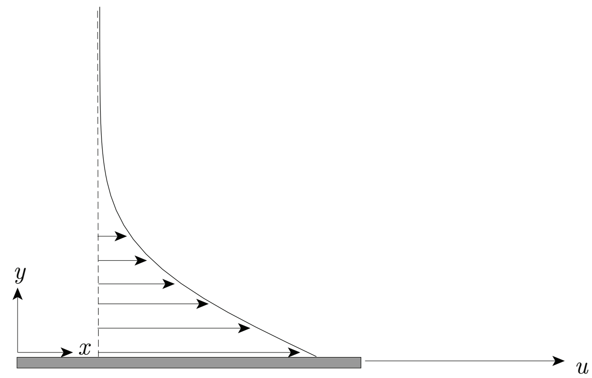


Figure 6.1: Schematic for Stokes' first problem of a suddenly accelerated plate diffusing linear momentum into a fluid at rest.

Using standard assumptions, the linear momentum principle reduces to

$$\underbrace{\rho\frac{\partial u}{\partial t}}_{\text{(mass)(acceleration)}} = \underbrace{\mu\frac{\partial^2 u}{\partial y^2}}_{\text{shear force}} . \tag{6.1}$$

Employing the momentum diffusivity definition $\nu = \mu/\rho$, we get the following partial differential equation, initial and boundary conditions:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{6.2}$$

$$u(y,0) = 0, \qquad u(0,t) = U, \qquad u(\infty,t) = 0. \tag{6.3}$$

We note that Eq. (6.2) which describes the diffusion of linear momentum is mathematically identical to the heat equation, Eq. (1.81) which describes the diffusion of energy.[2]

Now let us scale the equations. Choose

$$u_* = \frac{u}{U}, \qquad t_* = \frac{t}{t_c}, \qquad y_* = \frac{y}{y_c}. \tag{6.4}$$

We have yet to choose characteristic length, (y_c), and time, (t_c), scales. The equations become

$$\frac{U}{t_c}\frac{\partial u_*}{\partial t_*} = \frac{\nu U}{y_c^2}\frac{\partial^2 u_*}{\partial y_*^2}, \tag{6.5}$$

$$\frac{\partial u_*}{\partial t_*} = \frac{\nu t_c}{y_c^2}\frac{\partial^2 u_*}{\partial y_*^2}. \tag{6.6}$$

[2]The analog to temperature T is velocity u. The analog to Fourier's law, Eq. (1.77), $q_x = -k\ \partial T/\partial x$ is that of a Newtonian fluid, which in one dimension reduces to $\tau = \mu\ \partial u/\partial x$, where τ is the viscous shear stress. The analog to the energy equation, Eq. (1.75), $\rho\ \partial e/\partial t = -\partial q_x/\partial x$ is Newton's second law, which reduces to $\rho\ \partial u/\partial t = \mu\ \partial \tau/\partial x$. The analog to thermal diffusivity $\alpha = k/\rho c$ is momentum diffusivity $\nu = \mu/\rho$.

We choose

$$y_c \equiv \frac{\nu}{U} = \frac{\mu}{\rho U}. \tag{6.7}$$

Noting the SI units, we see $\mu/(\rho U)$ has units of length: $\frac{\mathrm{N}\ \mathrm{s}}{\mathrm{m}^2}\frac{\mathrm{m}^3}{\mathrm{kg}}\frac{\mathrm{s}}{\mathrm{m}} = \frac{\mathrm{kg}\ \mathrm{m}}{\mathrm{s}^2}\frac{\mathrm{s}}{\mathrm{m}^2}\frac{\mathrm{m}^3}{\mathrm{kg}}\frac{\mathrm{s}}{\mathrm{m}} = \mathrm{m}$. With this choice, we get

$$\frac{\nu t_c}{y_c^2} = \frac{\nu t_c U^2}{\nu^2} = \frac{t_c U^2}{\nu}. \tag{6.8}$$

This suggests we choose

$$t_c = \frac{\nu}{U^2}. \tag{6.9}$$

With all of these choices, the complete system can be written as

$$\frac{\partial u_*}{\partial t_*} = \frac{\partial^2 u_*}{\partial y_*^2}, \tag{6.10}$$

$$u_*(y_*,0) = 0, \qquad u_*(0,t_*) = 1, \qquad u_*(\infty,t_*) = 0. \tag{6.11}$$

Now for *self-similarity*, we seek transformation which reduce this partial differential equation, as well as its initial and boundary conditions, into an ordinary differential equation with suitable boundary conditions. If this transformation does not exist, no similarity solution exists. In this, but not all cases, the transformation does exist.

Let us first consider a general transformation from a y_*, t_* coordinate system to a new $\eta_*, \hat{t}_*$ coordinate system. We assume then a general transformation

$$\eta_* = \eta_*(y_*,t_*), \tag{6.12}$$

$$\hat{t}_* = \hat{t}_*(y_*,t_*). \tag{6.13}$$

We assume then that a general variable ψ_* which is a function of y_* and t_* also has the same value at the transformed point $\eta_*, \hat{t}_*$:

$$\psi_*(y_*,t_*) = \psi_*(\eta_*,\hat{t}_*). \tag{6.14}$$

The chain rule then gives expressions for derivatives:

$$\left.\frac{\partial \psi_*}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*}\left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} + \left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*}\left.\frac{\partial \hat{t}_*}{\partial t_*}\right|_{y_*}, \tag{6.15}$$

$$\left.\frac{\partial \psi_*}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*}\left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*} + \left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*}\left.\frac{\partial \hat{t}_*}{\partial y_*}\right|_{t_*}. \tag{6.16}$$

Now we will restrict ourselves to the transformation

$$\hat{t}_* = t_*, \tag{6.17}$$

so we have $\left.\frac{\partial \hat{t}_*}{\partial t_*}\right|_{y_*} = 1$ and $\left.\frac{\partial \hat{t}_*}{\partial y_*}\right|_{t_*} = 0$, so our rules for differentiation reduce to

$$\left.\frac{\partial \psi_*}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*} \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} + \left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*}, \tag{6.18}$$

$$\left.\frac{\partial \psi_*}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \psi_*}{\partial \eta_*}\right|_{\hat{t}_*} \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*}. \tag{6.19}$$

The next assumption is key for a similarity solution to exist. We restrict ourselves to transformations for which $\psi_* = \psi_*(\eta_*)$. That is we allow no dependence of ψ_* on $\hat{t}_*$. Hence we must require that $\left.\frac{\partial \psi_*}{\partial \hat{t}_*}\right|_{\eta_*} = 0$. Moreover, partial derivatives of ψ_* become total derivatives, giving us a final form of transformations for the derivatives

$$\left.\frac{\partial \psi_*}{\partial t_*}\right|_{y_*} = \frac{d\psi_*}{d\eta_*} \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*}, \tag{6.20}$$

$$\left.\frac{\partial \psi_*}{\partial y_*}\right|_{t_*} = \frac{d\psi_*}{d\eta_*} \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*}. \tag{6.21}$$

In terms of operators we can say

$$\left.\frac{\partial}{\partial t_*}\right|_{y_*} = \left.\frac{\partial \eta_*}{\partial t_*}\right|_{y_*} \frac{d}{d\eta_*}, \tag{6.22}$$

$$\left.\frac{\partial}{\partial y_*}\right|_{t_*} = \left.\frac{\partial \eta_*}{\partial y_*}\right|_{t_*} \frac{d}{d\eta_*}. \tag{6.23}$$

Now returning to Stokes' first problem, let us assume that a similarity solution exists of the form $u_*(y_*, t_*) = u_*(\eta_*)$. It is not always possible to find a similarity variable η_*. One of the more robust ways to find a similarity variable, if it exists, comes from group theory,[3] and

[3]Group theory has a long history in mathematics and physics. Its complicated origins generally include attribution to Évariste Galois, 1811-1832, a somewhat romantic figure, as well as Niels Henrick Abel, 1802-1829, the Norwegian mathematician. Critical developments were formalized by Marius Sophus Lie, 1842-1899, another Norwegian mathematician, in what today is known as Lie group theory. A modern variant, known as "renormalization group" (RNG) theory is an area for active research. The 1982 Nobel prize in physics went to Kenneth Geddes Wilson, 1936-, of Cornell University and The Ohio State University, for use of RNG in studying phase transitions, first done in the 1970s. The award citation refers to the possibilities of using RNG in studying the great unsolved problem of turbulence, a modern area of research in which Steven Alan Orszag, 1943-2011, made many contributions.

Quoting from the useful Eric Weisstein's World of Mathematics, available online at `https://mathworld.wolfram.com/Group.html`, "A group G is a finite or infinite set of elements together with a binary operation which together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. The operation with respect to which a group is defined is often called the 'group operation,' and a set is said to be a group 'under' this operation. Elements A, B, C, ... with binary operations A and B denoted AB form a group if

1. Closure: If A and B are two elements in G, then the product AB is also in G.

is explained in detail by Cantwell. Group theory, which is too detailed to explicate in full here, relies on a generalized *symmetry* of equations to find simpler forms. In the same sense that a snowflake, subjected to rotations of $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, or 2π, is transformed into a form which is indistinguishable from its original form, we seek transformations of the variables in our partial differential equation which map the equation into a form which is indistinguishable from the original. When systems are subject to such transformations, known as group operators, they are said to exhibit symmetry.

Let us subject our governing partial differential equation along with initial and boundary conditions to a particularly simple type of transformation, a simple stretching of space, time, and velocity:

$$\tilde{t} = e^a t_*, \qquad \tilde{y} = e^b y_*, \qquad \tilde{u} = e^c u_*. \tag{6.24}$$

Here the "$\sim$" variables are stretched variables, and a, b, and c are constant parameters. The exponential will be seen to be a convenience, which is not absolutely necessary. Note that for $a \in (-\infty, \infty)$, $b \in (-\infty, \infty)$, $c \in (-\infty, \infty)$, that $e^a \in (0, \infty)$, $e^b \in (0, \infty)$, $e^c \in (0, \infty)$. So the stretching does not change the direction of the variable; that is it is not a reflecting transformation. We note that with this stretching, the domain of the problem remains unchanged; that is $t_* \in [0, \infty)$ maps into $\tilde{t} \in [0, \infty)$; $y_* \in [0, \infty)$ maps into $\tilde{y} \in [0, \infty)$. The range is also unchanged if we allow $u_* \in [0, \infty)$, which maps into $\tilde{u} \in [0, \infty)$. Direct substitution of the transformation shows that in the stretched space, the system becomes

$$e^{a-c}\frac{\partial \tilde{u}}{\partial \tilde{t}} = e^{2b-c}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}, \tag{6.25}$$

$$e^{-c}\tilde{u}(\tilde{y}, 0) = 0, \qquad e^{-c}\tilde{u}(0, \tilde{t}) = 1, \qquad e^{-c}\tilde{u}(\infty, \tilde{t}) = 0. \tag{6.26}$$

In order that the stretching transformation map the system into a form indistinguishable from the original, that is for the transformation to exhibit symmetry, we must take

$$c = 0, \qquad a = 2b. \tag{6.27}$$

So our symmetry transformation is

$$\tilde{t} = e^{2b} t_*, \qquad \tilde{y} = e^b y_*, \qquad \tilde{u} = u_*, \tag{6.28}$$

2. Associativity: The defined multiplication is associative, i.e. for all $A, B, C \in G$, $(AB)C = A(BC)$.
3. Identity: There is an identity element I (a.k.a. $\mathbf{1}$, E, or e) such that $IA = AI = A$ for every element $A \in G$.
4. Inverse: There must be an inverse or reciprocal of each element. Therefore, the set must contain an element $B = A^{-1}$ such that $AA^{-1} = A^{-1}A = I$ for each element of G.

. . ., A map between two groups which preserves the identity and the group operation is called a homomorphism. If a homomorphism has an inverse which is also a homomorphism, then it is called an isomorphism and the two groups are called isomorphic. Two groups which are isomorphic to each other are considered to be 'the same' when viewed as abstract groups." For example, the group of 90 degree rotations of a square are isomorphic.

giving in transformed space

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}, \tag{6.29}$$

$$\tilde{u}(\tilde{y},0) = 0, \qquad \tilde{u}(0,\tilde{t}) = 1, \qquad \tilde{u}(\infty,\tilde{t}) = 0. \tag{6.30}$$

Now both the original and transformed systems are the same, and the remaining stretching parameter b does not enter directly into either formulation, so we cannot expect it in the solution of either form. That is we expect a solution to be independent of the stretching parameter b. This can be achieved if we take both u_* and $\tilde{u}$ to be functions of special combinations of the independent variables, combinations that are formed such that b does not appear. Eliminating b via

$$e^b = \frac{\tilde{y}}{y_*}, \tag{6.31}$$

we get

$$\frac{\tilde{t}}{t_*} = \left(\frac{\tilde{y}}{y_*}\right)^2, \tag{6.32}$$

or after rearrangement

$$\frac{y_*}{\sqrt{t_*}} = \frac{\tilde{y}}{\sqrt{\tilde{t}}}. \tag{6.33}$$

We thus expect $u_* = u_*\left(y_*/\sqrt{t_*}\right)$ or equivalently $\tilde{u} = \tilde{u}\left(\tilde{y}/\sqrt{\tilde{t}}\right)$. This form also allows $u_* = u_*\left(\beta y_*/\sqrt{t_*}\right)$, where β is any constant. Let us then define our similarity variable η_* as

$$\eta_* = \frac{y_*}{2\sqrt{t_*}}. \tag{6.34}$$

Here the factor of 1/2 is simply a convenience adopted so that the solution takes on a traditional form. We would find that any constant in the similarity transformation would induce a self-similar result.

Let us rewrite the differential equation, boundary, and initial conditions ($\partial u_*/\partial t_* = \partial^2 u_*/\partial y_*^2$, $u_*(y_*,0) = 0$, $u_*(0,t_*) = 1$, $u_*(\infty,t_*) = 0$) in terms of the similarity variable η_*. We first must use the chain rule to get expressions for the derivatives. Applying the general results just developed, we get

$$\frac{\partial u_*}{\partial t_*} = \frac{\partial \eta_*}{\partial t_*}\frac{du_*}{d\eta_*} = -\frac{1}{2}\frac{y_*}{2}t_*^{-3/2}\frac{du_*}{d\eta_*} = -\frac{\eta_*}{2t_*}\frac{du_*}{d\eta_*}, \tag{6.35}$$

$$\frac{\partial u_*}{\partial y_*} = \frac{\partial \eta_*}{\partial y_*}\frac{du_*}{d\eta_*} = \frac{1}{2\sqrt{t_*}}\frac{du_*}{d\eta_*}, \tag{6.36}$$

$$\frac{\partial^2 u_*}{\partial y_*^2} = \frac{\partial}{\partial y_*}\left(\frac{\partial u_*}{\partial y_*}\right) = \frac{\partial}{\partial y_*}\left(\frac{1}{2\sqrt{t_*}}\frac{du_*}{d\eta_*}\right), \tag{6.37}$$

$$= \frac{1}{2\sqrt{t_*}}\frac{\partial}{\partial y_*}\left(\frac{du_*}{d\eta_*}\right) = \frac{1}{2\sqrt{t_*}}\left(\frac{1}{2\sqrt{t_*}}\frac{d^2u_*}{d\eta_*^2}\right) = \frac{1}{4t_*}\frac{d^2u_*}{d\eta_*^2}. \tag{6.38}$$

Thus, applying these rules to our governing equation, Eq. (6.10), we recover

$$-\frac{\eta_*}{2t_*}\frac{du_*}{d\eta_*} = \frac{1}{4t_*}\frac{d^2u_*}{d\eta_*^2}, \tag{6.39}$$

$$\frac{d^2u_*}{d\eta_*^2} + 2\eta_*\frac{du_*}{d\eta_*} = 0. \tag{6.40}$$

Note our governing equation has a singularity at $t_* = 0$. As it appears on both sides of the equation, we cancel it on both sides, but we shall see that this point is associated with special behavior of the similarity solution. The important result is that the reduced equation has dependency on η_* only. If this did not occur, we could not have a similarity solution.

Now consider the initial and boundary conditions. They transform as follows:

$$y_* = 0, \Longrightarrow \eta_* = 0, \tag{6.41}$$

$$y_* \to \infty, \Longrightarrow \eta_* \to \infty, \tag{6.42}$$

$$t_* \to 0, \Longrightarrow \eta_* \to \infty. \tag{6.43}$$

Note that the three important points for t_* and y_* collapse into two corresponding points in η_*. This is also necessary for the similarity solution to exist. Consequently, our conditions in η_* space reduce to

$$u_*(0) = 1, \qquad \text{surface condition,} \tag{6.44}$$

$$u_*(\infty) = 0, \qquad \text{initial and far-field.} \tag{6.45}$$

We solve the second order differential equation by the method of reduction of order, noticing that it is really two first order equations in disguise:

$$\frac{d}{d\eta_*}\left(\frac{du_*}{d\eta_*}\right) + 2\eta_*\left(\frac{du_*}{d\eta_*}\right) = 0. \tag{6.46}$$

Multiply by the integrating factor $e^{\eta_*^2}$ to get

$$e^{\eta_*^2}\frac{d}{d\eta_*}\left(\frac{du_*}{d\eta_*}\right) + 2\eta_* e^{\eta_*^2}\left(\frac{du_*}{d\eta_*}\right) = 0. \tag{6.47}$$

$$\frac{d}{d\eta_*}\left(e^{\eta_*^2}\frac{du_*}{d\eta_*}\right) = 0, \tag{6.48}$$

$$e^{\eta_*^2}\frac{du_*}{d\eta_*} = A, \tag{6.49}$$

$$\frac{du_*}{d\eta_*} = Ae^{-\eta_*^2}, \tag{6.50}$$

$$u_* = B + A\int_0^{\eta_*} e^{-s^2}\, ds. \tag{6.51}$$

Now applying the condition $u_* = 1$ at $\eta_* = 0$ gives

$$1 = B + A \underbrace{\int_0^0 e^{-s^2} \, ds,}_{=0} \tag{6.52}$$

$$B = 1. \tag{6.53}$$

So we have

$$u_* = 1 + A \int_0^{\eta_*} e^{-s^2} \, ds. \tag{6.54}$$

Now applying the condition $u_* = 0$ at $\eta_* \to \infty$, we get

$$0 = 1 + A \underbrace{\int_0^\infty e^{-s^2} \, ds,}_{=\sqrt{\pi}/2} \tag{6.55}$$

$$0 = 1 + A \frac{\sqrt{\pi}}{2}, \tag{6.56}$$

$$A = -\frac{2}{\sqrt{\pi}}. \tag{6.57}$$

Though not immediately obvious, it can be shown by a simple variable transformation to a polar coordinate system that the above integral from 0 to ∞ has a finite value of $\sqrt{\pi}/2$. It is not surprising that this integral has finite value over the semi-infinite domain as the integrand is bounded between zero and one, and decays rapidly to zero as $s \to \infty$.

Let us divert to evaluate this integral. To do so, consider the related integral I_2 defined over the first quadrant in $s - t$ space, where

$$I_2 \equiv \int_0^\infty \int_0^\infty e^{-s^2 - t^2} \, ds \, dt, \tag{6.58}$$

$$= \int_0^\infty e^{-t^2} \int_0^\infty e^{-s^2} \, ds \, dt, \tag{6.59}$$

$$= \left(\int_0^\infty e^{-s^2} \, ds \right) \left(\int_0^\infty e^{-t^2} \, dt \right), \tag{6.60}$$

$$= \left(\int_0^\infty e^{-s^2} \, ds \right)^2, \tag{6.61}$$

$$\sqrt{I_2} = \int_0^\infty e^{-s^2} \, ds. \tag{6.62}$$

Now transform to polar coordinates with $s = r \cos\theta$, $t = r \sin\theta$. With this, we can easily show $ds \, dt = r \, dr \, d\theta$ and $s^2 + t^2 = r^2$. Substituting this into Eq. (6.58) and changing the limits of integration appropriately, we get

$$\begin{aligned} I_2 &= \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r \; dr \; d\theta, && (6.63)\\ &= \int_0^{\pi/2} \left(-\frac{1}{2} e^{-r^2} \right)_0^{\infty} \; d\theta, && (6.64)\\ &= \int_0^{\pi/2} \left(\frac{1}{2} \right) \; d\theta, && (6.65)\\ &= \frac{\pi}{4}. && (6.66) \end{aligned}$$

Comparing with Eq. (6.62), we deduce

$$\sqrt{I_2} = \int_0^{\infty} e^{-s^2} \; ds = \frac{\sqrt{\pi}}{2}. \qquad (6.67)$$

With this verified, we can return to our original analysis and say that the velocity profile can be written as

$$\begin{aligned} u_*(\eta_*) &= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta_*} e^{-s^2} \; ds, && (6.68)\\ u_*(y_*, t_*) &= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{y_*}{2\sqrt{t_*}}} e^{-s^2} \; ds, && (6.69)\\ u_*(y_*, t_*) &= \operatorname{erfc}\left(\frac{y_*}{2\sqrt{t_*}} \right). && (6.70) \end{aligned}$$

In the last form above, we have introduced the so-called error function complement, "erfc." Plots for the velocity profile in terms of both η_* and y_*, t_* are given in Figure 6.2. We see that in similarity space, the curve is a single curve that in which u_* has a value of unity at $\eta_* = 0$ and has nearly relaxed to zero when $\eta_* = 1$. In dimensionless physical space, we see that at early time, there is a thin momentum layer near the surface. At later time more momentum is present in the fluid. We can say in fact that momentum is diffusing into the fluid.

We define the momentum diffusion length as the length for which significant momentum has diffused into the fluid. This is well estimated by taking $\eta_* = 1$. In terms of physical variables, we have

$$\begin{aligned} \frac{y_*}{2\sqrt{t_*}} &= 1, && (6.71)\\ y_* &= 2\sqrt{t_*}, && (6.72)\\ \frac{y}{\frac{\nu}{U}} &= 2\sqrt{\frac{t}{\frac{\nu}{U^2}}}, && (6.73)\\ y &= \frac{2\nu}{U}\sqrt{\frac{U^2 t}{\nu}}, && (6.74)\\ y &= 2\sqrt{\nu t}. && (6.75) \end{aligned}$$

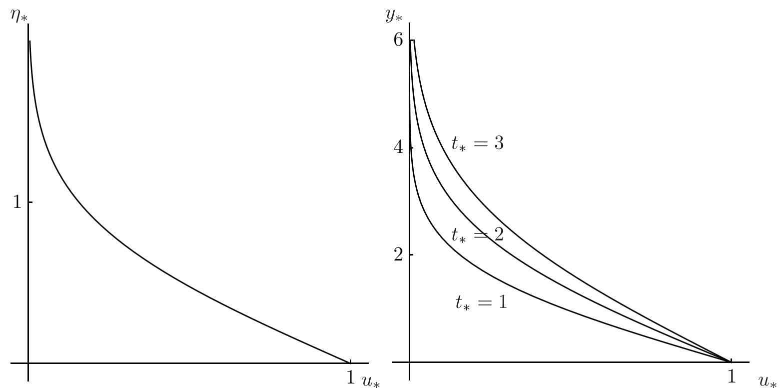


Figure 6.2: Sketch of velocity field solution for Stokes' first problem in both similarity coordinate η_* and primitive coordinates y_*, t_*.

We can in fact define this as a boundary layer thickness. That is to say the momentum boundary layer thickness in Stokes' first problem grows at a rate proportional to the square root of momentum diffusivity and time. This class of result is a hallmark of all diffusion processes, be it mass, momentum, or energy.

6.2 Taylor-Sedov solution

Here, we will study the Taylor[4]-Sedov[5] blast wave solution. We will follow most closely two papers of Taylor[6][7] from 1950. Taylor notes that the first of these was actually written in 1941, but was classified. Sedov's complementary study[8] is also of interest. One may also consult other articles by Taylor for background.[9][10] We shall follow Taylor's analysis and obtain what is known as *self-similar solutions.* Though there are more general approaches which may in fact expose more details of how self-similar solutions are obtained, we will confine ourselves to Taylor's approach and use his notation.

[4]Geoffrey Ingram Taylor, 1886-1975, English physicist.

[5]Leonid Ivanovitch Sedov, 1907-1999, Soviet physicist.

[6]Taylor, G. I., 1950, "The Formation of a Blast Wave by a Very Intense Explosion. I. Theoretical Discussion," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 201(1065): 159-174.

[7]Taylor, G. I., 1950, "The Formation of a Blast Wave by a Very Intense Explosion. II. The Atomic Explosion of 1945," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 201(1065): 175-186.

[8]Sedov, L. I., 1946, "Rasprostraneniya Sil'nykh Vzryvnykh Voln," *Prikladnaya Matematika i Mekhanika* 10: 241-250.

[9]Taylor, G. I., 1950, "The Dynamics of the Combustion Products Behind Plane and Spherical Detonation Fronts in Explosives," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 200(1061): 235-247.

[10]Taylor, G. I., 1946, "The Air Wave Surrounding an Expanding Sphere," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 186(1006): 273-292.

The self-similar solution will be enabled by studying the equations for the motion of a diffusion-free ideal compressible fluid in what is known as the *strong shock limit* for a spherical shock wave. Now, a shock wave will raise both the internal and kinetic energy of the ambient fluid into which it is propagating. We would like to consider a scenario in which the total energy, kinetic and internal, enclosed by the strong spherical shock wave is a constant. The ambient fluid, a calorically perfect ideal gas with gas constant R and ratio of specific heats γ, is initially at rest, and a point source of energy, E, exists at $r = 0$. For $t > 0$, this point source of energy is distributed to the mechanical and thermal energy of the surrounding fluid.

Let us follow now Taylor's analysis from his 1950 Part I "Theoretical Discussion" paper. We shall

- write the governing inert one-dimensional unsteady equations in spherical coordinates,
- reduce the partial differential equations in r and t to ordinary differential equations in an appropriate similarity variable,
- solve the ordinary differential equations numerically, and
- show our transformation guarantees constant total energy in the region $r \in [0, R(t)]$, where $R(t)$ is the locus of the moving shock wave.

We shall also refer to specific equations in Taylor's first 1950 paper.

6.2.1 Governing equations

The non-conservative formulation of the governing equations is as follows:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} = -\frac{2\rho u}{r}, \quad \text{mass conservation} \tag{6.76}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0, \quad \text{momentum conservation} \tag{6.77}$$

$$\left(\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r}\right) - \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) = 0, \quad \text{energy conservation} \tag{6.78}$$

$$e = \frac{1}{\gamma - 1}\frac{P}{\rho}, \quad \text{caloric state equation} \tag{6.79}$$

$$P = \rho \mathsf{R} T. \quad \text{thermal state equation} \tag{6.80}$$

The conservative version, not shown here, can also be written in the form of Eq. (4.1). The conservative form induces a set of shock jump equations in the form of Eq. (4.10). Taking the subscript s to denote the shocked state and the subscript o to denote the unshocked state, the shock velocity to be dR/dt, and the shock Mach number $M_s = (dR/dt)/\sqrt{\gamma P_0/\rho_0}$, their solution gives the jump over a shock discontinuity of so-called Rankine[11]-Hugoniot[12] jump conditions:

$$\frac{\rho_s}{\rho_0} = \frac{\gamma+1}{\gamma-1}\left(1+\frac{2}{(\gamma-1)}\frac{1}{M_s^2}\right)^{-1}, \tag{6.81}$$

$$\frac{P_s}{P_0} = \frac{2\gamma}{\gamma+1}M_s^2 - \frac{\gamma-1}{\gamma+1}, \tag{6.82}$$

$$\frac{dR}{dt} = \frac{\gamma+1}{4}u_s + \sqrt{\frac{\gamma P_0}{\rho_0} + u_s^2\left(\frac{\gamma+1}{4}\right)^2}. \tag{6.83}$$

Let us look at the energy equation, Eq. (6.78) in a little more detail. We define the material derivative as $d/dt = \partial/\partial t + u\,\partial/\partial r$, so Eq. (6.78) can be rewritten as

$$\frac{de}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} = 0. \tag{6.84}$$

As an aside, we recall that the specific volume v is defined as $v = 1/\rho$. Thus, we have $dv/dt = -(1/\rho^2)d\rho/dt$. Thus the energy equation can be rewritten as $de/dt + Pdv/dt = 0$, or $de/dt = -Pdv/dt$. In differential form, this is $de = -P\,dv$. This says the change in energy is solely due to reversible work done by a pressure force. We might recall the Gibbs equation from thermodynamics, $de = T\,ds - P\,dv$, where s is the entropy. For our system, we have $ds = 0$; thus, the flow is isentropic, at least behind the shock. It is isentropic because away from the shock, we have neglected all entropy-producing mechanisms like diffusion.

Let us now substitute the caloric energy equation, Eq. (6.79), into the energy equation, Eq. (6.84):

$$\frac{1}{\gamma-1}\frac{d}{dt}\left(\frac{P}{\rho}\right) - \frac{P}{\rho^2}\frac{d\rho}{dt} = 0, \tag{6.85}$$

$$-\frac{1}{\gamma-1}\frac{P}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\gamma-1}\frac{1}{\rho}\frac{dP}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} = 0, \tag{6.86}$$

$$-\frac{P}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\rho}\frac{dP}{dt} - (\gamma-1)\frac{P}{\rho^2}\frac{d\rho}{dt} = 0, \tag{6.87}$$

$$\frac{1}{\rho}\frac{dP}{dt} - \gamma\frac{P}{\rho^2}\frac{d\rho}{dt} = 0, \tag{6.88}$$

$$\frac{dP}{dt} - \gamma\frac{P}{\rho}\frac{d\rho}{dt} = 0, \tag{6.89}$$

$$\frac{1}{\rho^\gamma}\frac{dP}{dt} - \gamma\frac{P}{\rho^{\gamma+1}}\frac{d\rho}{dt} = 0, \tag{6.90}$$

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0, \tag{6.91}$$

[11] William John Macquorn Rankine, 1820-1872, Scottish engineer.
[12] Pierre Henri Hugoniot, 1851-1887, French mechanician.

$$\frac{\partial}{\partial t}\left(\frac{P}{\rho^\gamma}\right) + u\frac{\partial}{\partial r}\left(\frac{P}{\rho^\gamma}\right) = 0. \tag{6.92}$$

This says that following a fluid particle, P/ρ^γ is a constant. In terms of specific volume, this says $Pv^\gamma = C$, which is a well-known isentropic relation for a calorically perfect ideal gas.

6.2.2 Similarity transformation

We shall next make some non-intuitive and non-obvious choices for a transformed coordinate system and transformed dependent variables. These choices can be systematically studied with the techniques of group theory, not discussed here.

6.2.2.1 Independent variables

Let us transform the independent variables $(r, t) \to (\eta, \tau)$ with

$$\eta = \frac{r}{R(t)}, \tag{6.93}$$

$$\tau = t. \tag{6.94}$$

We will seek solutions such that the dependent variables are functions of η, the distance relative to the time-dependent shock, only. We will have little need for the transformed time τ because it is equivalent to the original time t.

6.2.2.2 Dependent variables

Let us also define new dependent variables as

$$\frac{P}{P_0} = y = R^{-3} f_1(\eta), \tag{6.95}$$

$$\frac{\rho}{\rho_0} = \psi(\eta), \tag{6.96}$$

$$u = R^{-3/2}\phi_1(\eta). \tag{6.97}$$

These amount to definitions of a scaled pressure f_1, a scaled density ψ, and a scaled velocity ϕ_1, with the assumption that each is a function of η only. Here, P_0, and ρ_0 are constant ambient values of pressure and density, respectively.

We also assume the shock velocity to be of the form

$$U(t) = \frac{dR}{dt} = AR^{-3/2}. \tag{6.98}$$

The constant A is to be determined.

6.2.2.3 Derivative transformations

By the chain rule we have

$$\frac{\partial}{\partial t} = \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau}. \tag{6.99}$$

Now, by Eq. (6.93) we get

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= -\frac{r}{R^2}\frac{dR}{dt}, && (6.100)\\
&= -\frac{\eta}{R(t)}\frac{dR}{dt}, && (6.101)\\
&= -\frac{\eta}{R}AR^{-3/2}, && (6.102)\\
&= -\frac{A\eta}{R^{5/2}}. && (6.103)
\end{aligned}$$

From Eq. (6.94) we simply get

$$\frac{\partial \tau}{\partial t} = 1. \tag{6.104}$$

Thus, the chain rule, Eq. (6.99), can be written as

$$\frac{\partial}{\partial t} = -\frac{A\eta}{R^{5/2}}\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \tau}. \tag{6.105}$$

As we are insisting the $\partial/\partial\tau = 0$, we get

$$\frac{\partial}{\partial t} = -\frac{A\eta}{R^{5/2}}\frac{d}{d\eta}. \tag{6.106}$$

In the same way, we get

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial \eta}{\partial r}\frac{\partial}{\partial \eta} + \frac{\partial \tau}{\partial r}\underbrace{\frac{\partial}{\partial \tau}}_{=0}, && (6.107)\\
&= \frac{1}{R}\frac{d}{d\eta}. && (6.108)
\end{aligned}$$

6.2.3 Transformed equations

Let us now apply our rules for derivative transformation, Eqs. (6.103,6.108), and our transformed dependent variables, Eqs. (6.95-6.97), to the governing equations.

6.2.3.1 Mass

First, we shall consider the mass equation, Eq. (6.76). We get

$$\underbrace{-\frac{A\eta}{R^{5/2}}\frac{d}{d\eta}}_{=\partial/\partial t}\underbrace{(\rho_0\psi)}_{=\rho}+\underbrace{R^{-3/2}\phi_1}_{=u}\underbrace{\frac{1}{R}\frac{d}{d\eta}}_{=\partial/\partial r}\underbrace{(\rho_0\psi)}_{=\rho}+\underbrace{\rho_0\psi}_{=\rho}\underbrace{\frac{1}{R}\frac{d}{d\eta}}_{=\partial/\partial r}\underbrace{\left(R^{-3/2}\phi_1\right)}_{=u}=-\underbrace{\frac{2}{r}}_{=2/(\eta R)}\underbrace{\rho_0\psi}_{=\rho}\underbrace{R^{-3/2}\phi_1}_{=u}. \tag{6.109}$$

Realizing that $R(t)=R(\tau)$ is not a function of η, canceling the common factor of ρ_0, and eliminating r with Eq. (6.93), we can write

$$-\frac{A\eta}{R^{5/2}}\frac{d\psi}{d\eta}+\frac{\phi_1}{R^{5/2}}\frac{d\psi}{d\eta}+\frac{\psi}{R^{5/2}}\frac{d\phi_1}{d\eta} = -\frac{2}{\eta}\frac{\psi\phi_1}{R^{5/2}}, \tag{6.110}$$

$$-A\eta\frac{d\psi}{d\eta}+\phi_1\frac{d\psi}{d\eta}+\psi\frac{d\phi_1}{d\eta} = -\frac{2}{\eta}\psi\phi_1, \tag{6.111}$$

$$-A\eta\frac{d\psi}{d\eta}+\phi_1\frac{d\psi}{d\eta}+\psi\left(\frac{d\phi_1}{d\eta}+\frac{2}{\eta}\phi_1\right) = 0, \qquad \text{mass.} \tag{6.112}$$

Equation (6.112) is number 9 in Taylor's paper, which we will call here Eq. T(9).

6.2.3.2 Linear momentum

Now, consider the linear momentum equation, Eq. (6.77), and apply the same transformations:

$$\frac{\partial}{\partial t}\underbrace{\left(R^{-3/2}\phi_1\right)}_{=u}+\underbrace{R^{-3/2}\phi_1}_{=u}\frac{\partial}{\partial r}\underbrace{\left(R^{-3/2}\phi_1\right)}_{=u}+\underbrace{\frac{1}{\rho_0\psi}}_{=1/\rho}\frac{\partial}{\partial r}\underbrace{\left(P_0R^{-3}f_1\right)}_{=P} = 0, \tag{6.113}$$

$$\underbrace{R^{-3/2}\frac{\partial\phi_1}{\partial t}-\frac{3}{2}R^{-5/2}\frac{dR}{dt}\phi_1}_{=\partial u/\partial t}+R^{-3/2}\phi_1\frac{\partial}{\partial r}\left(R^{-3/2}\phi_1\right)+\frac{1}{\rho_0\psi}\frac{\partial}{\partial r}\left(P_0R^{-3}f_1\right) = 0, \tag{6.114}$$

$$R^{-3/2}\left(-\frac{A\eta}{R^{5/2}}\right)\frac{d\phi_1}{d\eta}-\frac{3}{2}R^{-5/2}\left(AR^{-3/2}\right)\phi_1+R^{-3/2}\phi_1\frac{\partial}{\partial r}\left(R^{-3/2}\phi_1\right)$$
$$+\frac{1}{\rho_0\psi}\frac{\partial}{\partial r}\left(P_0R^{-3}f_1\right) = 0, \tag{6.115}$$

$$-\frac{A\eta}{R^4}\frac{d\phi_1}{d\eta}-\frac{3}{2}\frac{A}{R^4}\phi_1+R^{-3/2}\phi_1\frac{\partial}{\partial r}\left(R^{-3/2}\phi_1\right)+\frac{1}{\rho_0\psi}\frac{\partial}{\partial r}\left(P_0R^{-3}f_1\right) = 0, \tag{6.116}$$

$$-\frac{A\eta}{R^4}\frac{d\phi_1}{d\eta}-\frac{3}{2}\frac{A}{R^4}\phi_1+R^{-3/2}\phi_1\frac{1}{R}\frac{d}{d\eta}\left(R^{-3/2}\phi_1\right)+\frac{1}{\rho_0\psi}\frac{1}{R}\frac{d}{d\eta}\left(P_0R^{-3}f_1\right) = 0, \tag{6.117}$$

$$-\frac{A\eta}{R^4}\frac{d\phi_1}{d\eta}-\frac{3}{2}\frac{A}{R^4}\phi_1+\frac{\phi_1}{R^4}\frac{d\phi_1}{d\eta}+\frac{P_0}{\rho_0\psi}\frac{1}{R^4}\frac{df_1}{d\eta} = 0, \tag{6.118}$$

$$-A\eta\frac{d\phi_1}{d\eta}-\frac{3}{2}A\phi_1+\phi_1\frac{d\phi_1}{d\eta}+\frac{P_0}{\rho_0\psi}\frac{df_1}{d\eta} = 0. \tag{6.119}$$

Our final form is

$$-A\left(\frac{3}{2}\phi_1+\eta\frac{d\phi_1}{d\eta}\right)+\phi_1\frac{d\phi_1}{d\eta}+\frac{P_0}{\rho_0}\frac{1}{\psi}\frac{df_1}{d\eta}=0, \qquad \text{linear momentum.} \tag{6.120}$$

Equation (6.120) is T(7).

6.2.3.3 Energy

Let us now consider the energy equation. It is best to begin with a form in which the equation of state has already been imposed. So, we will start by expanding Eq. (6.89) in terms of partial derivatives:

$$\underbrace{\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial r}}_{=dP/dt}-\gamma\frac{P}{\rho}\underbrace{\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}\right)}_{=d\rho/dt} = 0, \tag{6.121}$$

$$\frac{\partial}{\partial t}\left(P_0R^{-3}f_1\right)+R^{-3/2}\phi_1\frac{\partial}{\partial r}\left(P_0R^{-3}f_1\right) -\gamma\frac{P_0R^{-3}f_1}{\rho_0\psi}\left(\frac{\partial}{\partial t}(\rho_0\psi)+R^{-3/2}\phi_1\frac{\partial}{\partial r}(\rho_0\psi)\right) = 0, \tag{6.122}$$

$$\frac{\partial}{\partial t}\left(R^{-3}f_1\right)+R^{-3/2}\phi_1\frac{\partial}{\partial r}\left(R^{-3}f_1\right) -\gamma\frac{R^{-3}f_1}{\psi}\left(\frac{\partial \psi}{\partial t}+R^{-3/2}\phi_1\frac{\partial \psi}{\partial r}\right) = 0, \tag{6.123}$$

$$R^{-3}\frac{\partial f_1}{\partial t}-3R^{-4}\frac{dR}{dt}f_1+R^{-3/2}\phi_1\frac{\partial}{\partial r}\left(R^{-3}f_1\right) -\gamma\frac{R^{-3}f_1}{\psi}\left(\frac{\partial \psi}{\partial t}+R^{-3/2}\phi_1\frac{\partial \psi}{\partial r}\right) = 0, \tag{6.124}$$

$$R^{-3}\left(-\frac{A\eta}{R^{5/2}}\right)\frac{df_1}{d\eta}-3R^{-4}(AR^{-3/2})f_1+R^{-3/2}\phi_1\frac{\partial}{\partial r}\left(R^{-3}f_1\right) -\gamma\frac{R^{-3}f_1}{\psi}\left(\frac{\partial \psi}{\partial t}+R^{-3/2}\phi_1\frac{\partial \psi}{\partial r}\right) = 0. \tag{6.125}$$

Carrying on, we have

$$-\frac{A\eta}{R^{11/2}}\frac{df_1}{d\eta}-3\frac{A}{R^{11/2}}f_1+R^{-3/2}\phi_1R^{-3}\frac{1}{R}\frac{df_1}{d\eta} -\gamma\frac{R^{-3}f_1}{\psi}\left(\left(-\frac{A\eta}{R^{5/2}}\right)\frac{d\psi}{d\eta}+R^{-3/2}\phi_1\frac{1}{R}\frac{d\psi}{d\eta}\right) = 0, \tag{6.126}$$

$$-\frac{A\eta}{R^{11/2}}\frac{df_1}{d\eta}-3\frac{A}{R^{11/2}}f_1+\frac{\phi_1}{R^{11/2}}\frac{df_1}{d\eta}-\gamma\frac{f_1}{\psi R^{11/2}}\left(-A\eta\frac{d\psi}{d\eta}+\phi_1\frac{d\psi}{d\eta}\right) = 0, \tag{6.127}$$

$$-A\eta\frac{df_1}{d\eta}-3Af_1+\phi_1\frac{df_1}{d\eta}-\gamma\frac{f_1}{\psi}\left(-A\eta\frac{d\psi}{d\eta}+\phi_1\frac{d\psi}{d\eta}\right) = 0. \tag{6.128}$$

Our final form is

$$A\left(3f_1+\eta\frac{df_1}{d\eta}\right)+\gamma\frac{f_1}{\psi}\left(-A\eta+\phi_1\right)\frac{d\psi}{d\eta}-\phi_1\frac{df_1}{d\eta}=0,\qquad \text{energy.} \tag{6.129}$$

Equation (6.129) is T(11), correcting for a typographical error replacing a r with γ.

6.2.4 Dimensionless equations

Let us now write our conservation principles in dimensionless form. We take the constant ambient sound speed c_0 to be defined for our gas as

$$c_0^2\equiv\gamma\frac{P_0}{\rho_0}. \tag{6.130}$$

Note, we have used our notation for sound speed here; Taylor uses a instead.

Let us also define

$$f \equiv \left(\frac{c_0}{A}\right)^2 f_1, \tag{6.131}$$

$$\phi \equiv \frac{\phi_1}{A}. \tag{6.132}$$

6.2.4.1 Mass

With these definitions, the mass equation, Eq. (6.112), becomes

$$-A\eta\frac{d\psi}{d\eta}+A\phi\frac{d\psi}{d\eta}+\psi\left(A\frac{d\phi}{d\eta}+\frac{2}{\eta}A\phi\right)=0, \tag{6.133}$$

$$-\eta\frac{d\psi}{d\eta}+\phi\frac{d\psi}{d\eta}+\psi\left(\frac{d\phi}{d\eta}+\frac{2}{\eta}\phi\right)=0, \tag{6.134}$$

$$\frac{d\psi}{d\eta}(\phi-\eta)=-\psi\left(\frac{d\phi}{d\eta}+\frac{2}{\eta}\phi\right), \tag{6.135}$$

$$\frac{1}{\psi}\frac{d\psi}{d\eta}=\frac{\frac{d\phi}{d\eta}+\frac{2\phi}{\eta}}{\eta-\phi},\qquad \text{mass.} \tag{6.136}$$

Equation (6.136) is T(9a).

6.2.4.2 Linear momentum

With the same definitions, the momentum equation, Eq. (6.120) becomes

$$-A\left(\frac{3}{2}A\phi+A\eta\frac{d\phi}{d\eta}\right)+A^2\phi\frac{d\phi}{d\eta}+\frac{P_0}{\rho_0}\frac{1}{\psi}\frac{A^2}{c_0^2}\frac{df}{d\eta}=0, \tag{6.137}$$

$$-\left(\frac{3}{2}\phi+\eta\frac{d\phi}{d\eta}\right)+\phi\frac{d\phi}{d\eta}+\frac{1}{\gamma}\frac{1}{\psi}\frac{df}{d\eta} = 0, \tag{6.138}$$

$$\frac{d\phi}{d\eta}(\phi-\eta)-\frac{3}{2}\phi+\frac{1}{\gamma\psi}\frac{df}{d\eta} = 0, \tag{6.139}$$

$$\frac{d\phi}{d\eta}(\eta-\phi) = \frac{1}{\gamma\psi}\frac{df}{d\eta}-\frac{3}{2}\phi, \text{ momentum.} \tag{6.140}$$

Equation (6.140) is T(7a).

6.2.4.3 Energy

The energy equation, Eq. (6.129) becomes

$$A\left(3\frac{A^2}{c_0^2}f+\eta\frac{A^2}{c_0^2}\frac{df}{d\eta}\right)+\gamma\frac{f}{\psi}\frac{A^2}{c_0^2}\left(-A\eta+A\phi\right)\frac{d\psi}{d\eta}-A\frac{A^2}{c_0^2}\phi\frac{df}{d\eta} = 0, \tag{6.141}$$

$$3f+\eta\frac{df}{d\eta}+\gamma\frac{f}{\psi}\left(-\eta+\phi\right)\frac{d\psi}{d\eta}-\phi\frac{df}{d\eta} = 0, \tag{6.142}$$

$$3f+\eta\frac{df}{d\eta}+\gamma\frac{1}{\psi}\frac{d\psi}{d\eta}f\left(-\eta+\phi\right)-\phi\frac{df}{d\eta} = 0, \quad \text{energy.} \tag{6.143}$$

Equation (6.143) is T(11a).

6.2.5 Reduction to nonautonomous form

Let us eliminate $d\psi/d\eta$ and $d\phi/d\eta$ from Eq. (6.143) with use of Eqs. (6.136,6.140).

$$3f+\eta\frac{df}{d\eta}+\gamma f\left(\frac{\frac{d\phi}{d\eta}+\frac{2\phi}{\eta}}{\eta-\phi}\right)\left(-\eta+\phi\right)-\phi\frac{df}{d\eta} = 0, \tag{6.144}$$

$$3f+\eta\frac{df}{d\eta}+\gamma f\left(\frac{\frac{\frac{1}{\gamma\psi}\frac{df}{d\eta}-\frac{3}{2}\phi}{\eta-\phi}+\frac{2\phi}{\eta}}{\eta-\phi}\right)\left(-\eta+\phi\right)-\phi\frac{df}{d\eta} = 0, \tag{6.145}$$

$$3f+(\eta-\phi)\frac{df}{d\eta}-\gamma f\left(\frac{\frac{1}{\gamma\psi}\frac{df}{d\eta}-\frac{3}{2}\phi}{\eta-\phi}+\frac{2\phi}{\eta}\right) = 0, \tag{6.146}$$

$$3f(\eta-\phi)+(\eta-\phi)^2\frac{df}{d\eta}-\gamma f\left(\frac{1}{\gamma\psi}\frac{df}{d\eta}-\frac{3}{2}\phi+\frac{2\phi}{\eta}(\eta-\phi)\right) = 0, \tag{6.147}$$

$$\left((\eta-\phi)^2-\frac{f}{\psi}\right)\frac{df}{d\eta}-f\left(-3(\eta-\phi)-\frac{3}{2}\gamma\phi+\frac{2\gamma\phi}{\eta}(\eta-\phi)\right) = 0, \tag{6.148}$$

$$\left((\eta-\phi)^2-\frac{f}{\psi}\right)\frac{df}{d\eta}+f\left(3\eta-3\phi+\frac{3}{2}\gamma\phi-2\gamma\phi+\frac{2\gamma\phi^2}{\eta}\right) = 0, \tag{6.149}$$

$$\left((\eta-\phi)^2-\frac{f}{\psi}\right)\frac{df}{d\eta}+f\left(3\eta-\phi\left(3+\frac{1}{2}\gamma\right)+\frac{2\gamma\phi^2}{\eta}\right) = 0. \tag{6.150}$$

Rearranging, we get

$$\left((\eta - \phi)^2 - \frac{f}{\psi}\right)\frac{df}{d\eta} = f\left(-3\eta + \phi\left(3 + \frac{1}{2}\gamma\right) - \frac{2\gamma\phi^2}{\eta}\right). \tag{6.151}$$

Equation (6.151) is T(14).

We can thus write an explicit nonautonomous ordinary differential equation for the evolution of f in terms of the state variables f, ψ, and ϕ, as well as the independent variable η.

$$\frac{df}{d\eta} = \frac{f\left(-3\eta + \phi\left(3 + \frac{1}{2}\gamma\right) - \frac{2\gamma\phi^2}{\eta}\right)}{(\eta - \phi)^2 - \frac{f}{\psi}}. \tag{6.152}$$

Eq. (6.152) can be directly substituted into the momentum equation, Eq. (6.140) to get

$$\frac{d\phi}{d\eta} = \frac{\frac{1}{\gamma\psi}\frac{df}{d\eta} - \frac{3}{2}\phi}{\eta - \phi}. \tag{6.153}$$

Then, Eq. (6.153) can be substituted into Eq. (6.136) to get

$$\frac{d\psi}{d\eta} = \psi\frac{\frac{d\phi}{d\eta} + \frac{2\phi}{\eta}}{\eta - \phi}. \tag{6.154}$$

Equations (6.152-6.154) form a nonautonomous system of first order differential equations of the form

$$\frac{df}{d\eta} = g_1(f, \phi, \psi, \eta), \tag{6.155}$$

$$\frac{d\phi}{d\eta} = g_2(f, \phi, \psi, \eta), \tag{6.156}$$

$$\frac{d\psi}{d\eta} = g_3(f, \phi, \psi, \eta). \tag{6.157}$$

They can be integrated with standard numerical software. One must of course provide conditions of all state variables at a particular point. We apply conditions not at $\eta = 0$, but at $\eta = 1$, the locus of the shock front. Following Taylor, the conditions are taken from the Rankine-Hugoniot equations, Eqs. (6.81-6.83), applied in the limit of a strong shock ($M_s \to \infty$). We omit the details of this analysis. We take the subscript s to denote the shock state at $\eta = 1$. For the density, one finds

$$\frac{\rho_s}{\rho_0} = \frac{\gamma + 1}{\gamma - 1}, \tag{6.158}$$

$$\frac{\rho_0\psi_s}{\rho_0} = \frac{\gamma + 1}{\gamma - 1}, \tag{6.159}$$

$$\psi_s = \psi(\eta = 1) = \frac{\gamma + 1}{\gamma - 1}. \tag{6.160}$$

For the pressure, leaving out details, one finds that

$$
\begin{aligned}
\frac{\frac{dR}{dt}^2}{c_0^2} &= \frac{\gamma+1}{2\gamma}\frac{P_s}{P_0}, &(6.161)\\
\frac{A^2 R^{-3}}{c_0^2} &= \frac{\gamma+1}{2\gamma} R^{-3} f_{1s}, &(6.162)\\
\frac{A^2 R^{-3}}{c_0^2} &= \frac{\gamma+1}{2\gamma} R^{-3} \frac{A^2}{c_0^2} f_s, &(6.163)\\
1 &= \frac{\gamma+1}{2\gamma} f_s, &(6.164)\\
f_s = f(\eta = 1) &= \frac{2\gamma}{\gamma+1}. &(6.165)
\end{aligned}
$$

For the velocity, leaving out details, one finds

$$
\begin{aligned}
\frac{u_s}{\frac{dR}{dt}} &= \frac{2}{\gamma+1}, &(6.166)\\
\frac{R^{-3/2}\phi_{1s}}{AR^{-3/2}} &= \frac{2}{\gamma+1}, &(6.167)\\
\frac{R^{-3/2}A\phi_s}{AR^{-3/2}} &= \frac{2}{\gamma+1}, &(6.168)\\
\phi_s = \phi(\eta = 1) &= \frac{2}{\gamma+1}. &(6.169)
\end{aligned}
$$

Equations (6.160, 6.165, 6.169) form the appropriate set of initial conditions for the integration of Eqs. (6.152-6.154).

6.2.6 Numerical solution

Solutions for $f(\eta)$, $\phi(\eta)$ and $\psi(\eta)$ are shown for $\gamma = 7/5$ in Figs. 6.3-6.5, respectively. So, we now have a similarity solution for the scaled variables. We need to relate this to physical dimensional quantities. Let us assign some initial conditions for $t = 0$, $r > 0$; that is, away from the point source. Take

$$u(r,0) = 0, \quad \rho(r,0) = \rho_0, \quad P(r,0) = P_0. \tag{6.170}$$

We also have from Eq. (6.80) that

$$T(r,0) = \frac{P_0}{\rho_0 \mathsf{R}} = T_0. \tag{6.171}$$

Using Eq. (6.79), we further have

$$e(r,0) = \frac{1}{\gamma-1}\frac{P_0}{\rho_0} = e_0. \tag{6.172}$$

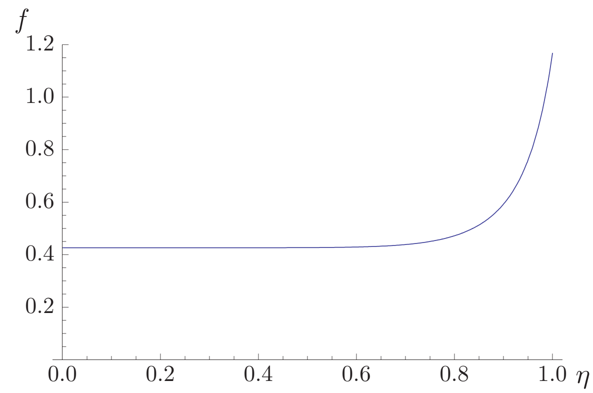


Figure 6.3: Scaled pressure f versus similarity variable η for $\gamma = 7/5$ in Taylor-Sedov blast wave.

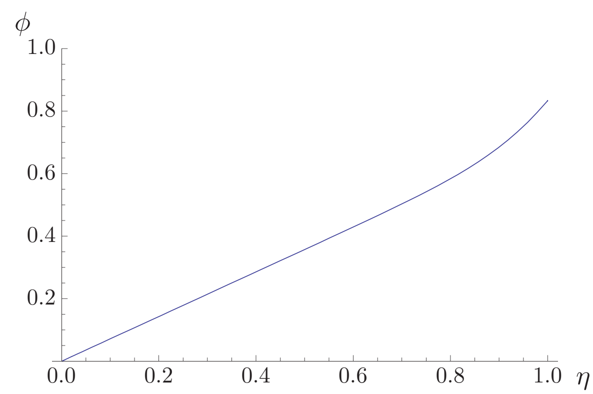


Figure 6.4: Scaled velocity ϕ versus similarity variable η for $\gamma = 7/5$ in Taylor-Sedov blast wave.

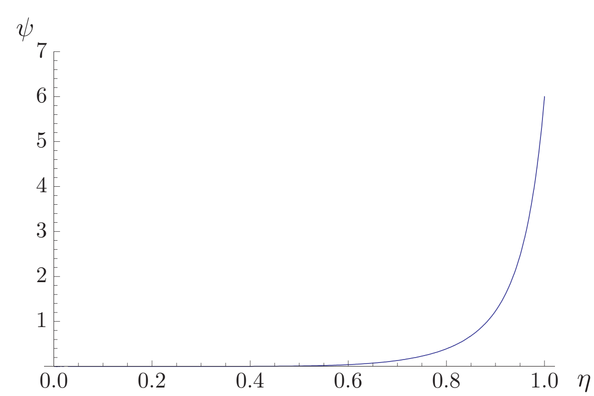


Figure 6.5: Scaled density ψ versus similarity variable η for $\gamma = 7/5$ in Taylor-Sedov blast wave.

6.2.6.1 Calculation of total energy

Now, as the point source expands, it will generate a strong shock wave. Material which has not been shocked is oblivious to the presence of the shock. Material which the shock wave has reached has been influenced by it. It stands to reason from energy conservation principles that we want the total energy, internal plus kinetic, to be *constant* in the shocked domain, $r \in (0, R(t)]$, where $R(t)$ is the shock front location.

Let us recall some spherical geometry so this energy conservation principle can be properly formulated. Consider a thin differential spherical shell of thickness dr located somewhere in the shocked region: $r \in (0, R(t)]$. The volume of the thin shell is

$$dV = \underbrace{4\pi r^2}_{\text{(surface area)}} \underbrace{dr}_{\text{(thickness)}} \tag{6.173}$$

The differential mass dm of this shell is

$$\begin{aligned} dm &= \rho\, dV, &(6.174)\\ &= 4\pi r^2 \rho\, dr. &(6.175)\end{aligned}$$

Now, recall the mass-specific internal energy is e and the mass-specific kinetic energy is $u^2/2$. So, the total differential energy, internal plus kinetic, in the differential shell is

$$\begin{aligned} dE &= \left(e + \frac{1}{2}u^2\right)\, dm, &(6.176)\\ &= 4\pi\rho\left(e + \frac{1}{2}u^2\right) r^2\, dr. &(6.177)\end{aligned}$$

Now, the total energy E within the shock is the integral through the entire sphere,

$$\begin{aligned} E = \int_0^{R(t)} dE &= \int_0^{R(t)} 4\pi\rho\left(e + \frac{1}{2}u^2\right) r^2\, dr, &(6.178)\\ &= \int_0^{R(t)} 4\pi\rho\left(\frac{1}{\gamma-1}\frac{P}{\rho} + \frac{1}{2}u^2\right) r^2\, dr, &(6.179)\\ &= \underbrace{\frac{4\pi}{\gamma-1}\int_0^{R(t)} Pr^2\, dr}_{\text{thermal energy}} + \underbrace{2\pi\int_0^{R(t)} \rho u^2 r^2\, dr}_{\text{kinetic energy}}. &(6.180)\end{aligned}$$

We introduce variables from our similarity transformations next:

$$\begin{aligned} E &= \frac{4\pi}{\gamma-1}\int_0^1 \underbrace{P_0 R^{-3} f_1}_{P} \underbrace{R^2\eta^2}_{r^2} \underbrace{R\, d\eta}_{dr} + 2\pi\int_0^1 \underbrace{\rho_0\psi}_{\rho} \underbrace{R^{-3}\phi_1^2}_{u^2} \underbrace{R^2\eta^2}_{r^2} \underbrace{R\, d\eta}_{dr}, &(6.181)\\ &= \frac{4\pi}{\gamma-1}\int_0^1 P_0 f_1 \eta^2\, d\eta + 2\pi\int_0^1 \rho_0\psi\phi_1^2\eta^2\, d\eta, &(6.182)\end{aligned}$$

$$
\begin{aligned}
&= \frac{4\pi}{\gamma-1}\int_0^1 \frac{P_0A^2}{c_0^2} f\eta^2\ d\eta + 2\pi\int_0^1 \rho_0\psi A^2\phi^2\eta^2\ d\eta, && (6.183)\\
&= 4\pi A^2\left(\frac{P_0}{c_0^2(\gamma-1)}\int_0^1 f\eta^2\ d\eta + \frac{\rho_0}{2}\int_0^1 \psi\phi^2\eta^2\ d\eta\right), && (6.184)\\
&= 4\pi A^2\rho_0 \underbrace{\left(\frac{1}{\gamma(\gamma-1)}\int_0^1 f\eta^2\ d\eta + \frac{1}{2}\int_0^1 \psi\phi^2\eta^2\ d\eta\right)}_{\text{dependent on }\gamma\text{ only}}. && (6.185)
\end{aligned}
$$

The term inside the parentheses is dependent on γ only. So, if we consider air with $\gamma = 7/5$, we can, using our knowledge of $f(\eta)$, $\psi(\eta)$, and $\phi(\eta)$, which only depend on γ, to calculate once and for all the value of the integrals. For $\gamma = 7/5$, we obtain via numerical quadrature

$$
\begin{aligned}
E &= 4\pi A^2\rho_0\left(\frac{1}{(7/5)(2/5)}(0.185194) + \frac{1}{2}(0.185168)\right), && (6.186)\\
&= 5.3192\rho_0 A^2. && (6.187)
\end{aligned}
$$

Now, from Eqs. (6.95, 6.130, 6.131, 6.187) with $\gamma = 7/5$, we get

$$
\begin{aligned}
P &= P_0R^{-3}f\frac{A^2}{c_0^2}, && (6.188)\\
&= P_0R^{-3}f\frac{\rho_0}{\gamma P_0}A^2, && (6.189)\\
&= R^{-3}f\frac{1}{\gamma}\rho_0A^2, && (6.190)\\
&= R^{-3}f\frac{1}{\frac{7}{5}}\frac{E}{5.3192}, && (6.191)\\
&= 0.1343R^{-3}Ef, && (6.192)\\
P(r,t) &= 0.1343\frac{E}{R^3(t)}f\left(\frac{r}{R(t)}\right). && (6.193)
\end{aligned}
$$

The peak pressure occurs at $\eta = 1$, where $r = R$, and where

$$f(\eta = 1) = \frac{2\gamma}{\gamma+1} = \frac{2(1.4)}{1.4+1} = 1.167. \qquad (6.194)$$

So, at $\eta = 1$, where $r = R$, we have

$$P = (0.1343)(1.167)R^{-3}E = 0.1567\frac{E}{R^3}. \qquad (6.195)$$

The peak pressure decays at a rate proportional to $1/R^3$ in the strong shock limit.

Now, from Eqs. (6.97, 6.132, 6.187) we get for u:

$$u = R^{-3/2}A\phi, \tag{6.196}$$

$$= R^{-3/2}\sqrt{\frac{E}{5.319\rho_0}}\ \phi, \tag{6.197}$$

$$u(r,t) = \sqrt{\frac{E}{5.319\rho_0}}\frac{1}{R^{3/2}(t)}\phi\left(\frac{r}{R(t)}\right). \tag{6.198}$$

Let us now explicitly solve for the shock position $R(t)$ and the shock velocity dR/dt. We have from Eqs. (6.98, 6.187) that

$$\frac{dR}{dt} = AR^{-3/2}, \tag{6.199}$$

$$= \sqrt{\frac{E}{5.319\rho_0}}\ \frac{1}{R^{3/2}(t)}, \tag{6.200}$$

$$R^{3/2}dR = \sqrt{\frac{E}{5.319\rho_0}}\ dt, \tag{6.201}$$

$$\frac{2}{5}R^{5/2} = \sqrt{\frac{E}{5.319\rho_0}}\ t + C. \tag{6.202}$$

Now, because $R(0) = 0$, we get $C = 0$, so

$$\frac{2}{5}R^{5/2} = \sqrt{\frac{E}{5.319\rho_0}}\ t, \tag{6.203}$$

$$t = \frac{2}{5}R^{5/2}\sqrt{5.319\rho_0}\ E^{-1/2}, \tag{6.204}$$

$$= 0.9225R^{5/2}\rho_0^{1/2}E^{-1/2}. \tag{6.205}$$

Equation (6.205) is T(38). Solving for R, we get

$$R^{5/2} = \frac{1}{0.9225}t\rho_0^{-1/2}E^{1/2}, \tag{6.206}$$

$$R(t) = 1.03279\rho_0^{-1/5}E^{1/5}t^{2/5}. \tag{6.207}$$

Thus, we have a prediction for the shock location as a function of time t, as well as point source energy E. If we know the position as a function of time, we can easily get the shock velocity by direct differentiation:

$$\frac{dR}{dt} = 0.4131\rho_0^{-1/5}E^{1/5}t^{-3/5}. \tag{6.208}$$

If we can make a measurement of the blast wave location R at a given known time t, and we know the ambient density ρ_0, we can estimate the point source energy E. Let us invert Eq. (6.207) to solve for E and get

$$E = \frac{\rho_0 R^5}{(1.03279)^5 t^2}, \tag{6.209}$$

$$= 0.85102 \frac{\rho_0 R^5}{t^2}. \tag{6.210}$$

6.2.6.2 Comparison with experimental data

Now, Taylor's Part II paper from 1950 gives data for the 19 July 1945 atomic explosion at the Trinity site in New Mexico. We choose one point from the photographic record which finds the shock from the blast to be located at $R = 185$ m when $t = 62$ ms. Let us assume the ambient air has a density of $\rho_0 = 1.161$ kg/m^3. Then, we can estimate the energy of the device by Eq. (6.210) as

$$E = 0.85102 \frac{\left(1.161 \ \frac{\text{kg}}{\text{m}^3}\right) (185 \text{ m})^5}{(0.062 \text{ s})^2}, \tag{6.211}$$

$$= 55.7 \times 10^{12} \text{ J}. \tag{6.212}$$

Now, 1 ton of the high explosive TNT[13] is known to contain 4.25×10^9 J of chemical energy. So, the estimated energy of the Trinity site device in terms of a TNT equivalent is

$$\text{TNT}_{equivalent} = \frac{55.7 \times 10^{12} \text{ J}}{4.25 \times 10^9 \ \frac{\text{J}}{\text{ton}}} = 13.1 \times 10^3 \text{ ton}. \tag{6.213}$$

In common parlance, the Trinity site device was a 13 kiloton bomb by this simple estimate. Taylor provides some nuanced corrections to this estimate. Modern estimates are now around 20 kiloton.

6.2.7 Contrast with acoustic limit

We saw in Eq. (6.195) that in the expansion associated with a strong shock, the pressure decays as $1/R^3$. Let us see how that compares with the decay of pressure in the limit of a weak shock.

Let us first rewrite the governing equations. Here, we 1) rewrite Eq. (6.76) in a conservative form, using the chain rule to absorb the source term inside the derivative, 2) repeat the linear momentum equation, Eq. (6.77), and 3) re-cast the energy equation for a calorically perfect ideal gas, Eq. (6.89) in terms of the full partial derivatives:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \rho u \right) = 0, \tag{6.214}$$

[13] More specifically, 2,4,6-trinitrotoluene, $C_6H_2(NO_2)_3CH_3$, first prepared in 1863.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0, \tag{6.215}$$

$$\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial r} - \gamma\frac{P}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) = 0. \tag{6.216}$$

Now, let us consider the acoustic limit, which corresponds to perturbations of a fluid at rest. Taking $0 < \epsilon \ll 1$, we recast the dependent variables ρ, P, and u as

$$\rho = \rho_0 + \epsilon\rho_1 + \ldots, \tag{6.217}$$

$$P = P_0 + \epsilon P_1 + \ldots, \tag{6.218}$$

$$u = \underbrace{u_0}_{=0} + \epsilon u_1 + \ldots. \tag{6.219}$$

Here, ρ_0 and P_0 are taken to be constants. The ambient velocity $u_0 = 0$. Strictly speaking, we should nondimensionalize the equations before we introduce an asymptotic expansion. However, so doing would not change the essence of the argument to be made.

We next introduce our expansions into the governing equations:

$$\frac{\partial}{\partial t}(\rho_0 + \epsilon\rho_1) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(\rho_0 + \epsilon\rho_1)(\epsilon u_1)\right) = 0, \tag{6.220}$$

$$\frac{\partial}{\partial t}(\epsilon u_1) + (\epsilon u_1)\frac{\partial}{\partial r}(\epsilon u_1) + \frac{1}{\rho_0 + \epsilon\rho_1}\frac{\partial}{\partial r}(P_0 + \epsilon P_1) = 0, \tag{6.221}$$

$$\begin{aligned}\frac{\partial}{\partial t}(P_0 + \epsilon P_1) &+ (\epsilon u_1)\frac{\partial}{\partial r}(P_0 + \epsilon P_1)\\ &-\gamma\frac{P_0 + \epsilon P_1}{\rho_0 + \epsilon\rho_1}\left(\frac{\partial}{\partial t}(\rho_0 + \epsilon\rho_1) + (\epsilon u_1)\frac{\partial}{\partial r}(\rho_0 + \epsilon\rho_1)\right) = 0.\end{aligned} \tag{6.222}$$

Now, derivatives of constants are all zero, and so at leading order the constant state satisfies the governing equations. At $\mathcal{O}(\epsilon)$, the equations reduce to

$$\frac{\partial \rho_1}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho_0 u_1) = 0, \tag{6.223}$$

$$\frac{\partial u_1}{\partial t} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial r} = 0, \tag{6.224}$$

$$\frac{\partial P_1}{\partial t} - \gamma\frac{P_0}{\rho_0}\frac{\partial \rho_1}{\partial t} = 0. \tag{6.225}$$

Now, adopt as before $c_0^2 = \gamma P_0/\rho_0$, so the energy equation, Eq. (6.225), becomes

$$\frac{\partial P_1}{\partial t} = c_0^2\frac{\partial \rho_1}{\partial t}. \tag{6.226}$$

Now, substitute Eq. (6.226) into the mass equation, Eq. (6.223), to get

$$\frac{1}{c_0^2}\frac{\partial P_1}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho_0 u_1) = 0. \tag{6.227}$$

We take the time derivative of Eq. (6.227) to get

$$\frac{1}{c_0^2}\frac{\partial^2 P_1}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho_0 u_1\right)\right) = 0, \tag{6.228}$$

$$\frac{1}{c_0^2}\frac{\partial^2 P_1}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho_0\frac{\partial u_1}{\partial t}\right) = 0. \tag{6.229}$$

We next use the momentum equation, Eq. (6.224), to eliminate $\partial u_1/\partial t$ in Eq. (6.229):

$$\frac{1}{c_0^2}\frac{\partial^2 P_1}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho_0\left(-\frac{1}{\rho_0}\frac{\partial P_1}{\partial r}\right)\right) = 0, \tag{6.230}$$

$$\frac{1}{c_0^2}\frac{\partial^2 P_1}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P_1}{\partial r}\right) = 0, \tag{6.231}$$

$$\frac{1}{c_0^2}\frac{\partial^2 P_1}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P_1}{\partial r}\right). \tag{6.232}$$

This second-order linear partial differential equation has a well-known solution of the d'Alembert form:

$$P_1 = \frac{1}{r}g\left(t - \frac{r}{c_0}\right) + \frac{1}{r}h\left(t + \frac{r}{c_0}\right). \tag{6.233}$$

Here, g and h are arbitrary functions which are chosen to match the initial conditions. Let us check this solution for g; the procedure can easily be repeated for h.

If $P_1 = (1/r)g(t - r/c_0)$, then

$$\frac{\partial P_1}{\partial t} = \frac{1}{r}g'\left(t - \frac{r}{c_0}\right), \tag{6.234}$$

$$\frac{\partial^2 P_1}{\partial t^2} = \frac{1}{r}g''\left(t - \frac{r}{c_0}\right), \tag{6.235}$$

and

$$\frac{\partial P_1}{\partial r} = -\frac{1}{c_0}\frac{1}{r}g'\left(t - \frac{r}{c_0}\right) - \frac{1}{r^2}g\left(t - \frac{r}{c_0}\right). \tag{6.236}$$

With these results, let us substitute into Eq. (6.232) to see if it is satisfied:

$$\frac{1}{c_0^2}\frac{1}{r}g''\left(t - \frac{r}{c_0}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{1}{c_0}\frac{1}{r}g'\left(t - \frac{r}{c_0}\right) - \frac{1}{r^2}g\left(t - \frac{r}{c_0}\right)\right)\right), \tag{6.237}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r}{c_0}g'\left(t - \frac{r}{c_0}\right) + g\left(t - \frac{r}{c_0}\right)\right), \tag{6.238}$$

$$= -\frac{1}{r^2}\left(-\frac{r}{c_0^2}g''\left(t - \frac{r}{c_0}\right) + \frac{1}{c_0}g'\left(t - \frac{r}{c_0}\right) - \frac{1}{c_0}g'\left(t - \frac{r}{c_0}\right)\right), \tag{6.239}$$

$$= \frac{1}{r^2}\left(\frac{r}{c_0^2}g''\left(t - \frac{r}{c_0}\right)\right), \tag{6.240}$$

$$= \frac{1}{c_0^2}\frac{1}{r}g''\left(t - \frac{r}{c_0}\right). \tag{6.241}$$

Indeed, our form of $P_1(r,t)$ satisfies the governing partial differential equation. Moreover, we can see by inspection of Eq. (6.233) that the pressure decays as does $1/r$ in the limit of acoustic disturbances. This is a much slower rate of decay than for the blast wave, which goes as the inverse cube of radius.

Problems

Chapter 7

Monoscale and multiscale features

Let us consider some simple model linear partial differential equations to consider the notion of scales. We consider a model system motivated by combustion. The first example will be "monoscale" in that a single state variable will be driven by a single source term. The second example will be "multiscale" in that more than one state variable will be driven by a more complicated linear source term, inducing evolution on more than one scale. Such multiscale effects are endemic in nature and render the computational solution of associated mathematical model problems to be difficult. The discussion is drawn from Powers.[1]

7.1 Monoscale problem

Consider the following linear advection-reaction-diffusion problem motivated by combustion:

$$\frac{\partial}{\partial t}Y(x,t) + u\frac{\partial}{\partial x}Y(x,t) = \mathcal{D}\frac{\partial^2}{\partial x^2}Y(x,t) - a(Y(x,t) - Y_{eq}), \tag{7.1}$$

$$Y(x,0) = Y_0, \quad Y(0,t) = Y_0, \quad \frac{\partial Y}{\partial x}(\infty,t) \to 0, \tag{7.2}$$

where the independent variables are time $t > 0$ and distance $x \in (0,\infty)$. Here, $Y(x,t) > 0$ is a scalar that can be loosely considered to be a mass fraction, $u > 0$ is a constant advective wave speed, $\mathcal{D} > 0$ is a constant diffusion coefficient, $a > 0$ is the chemical consumption rate constant, $Y_0 > 0$ is a constant, as is $Y_{eq} > 0$. We note that $Y(x,t) = Y_{eq}$ is a solution iff $Y_0 = Y_{eq}$. For $Y_0 \neq Y_{eq}$, we may expect a boundary layer in which Y adjusts from its value at $x = 0$ to Y_{eq}, the equilibrium value.

[1] J. M. Powers, *Combustion Thermodynamics and Dynamics*, Cambridge University Press, New York, 2016.

7.1.1 Spatially homogeneous solution

The spatially homogeneous version of Eqs. (7.1-7.2) is

$$\frac{dY(t)}{dt} = -a(Y(t) - Y_{eq}), \qquad Y\big|_{t=0} = Y_0, \tag{7.3}$$

that has solution

$$Y(t) = Y_{eq} + (Y_0 - Y_{eq})e^{-at}. \tag{7.4}$$

The time scale τ over which Y evolves is

$$\tau = 1/a. \tag{7.5}$$

This time scale serves as an upper bound for the required time step to capture the dynamics in a numerical simulation. Because there is only one dependent variable in this problem, the temporal spectrum contains only one time scale. Consequently, this formulation of the system is not temporally stiff.

Example 7.1

For a spatially homogeneous solution, plot the solution $Y(t)$ to Eq. (7.3) if $a = 10^8\ \mathrm{s}^{-1}$, $Y_0 = 0.1$, and $Y_{eq} = 0.001$.

For these parameters, the solution from Eq. (7.4) is

$$Y(t) = 0.001 + 0.099e^{-(10^8\ \mathrm{s}^{-1})t}. \tag{7.6}$$

The time scale of relaxation is given by Eq. (7.5) and is

$$\tau = 1/a = 1/(10^8\ \mathrm{s}^{-1}) = 10^{-8}\ \mathrm{s}. \tag{7.7}$$

A plot of $Y(t)$ is given in Fig. 7.1. It is seen that for early time, $t \ll \tau$, that Y is near Y_0. Significant relaxation of Y occurs when $t \approx \tau$. For $t \gg \tau$, we see $Y \to Y_{eq}$. The plot is presented on a log-log scale that better highlights the dynamics. In particular, when examined over orders of magnitude, the reaction event is seen in perspective as a sharp change from one state to another. Reaction dynamics are typically characterized by a near constant,"frozen" state, seemingly in equilibrium. This pseudo-equilibrium is punctuated by a reaction event, during which the system relaxes to a final true equilibrium. The notion of "punctuated equilibrium" is also well-known in modern evolutionary biology[2], usually for far longer time scale events, and has analog with our chemical reaction dynamics.

[2] N. Eldredge and S. J. Gould, 1972, Punctuated equilibria: an alternative to phyletic gradualism, in *Models in Paleobiology*, T. J. M. Schopf, ed., Freeman-Cooper, San Francisco, pp. 82-115.

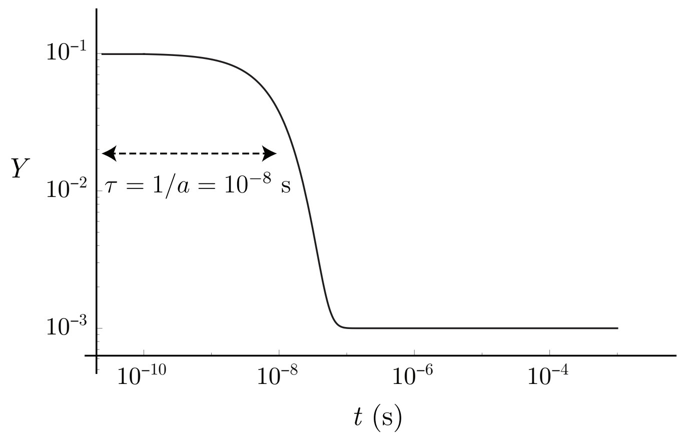


Figure 7.1: Mass fraction versus time for spatially homogeneous problem with simple one-step linear kinetics.

7.1.2 Steady solution

A simple means to determine the relevant length scales, and consequently, an upper bound for the required spatial grid resolution, is to obtain the steady structure $Y(x)$, that is governed by the time-independent version of Eqs. (7.1):

$$u\frac{dY(x)}{dx} = \mathcal{D}\frac{d^2Y(x)}{dx^2} - a(Y(x) - Y_{eq}), \quad Y|_{x=0} = Y_0, \quad \left.\frac{dY}{dx}\right|_{x\to\infty} \to 0. \tag{7.8}$$

Assuming solutions of the form $Y(x) = Y_{eq} + Ce^{rx}$, we rewrite Eq. (7.8) as

$$uCre^{rx} = \mathcal{D}Cr^2e^{rx} - aCe^{rx}, \tag{7.9}$$

and through simplification are led to a characteristic polynomial of

$$ur = \mathcal{D}r^2 - a, \tag{7.10}$$

that has roots

$$r = \frac{u}{2\mathcal{D}}\left(1 \pm \sqrt{1 + \frac{4a\mathcal{D}}{u^2}}\right). \tag{7.11}$$

Taking r_1 to denote the "plus" root, for which $r_1 > 0$, and r_2 to denote the "minus" root, for which $r_2 < 0$, the two solutions can be linearly combined to take the form

$$Y(x) = Y_{eq} + C_1e^{r_1x} + C_2e^{r_2x}, \tag{7.12}$$

where C_1 and C_2 are constants. Taking the spatial derivative of Eq. (7.12), we get

$$\frac{dY}{dx} = C_1r_1e^{r_1x} + C_2r_2e^{r_2x}. \tag{7.13}$$

In the limit of large positive x, the boundary condition at infinity in Eq. (7.8) requires the derivative to vanish giving

$$\lim_{x\to\infty} \frac{dY}{dx} = 0 = \lim_{x\to\infty} \left(C_1 r_1 e^{r_1 x} + C_2 r_2 e^{r_2 x}\right). \tag{7.14}$$

Because $r_1 > 0$, we must insist that $C_1 = 0$. Then, enforcing that $Y(0) = Y_0$, we find the solution of Eq. (7.8) is

$$Y(x) = Y_{eq} + (Y_0 - Y_{eq})e^{r_2 x}, \tag{7.15}$$

where

$$r_2 = \frac{u}{2\mathcal{D}}\left(1 - \sqrt{1 + \frac{4a\mathcal{D}}{u^2}}\right). \tag{7.16}$$

Hence, there is one length scale in the system, $\ell \equiv 1/|r_2|$; this formulation of the system is not spatially stiff. By examining Eq. (7.16) in the limit $a\mathcal{D}/u^2 \gg 1$, one finds that

$$r_2 \approx \frac{u}{2\mathcal{D}}\left(-\sqrt{\frac{4a\mathcal{D}}{u^2}}\right) = -\sqrt{\frac{a}{\mathcal{D}}}. \tag{7.17}$$

Thus solving for the length scale ℓ in this limit, we get

$$\ell = \frac{1}{|r_2|} \approx \sqrt{\frac{\mathcal{D}}{a}} = \sqrt{\mathcal{D}\tau}, \tag{7.18}$$

where $\tau = 1/a$ is the time scale from spatially homogeneous reaction, Eq. (7.5). So, this length scale ℓ reflects the inherent physics of coupled advection-reaction-diffusion. In the limit of $a\mathcal{D}/u^2 \ll 1$, one finds $r_2 \to 0$, $\ell \to \infty$, and $Y(x) \to Y_0$, a constant.

Example 7.2

For a steady solution, plot $Y(x)$ if $a = 10^8\ \mathrm{s}^{-1}$, $u = 10^2$ cm/s, $\mathcal{D} = 10^1\ \mathrm{cm}^2/\mathrm{s}$, $Y_0 = 10^{-1}$, and $Y_{eq} = 10^{-3}$.

For this system, we have from Eq. (7.16) that

$$r_2 = \frac{u}{2\mathcal{D}}\left(1 - \sqrt{1 + \frac{4a\mathcal{D}}{u^2}}\right), \tag{7.19}$$

$$= \frac{10^2\ \frac{\mathrm{cm}}{\mathrm{s}}}{2\left(10^1\ \frac{\mathrm{cm}^2}{\mathrm{s}}\right)}\left(1 - \sqrt{1 + \frac{4\left(10^8\ \mathrm{s}^{-1}\right)\left(10^1\ \frac{\mathrm{cm}^2}{\mathrm{s}}\right)}{\left(10^2\ \frac{\mathrm{cm}}{\mathrm{s}}\right)^2}}\right) = -3.2 \times 10^3\ \mathrm{cm}^{-1}. \tag{7.20}$$

Because $a\mathcal{D}/u^2 = 10^5 \gg 1$, r_2 is well estimated by Eq. (7.17):

$$r_2 \approx -\sqrt{\frac{a}{\mathcal{D}}} = -\sqrt{\frac{10^8\ \mathrm{s}^{-1}}{10^1\ \frac{\mathrm{cm}^2}{\mathrm{s}}}} = -3.2 \times 10^3\ \mathrm{cm}^{-1}. \tag{7.21}$$

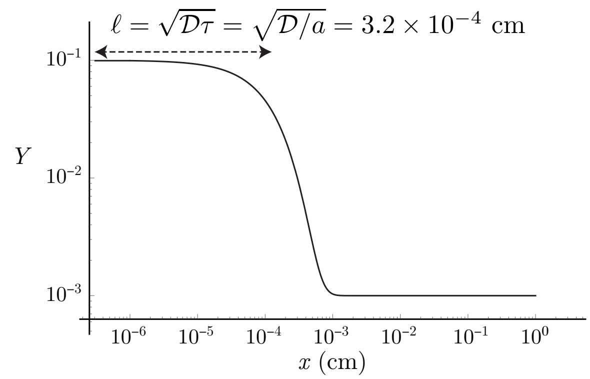


Figure 7.2: Mass fraction versus distance for steady advection-reaction-diffusion problem with simple one-step linear kinetics.

Then from Eq. (7.15), the solution is

$$Y(x) = Y_{eq} + (Y_0 - Y_{eq})e^{r_2 x} = 0.001 + 0.099e^{-(3.2\times 10^3 \text{ cm}^{-1})x}. \tag{7.22}$$

The length scale of reaction is estimated by Eq. (7.18):

$$\ell = \frac{1}{|r_2|} \approx \sqrt{\frac{\mathcal{D}}{a}} = \sqrt{\mathcal{D}\tau} = \sqrt{\left(10^1 \ \frac{\text{cm}^2}{\text{s}}\right)(10^{-8} \text{ s})} = 3.2 \times 10^{-4} \text{ cm}. \tag{7.23}$$

A plot of $Y(x)$ is given in Fig. 7.2.

7.1.3 Spatio-temporal solution

Now, for Eq. (7.1), it is possible to find a simple analytic expression for the continuous spectrum of time scales τ as a function of a particular linearly independent Fourier mode's wave number $\hat{k}$. A Fourier mode with wave number $\hat{k}$ has wavelength $\lambda = 2\pi/\hat{k}$. Assume a solution of the form

$$Y(x,t) = Y_{eq} + B(t)e^{i\hat{k}x}, \tag{7.24}$$

where $B(t)$ is the time-dependent amplitude of the chosen mode. Recall that $e^{i\hat{k}x} = \cos\hat{k}x + i\sin\hat{k}x$. We thus see spatial oscillations are built into our assumed functional form for Y. The fact that our chosen form also contains an imaginary part is inconsequential. It does simplify some of the notation, and one can always confine attention to the real part of the solution.

For this problem that considers a single Fourier mode, it does not make sense to impose the initial condition of Eq. (7.2). Substituting Eq. (7.24) into Eq. (7.1) gives

$$\frac{dB}{dt}e^{i\hat{k}x} + i\hat{k}uBe^{i\hat{k}x} = -\mathcal{D}\hat{k}^2Be^{i\hat{k}x} - aBe^{i\hat{k}x}, \tag{7.25}$$

$$\frac{dB}{dt} + i\hat{k}uB = -\mathcal{D}\hat{k}^2B - aB. \tag{7.26}$$

This takes the form

$$\frac{dB(t)}{dt} = -\beta B(t), \qquad B(0) = B_0, \tag{7.27}$$

where

$$\beta = a\left(1 + \frac{\mathcal{D}\hat{k}^2}{a} + \frac{i\hat{k}u}{a}\right), \tag{7.28}$$

and we have imposed B_0 as an initial value. This has solution

$$B(t) = B_0 e^{-\beta t}. \tag{7.29}$$

The complete solution is easily shown to be

$$Y(x,t) = Y_{eq} + B_0 e^{i\hat{k}(x-ut)-\mathcal{D}\hat{k}^2 t - at}. \tag{7.30}$$

The continuous time scale spectrum for amplitude growth or decay is given by

$$\tau = \frac{1}{|\mathrm{Re}\,(\beta)|} = \frac{1}{a\left(1+\frac{\mathcal{D}\hat{k}^2}{a}\right)}, \qquad 0 < \hat{k} \in \mathbb{R}. \tag{7.31}$$

From Eq. (7.31), it is clear that for $\mathcal{D}\hat{k}^2/a \ll 1$, i.e. for sufficiently small wave numbers, the time scales of amplitude growth or decay will be dominated by reaction:

$$\lim_{\hat{k}\to 0} \tau = 1/a. \tag{7.32}$$

However, for $\mathcal{D}\hat{k}^2/a \gg 1$, i.e. for sufficiently large wave numbers or small wavelengths, the amplitude growth/decay time scales are dominated by diffusion:

$$\lim_{\hat{k}\to\infty} \tau = \frac{1}{\mathcal{D}\hat{k}^2} = \frac{1}{\mathcal{D}}\left(\frac{\lambda}{2\pi}\right)^2. \tag{7.33}$$

From Eq. (7.31), we see that a balance between reaction and diffusion exists for $\hat{k} = \sqrt{a/\mathcal{D}}$. In terms of wavelength, and recalling Eq. (7.18), we see the balance at

$$\lambda/(2\pi) = 1/\hat{k} = \sqrt{\mathcal{D}/a} = \sqrt{\mathcal{D}\tau} = \ell, \tag{7.34}$$

where $\ell = 1/\hat{k}$ is proportional to the wavelength.

The oscillatory behavior is of lesser importance. The continuous time scale spectrum for oscillatory mode, τ_O is given by

$$\tau_O = 1/|\mathrm{Im}(\beta)| = 1/(\hat{k}u). \tag{7.35}$$

As $\hat{k} \to 0$, $\tau_O \to \infty$. While $\tau_O \to 0$ as $\hat{k} \to \infty$, it approaches at a rate $\sim 1/\hat{k}$, in contrast to the more demanding time scale of diffusion that approaches zero at a faster rate $\sim 1/\hat{k}^2$.

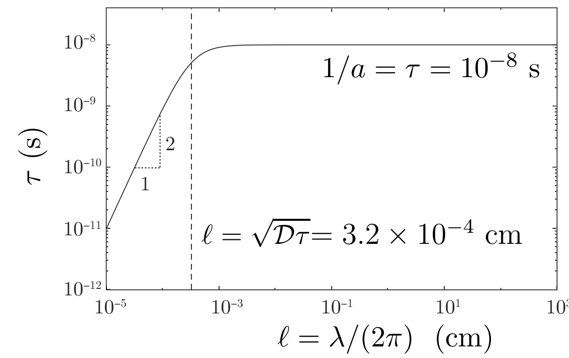


Figure 7.3: Time scale spectrum versus length scale for the simple advection-reaction-diffusion model.

Thus, it is clear that advection does not play a role in determining the limiting values of the time scale spectrum; reaction and diffusion are the major players. Lastly, it is easy to show in the absence of diffusion, that the length scale where reaction effects balance advection effects is found at

$$\ell = u/a = u\tau, \tag{7.36}$$

where $\tau = 1/a$ is the time scale from spatially homogeneous chemistry.

Example 7.3

Examine the behavior of the time scales as a function of the length scales for the linear advective-reactive-diffusive system characterized by $a = 10^8$ 1/s, $\mathcal{D} = 10^1$ cm^2/s, $u = 10^2$ cm/s.

These values are loosely motivated by values for gas phase kinetics of physical systems. For these values, we find the estimate from Eq. (7.18) for the length scale where reaction balances diffusion as

$$\ell = \sqrt{\mathcal{D}\tau} = \sqrt{\frac{\mathcal{D}}{a}} = \sqrt{\left(10^1\ \frac{\text{cm}^2}{\text{s}}\right)(10^{-8}\ \text{s})} = 3.16228 \times 10^{-4}\ \text{cm}. \tag{7.37}$$

A plot of τ versus $\ell = \lambda/(2\pi)$ from Eq. (7.31)

$$\tau = \frac{1}{a\left(1 + \dfrac{\mathcal{D}\hat{k}^2}{a}\right)} = \frac{1}{a + \dfrac{\mathcal{D}}{\ell^2}} \tag{7.38}$$

is given in Fig. 7.3. For long wavelengths, the time scales are determined by reaction; for fine wavelengths, the time scale's falloff is dictated by diffusion, and our simple formula for the critical $\ell = \sqrt{\mathcal{D}\tau}$, illustrated as a dashed line, predicts the transition well. For small ℓ, it is seen that a one decade decrease in ℓ induces a two decade decrease in τ, consistent with the prediction of Eq. (7.33): $\lim_{\hat{k}\to\infty} (\ln \tau) \sim 2\ln(\ell) - \ln(\mathcal{D})$. Lastly, over the same range of ℓ, the oscillatory time scales induced by advection are orders of magnitude less demanding, and are thus not included in the plot.

The results of this simple analysis can be summarized as follows:

- *Long wavelength spatial disturbances have time dynamics that are dominated by chemistry; each spatial point behaves as an isolated spatially homogeneous reactor.*
- *Short wavelength spatial disturbances have time dynamics that are dominated by diffusion.*
- *Intermediate wavelength spatial disturbances have time dynamics determined by fully coupled combination diffusion and chemistry. The critical intermediate length scale where this balance exists is given by* $\ell = \sqrt{\mathcal{D}\tau}$.
- *A so-called "Direct Numerical Simulation" (DNS) of a combustion process with advection-reaction-diffusion requires*

$$\Delta t < \tau, \qquad \Delta x < \sqrt{\mathcal{D}\tau}. \tag{7.39}$$

  Less restrictive choices will not capture time dynamics and spatial structures inherent in the continuum model. Advection usually plays a secondary role in determining time dynamics.

This argument is by no means new, and is effectively the same given by Landau and Lifshitz,[3] in their chapter on combustion.

7.2 Multiscale problem

Let us next consider a multiple reaction extension to Eqs. (7.1-7.2):

$$\frac{\partial}{\partial t}\mathbf{Y}(x,t) + u\frac{\partial}{\partial x}\mathbf{Y}(x,t) = \mathcal{D}\frac{\partial^2}{\partial x^2}\mathbf{Y}(x,t) - \mathbf{A}\cdot(\mathbf{Y}(x,t) - \mathbf{Y}_{eq}), \tag{7.40}$$

$$\mathbf{Y}(x,0) = \mathbf{Y}_0, \quad \mathbf{Y}(0,t) = \mathbf{Y}_0, \quad \frac{\partial \mathbf{Y}}{\partial x}(\infty,t) \to \mathbf{0}. \tag{7.41}$$

Here all variables are as before, except we take $\mathbf{Y}$ to be a vector of length N and $\mathbf{A}$ to be a constant full rank matrix of dimension $N \times N$ with real and positive eigenvalues, with N linearly independent eigenvectors , not necessarily symmetric.

7.2.1 Spatially homogeneous solution

The spatially homogeneous version of Eqs. (7.40-7.41) is

$$\frac{d\mathbf{Y}}{dt} = -\mathbf{A}\cdot(\mathbf{Y} - \mathbf{Y}_{eq}), \qquad \mathbf{Y}(0) = \mathbf{Y}_0. \tag{7.42}$$

[3] L. D. Landau and E. M. Lifshitz, 1959, *Fluid Mechanics*, Pergamon Press, London, p. 475.

Because of the way $\mathbf{A}$ has been defined, it can be decomposed as

$$\mathbf{A} = \mathbf{S} \cdot \boldsymbol{\sigma} \cdot \mathbf{S}^{-1}, \tag{7.43}$$

where $\mathbf{S}$ is an $N \times N$ matrix whose columns are populated by the N linearly independent eigenvectors of $\mathbf{A}$, and $\boldsymbol{\sigma}$ is the diagonal matrix with the N positive eigenvalues, $\sigma_1, \ldots, \sigma_N$, of $\mathbf{A}$ on its diagonal. Substitute Eq. (7.43) into Eq. (7.40), take advantage of the fact that $d\mathbf{Y}_{eq}/dt = \mathbf{0}$, and operate to find

$$\begin{aligned}
\frac{d}{dt}(\mathbf{Y} - \mathbf{Y}_{eq}) &= -\underbrace{\mathbf{S} \cdot \boldsymbol{\sigma} \cdot \mathbf{S}^{-1}}_{\mathbf{A}} \cdot (\mathbf{Y} - \mathbf{Y}_{eq}), &(7.44)\\
\mathbf{S}^{-1} \cdot \frac{d}{dt}(\mathbf{Y} - \mathbf{Y}_{eq}) &= -\mathbf{S}^{-1} \cdot \mathbf{S} \cdot \boldsymbol{\sigma} \cdot \mathbf{S}^{-1} \cdot (\mathbf{Y} - \mathbf{Y}_{eq}), &(7.45)\\
\frac{d}{dt}\left(\mathbf{S}^{-1} \cdot (\mathbf{Y} - \mathbf{Y}_{eq})\right) &= -\boldsymbol{\sigma} \cdot \mathbf{S}^{-1} \cdot (\mathbf{Y} - \mathbf{Y}_{eq}). &(7.46)
\end{aligned}$$

Take now

$$\mathbf{Z} = \mathbf{S}^{-1} \cdot (\mathbf{Y} - \mathbf{Y}_{eq}), \tag{7.47}$$

so that

$$\frac{d\mathbf{Z}}{dt} = -\boldsymbol{\sigma} \cdot \mathbf{Z}. \tag{7.48}$$

Our initial condition becomes

$$\mathbf{Z}(0) = \mathbf{S}^{-1} \cdot (\mathbf{Y}_0 - \mathbf{Y}_{eq}) = \mathbf{Z}_0. \tag{7.49}$$

The solution is

$$\begin{aligned}
\mathbf{Z}(t) &= e^{-\boldsymbol{\sigma} t} \cdot \mathbf{Z}_0, &(7.50)\\
\mathbf{S}^{-1} \cdot (\mathbf{Y}(t) - \mathbf{Y}_{eq}) &= e^{-\boldsymbol{\sigma} t} \cdot \mathbf{S}^{-1} \cdot (\mathbf{Y}_0 - \mathbf{Y}_{eq}), &(7.51)\\
\mathbf{Y}(t) &= \mathbf{Y}_{eq} + \mathbf{S} \cdot e^{-\boldsymbol{\sigma} t} \cdot \mathbf{S}^{-1} \cdot (\mathbf{Y}_0 - \mathbf{Y}_{eq}). &(7.52)
\end{aligned}$$

Expanded, one can say

$$\begin{aligned}
\begin{pmatrix} Y_1(t) \\ \vdots \\ Y_N(t) \end{pmatrix} &= \begin{pmatrix} Y_{1e} \\ \vdots \\ Y_{Ne} \end{pmatrix} + \begin{pmatrix} \vdots & \vdots & \vdots \\ \mathbf{s}_1 & \vdots & \mathbf{s}_N \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} e^{-\sigma_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{-\sigma_N t} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \cdots & \mathbf{s}_1^{-1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \mathbf{s}_N^{-1} & \cdots \end{pmatrix} \begin{pmatrix} Y_{10} - Y_{1eq} \\ \vdots \\ Y_{N0} - Y_{Neq} \end{pmatrix}. &(7.53)
\end{aligned}$$

Here $\mathbf{s}_i$, $i = 1, \ldots N$, are eigenvectors of $\mathbf{A}$. There are N time scales $\tau_i = 1/\sigma_i$, $i = 1, \ldots, N$, on which the solution evolves. Each dependent variable $Y_i(t)$, $i = 1, \ldots, N$, can evolve on each of the time scales.

Example 7.4

For a case where $N = 2$, examine the solution to Eqs. (7.42) if

$$\mathbf{A} = \begin{pmatrix} 1000000 \text{ s}^{-1} & -99000000 \text{ s}^{-1} \\ -99000000 \text{ s}^{-1} & 99010000 \text{ s}^{-1} \end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} 10^{-2} \\ 10^{-1} \end{pmatrix}, \quad \mathbf{Y}_{eq} = \begin{pmatrix} 10^{-5} \\ 10^{-6} \end{pmatrix}. \tag{7.54}$$

Thus, solve

$$\frac{dY_1}{dt} = -(1000000 \text{ s}^{-1})(Y_1 - 10^{-5}) + (99000000 \text{ s}^{-1})(Y_2 - 10^{-6}), \quad Y_1(0) = 10^{-2}, \tag{7.55}$$

$$\frac{dY_2}{dt} = (99000000 \text{ s}^{-1})(Y_1 - 10^{-5}) - (99010000 \text{ s}^{-1})(Y_2 - 10^{-6}), \quad Y_2(0) = 10^{-1}. \tag{7.56}$$

Straightforward calculation reveals the eigenvalues of $\mathbf{A}$ to be

$$\sigma_1 = 10^8 \text{ s}^{-1}, \qquad \sigma_2 = 10^4 \text{ s}^{-1}. \tag{7.57}$$

Thus the time scales of reaction $\tau_i = 1/\sigma_i$ are

$$\tau_1 = 10^{-8} \text{ s}, \qquad \tau_2 = 10^{-4} \text{ s}. \tag{7.58}$$

Clearly the ratio of time scales is large with a stiffness ratio of 10^4; thus, this is obviously a multiscale problem. It is not easy to infer either the time scales or the stiffness ratio from simple examination of the numerical values of $\mathbf{A}$. Instead, one must perform the eigenvalue calculation.

It is easily shown that a diagonal decomposition of $\mathbf{A}$ is given by

$$\mathbf{A} = \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & \frac{1}{100} \end{pmatrix}}_{\mathbf{S}} \underbrace{\begin{pmatrix} 10^8 & 0 \\ 0 & 10^4 \end{pmatrix}}_{\boldsymbol{\sigma}} \underbrace{\begin{pmatrix} -\frac{1}{101} & \frac{100}{101} \\ \frac{100}{101} & \frac{100}{101} \end{pmatrix}}_{\mathbf{S}^{-1}}. \tag{7.59}$$

Detailed calculation as given in the preceding section shows that the exact solution is given by

$$Y_1(t) = -\frac{9891 e^{-(10^8 \text{ s}^{-1})t}}{100000} + \frac{1089 e^{-(10^4 \text{ s}^{-1})t}}{10000} + 10^{-5}, \tag{7.60}$$

$$Y_2(t) = \frac{9891 e^{-(10^8 \text{ s}^{-1})t}}{100000} + \frac{1089 e^{-(10^4 \text{ s}^{-1})t}}{1000000} + 10^{-6}. \tag{7.61}$$

A plot of $Y_1(t)$ and $Y_2(t)$ is given in Fig. 7.4. Clearly, for $t < \tau_1 = 10^{-8}$ s, both $Y_1(t)$ and $Y_2(t)$ are frozen at the initial values. When $t \approx \tau_1 = 10^{-8}$ s, the first reaction mode begins to have an effect. Both Y_1 and Y_2 then maintain intermediate pseudo-equilibrium values for $t \in [\tau_1, \tau_2]$. When $t \approx \tau_2 = 10^{-4}$ s, both Y_1 and Y_2 rapidly approach their true equilibrium values.

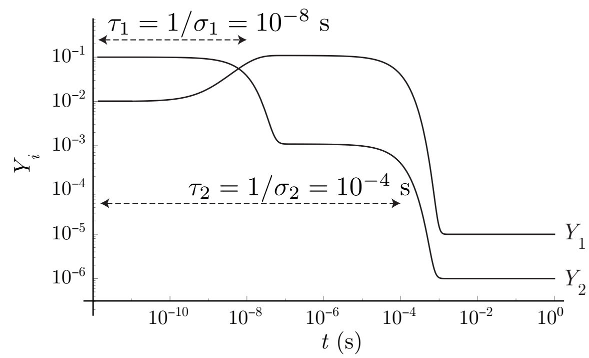


Figure 7.4: Mass fraction versus time for spatially homogeneous problem with simple two-step linear kinetics.

7.2.2 Steady solution

The time-independent version of Eqs. (7.40-7.41) is

$$u\frac{d\mathbf{Y}}{dx} = \mathcal{D}\frac{d^2\mathbf{Y}}{dx^2} - \mathbf{A}\cdot(\mathbf{Y}-\mathbf{Y}_{eq}), \quad \mathbf{Y}(0) = \mathbf{Y}_0, \quad \lim_{x\to\infty}\frac{d\mathbf{Y}}{dx} \to \mathbf{0}. \tag{7.62}$$

Let us again employ Eq. (7.43) and the fact that $d\mathbf{Y}_{eq}/dx = \mathbf{0}$ to recast Eq. (7.62) as

$$u\frac{d}{dx}(\mathbf{Y}-\mathbf{Y}_{eq}) = \mathcal{D}\frac{d^2}{dx^2}(\mathbf{Y}-\mathbf{Y}_{eq}) - \mathbf{S}\cdot\boldsymbol{\sigma}\cdot\mathbf{S}^{-1}\cdot(\mathbf{Y}-\mathbf{Y}_{eq}). \tag{7.63}$$

Next operate on both sides of Eq. (7.63) with the constant matrix $\mathbf{S}^{-1}$ and then use Eq. (7.47) to get

$$\begin{aligned}
u\mathbf{S}^{-1}\cdot\frac{d}{dx}(\mathbf{Y}-\mathbf{Y}_{eq}) &= \mathcal{D}\mathbf{S}^{-1}\cdot\frac{d^2}{dx^2}(\mathbf{Y}-\mathbf{Y}_{eq}) \\
&\quad -\mathbf{S}^{-1}\cdot\mathbf{S}\cdot\boldsymbol{\sigma}\cdot\mathbf{S}^{-1}\cdot(\mathbf{Y}-\mathbf{Y}_{eq}), &(7.64)\\
u\frac{d}{dx}\left(\mathbf{S}^{-1}\cdot(\mathbf{Y}-\mathbf{Y}_{eq})\right) &= \mathcal{D}\frac{d^2}{dx^2}\left(\mathbf{S}^{-1}\cdot(\mathbf{Y}-\mathbf{Y}_{eq})\right) \\
&\quad -\boldsymbol{\sigma}\cdot\mathbf{S}^{-1}\cdot(\mathbf{Y}-\mathbf{Y}_{eq}), &(7.65)\\
u\frac{d\mathbf{Z}}{dx} &= \mathcal{D}\frac{d^2\mathbf{Z}}{dx^2} - \boldsymbol{\sigma}\cdot\mathbf{Z}, &(7.66)
\end{aligned}$$

Similar to Eq. (7.49), the boundary conditions become

$$\mathbf{Z}(0) = \mathbf{Z}_0, \quad \lim_{x\to\infty}\frac{d\mathbf{Z}}{dx} \to \mathbf{0}. \tag{7.67}$$

Importantly, these equations are now uncoupled. For example, the i^{th} equation and boundary conditions become

$$u\frac{dZ_i(x)}{dx} = \mathcal{D}\frac{d^2Z_i(x)}{dx^2} - \sigma_i Z_i(x), \qquad Z_i\big|_{x=0} = Z_{io}, \quad \left.\frac{dZ_i}{dx}\right|_{x\to\infty} \to 0. \tag{7.68}$$

The solution can then be directly inferred from Eqs. (7.15, 7.16) to be

$$Z_i(x) = Z_{i,0}e^{r_{2,i}x}, \qquad i = 1, \ldots, N, \tag{7.69}$$

where

$$r_{2,i} = \frac{u}{2\mathcal{D}}\left(1 - \sqrt{1 + \frac{4\sigma_i\mathcal{D}}{u^2}}\right), \qquad i = 1, \ldots, N. \tag{7.70}$$

Analogously, in the limit where $\sigma_i\mathcal{D}/u^2 \gg 1$, we can infer

$$\ell_i = \sqrt{\mathcal{D}\tau_i}, \qquad i = 1, \ldots, N, \tag{7.71}$$

with the reaction time scale τ_i taken as

$$\tau_i = 1/\sigma_i, \qquad i = 1, \ldots, N. \tag{7.72}$$

Then knowing $\mathbf{Z}(x)$, one can use Eq. (7.47) to form

$$\mathbf{Y}(x) = \mathbf{Y}_{eq} + \mathbf{S}\cdot\mathbf{Z}(x). \tag{7.73}$$

Thus, any $Y_i(x)$ can be expected to relax over all N values of length scales ℓ_i.

Example 7.5

For a case where $N = 2$, $\mathcal{D} = 10^1$ cm^2/s, $u = 10^2$ cm/s, examine the solution to Eqs. (7.62) if

$$\mathbf{A} = \begin{pmatrix} 1000000\ \text{s}^{-1} & -99000000\ \text{s}^{-1} \\ -99000000\ \text{s}^{-1} & 99010000\ \text{s}^{-1}\end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} 10^{-2} \\ 10^{-1}\end{pmatrix}, \quad \mathbf{Y}_{eq} = \begin{pmatrix} 10^{-5} \\ 10^{-6}\end{pmatrix}. \tag{7.74}$$

Thus, solve

$$\left(10^2\ \frac{\text{cm}}{\text{s}}\right)\frac{dY_1}{dx} = \left(10^1\ \frac{\text{cm}^2}{\text{s}}\right)\frac{d^2Y_1}{dx^2} - (1000000\ \text{s}^{-1})(Y_1 - 10^{-5}) + (99000000\ \text{s}^{-1})(Y_2 - 10^{-6}), \tag{7.75}$$

$$\left(10^2\ \frac{\text{cm}}{\text{s}}\right)\frac{dY_2}{dx} = \left(10^1\ \frac{\text{cm}^2}{\text{s}}\right)\frac{d^2Y_2}{dx^2} + (99000000\ \text{s}^{-1})(Y_1 - 10^{-5}) - (99010000\ \text{s}^{-1})(Y_2 - 10^{-6}), \tag{7.76}$$

$$Y_1(0) = 10^{-2}, \quad Y_2(0) = 10^{-1}, \quad \lim_{x\to\infty}\frac{dY_1}{dx} = 0, \quad \lim_{x\to\infty}\frac{dY_2}{dx} = 0. \tag{7.77}$$

Employing the transformation from $\mathbf{Y}$ to $\mathbf{Z}$ along with $\mathbf{S}^{-1}$ as given in Eq. (7.59), our system can be rewritten as

$$\left(10^2\ \frac{\text{cm}}{\text{s}}\right)\frac{dZ_1}{dx} = \left(10^1\ \frac{\text{cm}^2}{\text{s}}\right)\frac{d^2Z_1}{dx^2} - (10^8\ \text{s}^{-1})Z_1, \tag{7.78}$$

$$\left(10^2\ \frac{\text{cm}}{\text{s}}\right)\frac{dZ_2}{dx} = \left(10^1\ \frac{\text{cm}^2}{\text{s}}\right)\frac{d^2Z_2}{dx^2} - (10^4\ \text{s}^{-1})Z_2, \tag{7.79}$$

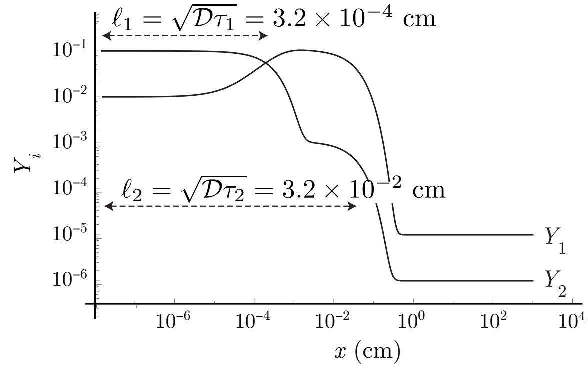


Figure 7.5: Mass fraction versus distance for advection-reaction-diffusion problem with simple two-step linear kinetics.

$$Z_1(0) = \frac{9891}{100000}, \quad Z_2(0) = \frac{1089}{10000}, \quad \lim_{x\to\infty} \frac{dZ_1}{dx} = 0, \quad \lim_{x\to\infty} \frac{dZ_2}{dx} = 0. \tag{7.80}$$

These have solution

$$Z_1(x) = \frac{9891 e^{\left(\left(5-5\sqrt{400001}\right)\text{cm}^{-1}\right)x}}{100000} = 0.0981 e^{-(3.2\times 10^3 \text{ cm}^{-1})x}, \tag{7.81}$$

$$Z_2(x) = \frac{1089 e^{\left(\left(5-5\sqrt{41}\right)\text{cm}^{-1}\right)x}}{10000} = 0.1089 e^{-(2.7\times 10^1 \text{ cm}^{-1})x}. \tag{7.82}$$

The relevant length scales are

$$\ell_1 = \frac{1}{\left(5-5\sqrt{400001}\right)\text{cm}^{-1}} = 3.2 \times 10^{-4} \text{ cm}, \tag{7.83}$$

$$\ell_2 = \frac{1}{\left(5-5\sqrt{41}\right)\text{cm}^{-1}} = 3.7 \times 10^{-2} \text{ cm}. \tag{7.84}$$

Especially for ℓ_1, these are both well estimated by the simple formulæ of Eq. (7.71):

$$\ell_1 \approx \sqrt{\mathcal{D}\tau_1} = \sqrt{\left(10^1 \frac{\text{cm}^2}{\text{s}}\right)(10^{-8} \text{ s})} = 3.2 \times 10^{-4} \text{ cm}, \tag{7.85}$$

$$\ell_2 \approx \sqrt{\mathcal{D}\tau_2} = \sqrt{\left(10^1 \frac{\text{cm}^2}{\text{s}}\right)(10^{-4} \text{ s})} = 3.2 \times 10^{-2} \text{ cm}. \tag{7.86}$$

For the slower reaction 2, advection plays a larger role, rendering the diffusion-based estimate to have a small but noticeable error.

Forming $\mathbf{Y}$ via $\mathbf{Y} = \mathbf{Y}_{eq} + \mathbf{S}\cdot\mathbf{Z}$, we find the steady solution to be

$$Y_1(x) = 10^{-5} - 0.09891 e^{-(3.2\times 10^3 \text{ cm}^{-1})x} + 0.1089 e^{-(2.7\times 10^1 \text{ cm}^{-1})x}, \tag{7.87}$$

$$Y_2(x) = 10^{-6} + 0.09891 e^{-(3.2\times 10^3 \text{ cm}^{-1})x} + 0.001089 e^{-(2.7\times 10^1 \text{ cm}^{-1})x} \tag{7.88}$$

Both variables evolve over two distinct length scales as they relax to their distinct equilibria. A plot of $Y_1(t)$ and $Y_2(t)$ is given in Fig. 7.5. Similar to the time-dependent version of this system, a frozen state near $x = 0$ first undergoes a reaction to a pseudo-equilibrium state near $x = \ell_1$. Near $x = \ell_2$, the system relaxes to its true equilibrium.

7.2.3 Spatio-temporal solution

Let us next study solutions with dependency on both time and distance. We extend the analysis and nomenclature of Sec. 7.1.3 so as to take

$$\mathbf{Y}(x,t) = \mathbf{Y}_{eq} + \mathbf{B}(t)e^{i\hat{k}x}, \tag{7.89}$$

so that Eq. (7.40) becomes

$$\begin{aligned} \frac{d\mathbf{B}}{dt}e^{i\hat{k}x} + i\hat{k}u\mathbf{B}e^{i\hat{k}x} &= -\mathcal{D}\hat{k}^2\mathbf{B}e^{i\hat{k}x} - \mathbf{A}\cdot\mathbf{B}e^{i\hat{k}x}, && (7.90)\\ \frac{d\mathbf{B}}{dt} + i\hat{k}u\mathbf{B} &= -\mathcal{D}\hat{k}^2\mathbf{B} - \mathbf{A}\cdot\mathbf{B}, && (7.91)\\ \frac{d\mathbf{B}}{dt} &= -\left(\left(i\hat{k}u + \mathcal{D}\hat{k}^2\right)\mathbf{I} + \mathbf{A}\right)\cdot\mathbf{B}. && (7.92) \end{aligned}$$

Here $\mathbf{I}$ is the identity matrix. Now it is the real part of the eigenvalues of the matrix $-\left(\left(i\hat{k}u + \mathcal{D}\hat{k}^2\right)\mathbf{I} + \mathbf{A}\right)$ that dictates whether the amplitudes grow or decay. With the operator "eig" operating on a matrix to yield its eigenvalues, it is a well-known result from linear algebra that

$$\mathrm{eig}\,(\alpha\mathbf{I} + \mathbf{A}) = \alpha + \mathrm{eig}\,\mathbf{A}. \tag{7.93}$$

Now, $\mathbf{A}$ is dictated by chemical kinetics alone, and is known to have N real and positive eigenvalues, σ_i, $i = 1, \ldots, N$. Our eigenvalues β_i are thus seen to be

$$\beta_i = i\hat{k}u + \mathcal{D}\hat{k}^2 + \sigma_i, \qquad i = 1, \ldots, N. \tag{7.94}$$

It is only the real part of β_i that dictates growth or decay of a mode. Because

$$\mathrm{Re}\,(\beta_i) = \mathcal{D}\hat{k}^2 + \sigma_i > 0, \quad \forall i = 1, \ldots, N, \tag{7.95}$$

we see that all modes are decaying, and that diffusion induces them to decay more rapidly. The time scales of decay τ_i are again given by the reciprocals of the eigenvalues, and are seen to be

$$\tau_i = \frac{1}{\mathrm{Re}\,(\beta_i)} = \frac{1}{\sigma_i\left(1 + \dfrac{\mathcal{D}\hat{k}^2}{\sigma_i}\right)} = \frac{1}{\sigma_i + \dfrac{\mathcal{D}}{\ell^2}}, \quad i = 1, \ldots, N, \tag{7.96}$$

using $\ell = 1/\hat{k}$ from Eq. (7.34).

From Eq. (7.96), it is clear that for $\mathcal{D}\hat{k}^2/\sigma_i \ll 1$, i.e. for sufficiently small wave numbers or long wavelengths, the time scales of amplitude growth or decay will be dominated by reaction:

$$\lim_{\hat{k}\to 0} \tau_i = 1/\sigma_i. \tag{7.97}$$

However, for $\mathcal{D}\hat{k}^2/\sigma_i \gg 1$, i.e. for sufficiently large wave numbers or small wavelengths, the amplitude growth/decay time scales are dominated by diffusion:

$$\lim_{\hat{k}\to\infty} \tau_i = \frac{1}{\mathcal{D}\hat{k}^2} = \frac{1}{\mathcal{D}}\left(\frac{\lambda}{2\pi}\right)^2. \tag{7.98}$$

From Eq. (7.96), we see that a balance between reaction and diffusion exists for $\hat{k} = \hat{k}_i = \sqrt{\sigma_i/\mathcal{D}}$. In terms of wavelength, and recalling Eq. (7.72), we see the balance at

$$\lambda/(2\pi) = 1/\hat{k}_i = \sqrt{\mathcal{D}/\sigma_i} = \sqrt{\mathcal{D}\tau_i} = \ell_i. \tag{7.99}$$

Here ℓ_i is the ℓ for which the balance exists.

Example 7.6

For a case where $N = 2$, $\mathcal{D} = 10^1$ cm^2/s, $u = 10^2$ cm/s, examine time scales as a function of the length scales when considering solutions to Eq. (7.40) if

$$\mathbf{A} = \begin{pmatrix} 1000000 \text{ s}^{-1} & -99000000 \text{ s}^{-1} \\ -99000000 \text{ s}^{-1} & 99010000 \text{ s}^{-1} \end{pmatrix}. \tag{7.100}$$

We have examined this matrix earlier and know from Eqs. (7.57, 7.58) that the eigenvalues and spatially homogeneous reaction time scales are

$$\sigma_1 = 10^8 \text{ s}^{-1}, \quad \tau_1 = 10^{-8} \text{ s}, \qquad \sigma_2 = 10^4 \text{ s}^{-1}, \quad \tau_2 = 10^{-4} \text{ s}. \tag{7.101}$$

From Eq. (7.96), we get expressions for the effects of diffusion on the two time scales:

$$\tau_1 = \frac{1}{\sigma_1 + \frac{\mathcal{D}}{\ell^2}} = \frac{1}{(10^8 \text{ s}^{-1}) + \frac{10^1 \frac{\text{cm}^2}{\text{s}}}{\ell^2}}, \tag{7.102}$$

$$\tau_2 = \frac{1}{\sigma_2 + \frac{\mathcal{D}}{\ell^2}} = \frac{1}{(10^4 \text{ s}^{-1}) + \frac{10^1 \frac{\text{cm}^2}{\text{s}}}{\ell^2}}. \tag{7.103}$$

The length scales where reaction and diffusion balance are given by Eq. (7.99):

$$\ell_1 = \sqrt{\mathcal{D}\tau_1} = \sqrt{\left(10^1 \frac{\text{cm}^2}{\text{s}}\right)(10^{-8} \text{ s})} = 3.2 \times 10^{-4} \text{ cm}, \tag{7.104}$$

$$\ell_2 = \sqrt{\mathcal{D}\tau_2} = \sqrt{\left(10^1 \frac{\text{cm}^2}{\text{s}}\right)(10^{-4} \text{ s})} = 3.2 \times 10^{-2} \text{ cm}. \tag{7.105}$$

This behavior is displayed in Fig. 7.6. We see that for large ℓ, the time scales are dictated by those given by a spatially homogeneous theory. As ℓ is reduced, diffusion first plays a role in modulating the time scale of the slow reaction. As ℓ is further reduced, diffusion also modulates the time scale of the fast reaction. It is the fast reaction that dictates the time scale that needs to be considered to capture the advection-reaction-diffusion dynamics.

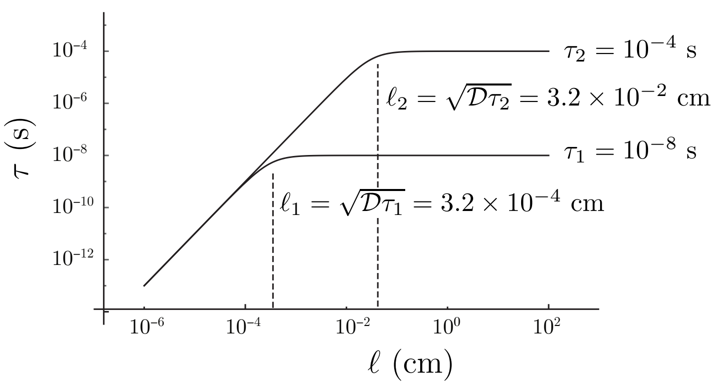


Figure 7.6: Time scale spectrum versus length scale for the simple advection-reaction-diffusion model with two-step linear kinetics.

Chapter 8

Complex variable methods

see Mei, Chapters 9, 11.

Here we consider complex variable methods. We will give a brief physical motivation in the context of Laplace's equation in two dimensions, whose solution can be elegantly described with the use of the methods of this chapter. Solutions we find can be applied in highly disparate fields, as fluid mechanics, heat transfer, mass transfer, and electromagnetism. Much of this theory was developed throughout the nineteenth century. The surprising dexterity of Laplace's equation in describing so much of nature did not escape broader notice. One even finds in Tolstoy's great novel[1] the following musing from his character Levin, who is depicted reading Tyndall[2]

> *He took up his book again. 'Very good, electricity and heat are the same thing; but is it possible to substitute the one quantity for the other in the equation for the solution of any problem?'*

8.1 Laplace's equation in engineering

We have seen in Sec. (1.3) a derivation of Laplace's equation to describe the diffusion of heat in a material whose temperature varies in two spatial dimensions and is constant in time. The analysis yields Eq. (1.104), repeated below

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{8.1}$$

where T is the temperature, with x and y as spatial variables. Also relevant in the derivation was the heat flux vector, Eq. (1.96), $\mathbf{q} = (q_x, q_y)^T$, the steady state limit of the energy

[1]Leo Tolstoy, 1877, *Anna Karenina*, Part 1, Chapter 27, in English translation by C. Garnett with L. J. Kent and N. Berberova, Modern Library Classics, New York, 2000. Also see e-book format from Project Gutenberg

[2]John Tyndall, 1820-1893, Anglo-Irish physicist.

conservation principle in which Eq. (1.98) reduces to $\nabla^T \cdot \mathbf{q} = 0$, or $\partial q_x/\partial x + \partial q_y/\partial y = 0$, and the two-dimensional limit of Fourier's law, Eq. (1.99), $\mathbf{q} = -k\nabla T$, or $q_x = -k\partial T/\partial x$; $q_y = -k\partial T/\partial y$. We give analogs from many branches of engineering science in Table 8.1, which include Fick's,[3] Newton's,[4] and Gauss's laws.

Table 8.1: Notations from branches of engineering in which Laplace's equation arises.

	heat diffusion	mass diffusion	fluid mechanics	dynamics	electrostatics
Laplace's equation	$\nabla^2 T = 0$	$\nabla^2 Y = 0$	$\nabla^2 \phi = 0$	$\nabla^2 \phi = 0$	$\nabla^2 \Phi = 0$
relevant vector	$\mathbf{q} = -k\nabla T$ Fourier's law	$\mathbf{j} = -D\nabla Y$ Fick's law	$\mathbf{u} = \nabla\phi$ irrotationality	$\mathbf{g} = -\nabla\phi$ gravitational potential	$\mathbf{E} = -\nabla\Phi$ electrical potential
divergence condition	$\nabla^T \cdot \mathbf{q} = 0$ energy conservation	$\nabla^T \cdot \mathbf{j} = 0$ mass conservation	$\nabla^T \cdot \mathbf{u} = 0$ incompressibility	$\nabla^T \cdot \mathbf{g} = 0$ Newton's law in a vacuum	$\nabla^T \cdot \mathbf{E} = 0$ Gauss's law in a vacuum

8.2 Velocity potential and stream function

We choose here to loosely focus on Laplace's equation as it arises in two-dimensional, incompressible, irrotational, inviscid fluid mechanics. One can easily use analogs from Table 8.1 to extend the same mathematical analysis to other fields. We first consider the so-called velocity potential and stream function. We consider $\mathbf{u}$ to be a velocity vector, confined to nonzero values in two dimensions:

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}. \tag{8.2}$$

Recall if a vector $\mathbf{u}$ is confined to the $x-y$ plane, and there is no variation of $\mathbf{u}$ with z ($\partial/\partial z = 0$), then the curl of that vector, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, is confined to the z direction and takes the form

$$\boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}. \tag{8.3}$$

[3]Adolf Eugen Fick, 1829-1901, German physician.
[4]Isaac Newton, 1642-1726, English physicist and mathematician.

Now if the field is two-dimensional and curl-free, we have $\boldsymbol{\omega} = \mathbf{0}$ and thus

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{8.4}$$

Moreover, because $\nabla \times \mathbf{u} = \mathbf{0}$, we can express $\mathbf{u}$ as the gradient of a potential ϕ, the *velocity potential*:

$$\mathbf{u} = \nabla\phi. \tag{8.5}$$

Note that with this definition, the velocity vector points in the direction of maximum increase of ϕ. Expanding, we can say

$$u = \frac{\partial \phi}{\partial x}, \tag{8.6}$$

$$v = \frac{\partial \phi}{\partial y}. \tag{8.7}$$

We see by substitution of Eqs. (8.6, 8.7) into Eq. (8.4) that the curl-free condition is true identically:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0. \tag{8.8}$$

This holds as long as ϕ is continuous and sufficiently differentiable. In short, we may recall that any vector field that is curl-free may be expressed as the gradient of a potential.

Now it can be shown that the physics of incompressible flows is such that

$$\nabla^T \cdot \mathbf{u} = 0. \tag{8.9}$$

Restricting to two dimensions, Eq. (8.9) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{8.10}$$

Substituting from Eqs. (8.6, 8.7) for u and v in favor of ϕ, we see Eq. (8.10) reduces to

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0, \tag{8.11}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{8.12}$$

$$\nabla^2 \phi = 0. \tag{8.13}$$

Now if Eq. (8.10) holds, we find it useful to define the *stream function* ψ as follows:

$$u = \frac{\partial \psi}{\partial y}, \tag{8.14}$$

$$v = -\frac{\partial \psi}{\partial x}. \tag{8.15}$$

Direct substitution of Eqs. (8.14, 8.15) into Eq. (8.10) shows that this yields an identity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0. \tag{8.16}$$

Now, in an equation which will be critically important soon, we can set our definitions of u and v in terms of ϕ and ψ equal to each other, as they must be. Thus combining Eqs. (8.6, 8.7, 8.14, 8.15), we see

$$\underbrace{\frac{\partial \phi}{\partial x}}_{u} = \underbrace{\frac{\partial \psi}{\partial y}}_{u}, \tag{8.17}$$

$$\underbrace{\frac{\partial \phi}{\partial y}}_{v} = \underbrace{-\frac{\partial \psi}{\partial x}}_{v}. \tag{8.18}$$

If we differentiate Eq. (8.17) with respect to y and Eq. (8.18) with respect to x, we see

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial y^2}, \tag{8.19}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2 \psi}{\partial x^2}. \tag{8.20}$$

Now subtract the Eq. (8.20) from Eq. (8.19) to get

$$0 = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}, \tag{8.21}$$

$$\nabla^2 \psi = 0. \tag{8.22}$$

Laplace's equation holds not only for ϕ but also for ψ.

Let us now examine lines of constant ϕ (equipotential lines) and lines of constant ψ (which we call streamlines). So take $\phi = C_1$, $\psi = C_2$. For $\phi = \phi(x, y)$, we can differentiate to get

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = 0, \tag{8.23}$$

$$d\phi = u\, dx + v\, dy = 0, \tag{8.24}$$

$$\left.\frac{dy}{dx}\right|_{\phi = C_1} = -\frac{u}{v}. \tag{8.25}$$

Now for $\psi = \psi(x, y)$ we similarly get

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy = 0, \tag{8.26}$$

$$d\psi = -v\, dx + u\, dy = 0, \tag{8.27}$$

$$\left.\frac{dy}{dx}\right|_{\psi = C_2} = \frac{v}{u}. \tag{8.28}$$

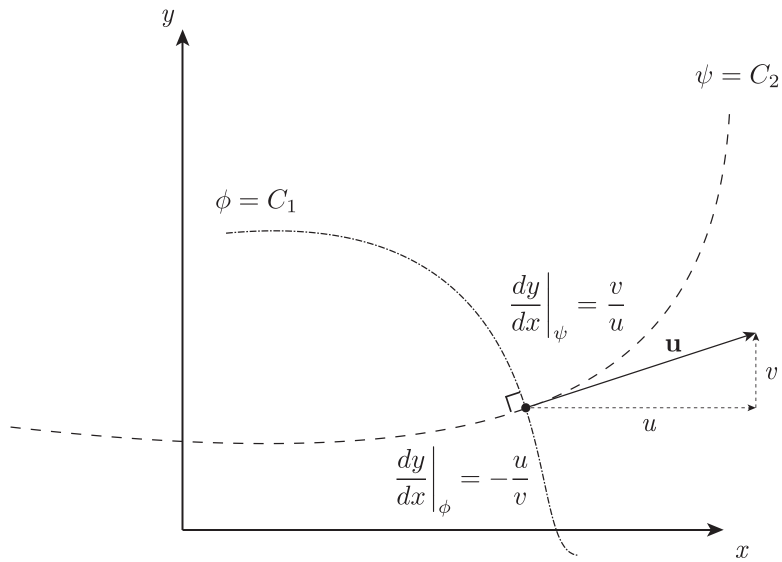


Figure 8.1: Sketch of lines of constant ψ and ϕ.

We note

$$\left.\frac{dy}{dx}\right|_{\phi=C_1} = -\frac{1}{\left.\frac{dy}{dx}\right|_{\psi=C_2}}. \tag{8.29}$$

Hence, lines of constant ϕ are orthogonal to lines of constant ψ. Furthermore, we see that

$$\frac{dx}{u} = \frac{dy}{v} \qquad \text{on lines for which} \qquad \psi = C_2. \tag{8.30}$$

As a result, we have

$$\left.\frac{dy}{dx}\right|_{\psi=C_2} = \frac{v}{u}, \tag{8.31}$$

which amounts to saying the vector $\mathbf{u}$ is tangent to the curve for which $\psi = C_2$. These notions are sketched in in Fig. 8.1.

Now solutions to the two key equations of potential flow $\nabla^2\phi = 0, \nabla^2\psi = 0$, are most efficiently studied using methods involving complex variables. We will delay discussing solutions until we have reviewed the necessary mathematics.

8.3 Mathematics of complex variables

Here we briefly introduce relevant elements of complex variable theory. Recall that the imaginary number i is defined such that

$$i^2 = -1, \qquad i = \sqrt{-1}. \tag{8.32}$$

8.3.1 Euler's formula

We can arrive at the useful *Euler's formula*, by considering the following Taylor expansions of common functions about $t = 0$:

$$\begin{aligned} e^t &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 \ldots, && (8.33)\\ \sin t &= 0 + t + 0\frac{1}{2!}t^2 - \frac{1}{3!}t^3 + 0\frac{1}{4!}t^4 + \frac{1}{5!}t^5 \ldots, && (8.34)\\ \cos t &= 1 + 0t - \frac{1}{2!}t^2 + 0\frac{1}{3!}t^3 + \frac{1}{4!}t^4 + 0\frac{1}{5!}t^5 \ldots && (8.35)\end{aligned}$$

With these expansions now consider the following combinations: $(\cos t + i \sin t)_{t=\theta}$ and $e^t|_{t=i\theta}$:

$$\begin{aligned} \cos\theta + i\sin\theta &= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 + \ldots, && (8.36)\\ e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots, && (8.37)\\ &= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 + \ldots && (8.38)\end{aligned}$$

As the two series are identical, we have Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{8.39}$$

8.3.2 Polar and Cartesian representations

We take $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ and define the complex number z to be

$$z = x + iy. \tag{8.40}$$

We say that $z \in \mathbb{C}^1$. We define the operator $\Re$ as selecting the real part of a complex number and $\Im$ as selecting the imaginary part of a complex number. For Eq. (8.40), we see

$$\begin{aligned} \Re(z) &= x, && (8.41)\\ \Im(z) &= y. && (8.42)\end{aligned}$$

Both operators $\Re$ and $\Im$ take $\mathbb{C}^1 \to \mathbb{R}^1$. We can multiply and divide Eq. (8.40) by $\sqrt{x^2+y^2}$ to obtain

$$z = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right). \tag{8.43}$$

Noting the similarities between this and the transformation between Cartesian and polar coordinates suggests we adopt

$$r = \sqrt{x^2+y^2}, \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}. \tag{8.44}$$

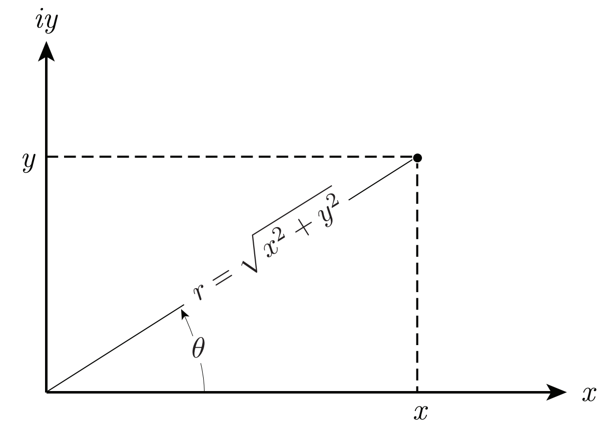


Figure 8.2: Polar and Cartesian representation of a complex number z.

Thus we have

$$\begin{aligned} z &= r\left(\cos\theta + i\sin\theta\right), && (8.45)\\ z &= re^{i\theta}. && (8.46) \end{aligned}$$

We often say that a complex number can be characterized by its magnitude $|z|$ and its argument, θ; we say then

$$\begin{aligned} r &= |z|, && (8.47)\\ \theta &= \arg z. && (8.48) \end{aligned}$$

Here, $r \in \mathbb{R}^1$ and $\theta \in \mathbb{R}^1$. Note that $|e^{i\theta}| = 1$. If $x > 0$, the function $\arg z$ is identical to $\arctan(y/x)$ and is suggested by the polar and Cartesian representation of z as shown in Fig. 8.2. However, we recognize that the ordinary arctan (also known as $\tan^{-1}$) function maps onto the range $[-\pi/2, \pi/2]$, while we would like arg to map onto $[-\pi, \pi]$. For example, to capture the entire unit circle if $r = 1$, we need $\theta \in [-\pi, \pi]$. This can be achieved if we define arg, also known as Tan^{-1} as follows:

$$\arg z = \arg(x + iy) = \mathrm{Tan}^{-1}(x, y) = 2\arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right). \qquad (8.49)$$

Iff $x > 0$, this reduces to the more typical

$$\arg z = \arg(x + iy) = \mathrm{Tan}^{-1}(x, y) = \arctan\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right), \qquad x > 0. \qquad (8.50)$$

The preferred and more general form is Eq. (8.49). We give simple function evaluations involving arctan and Tan^{-1} for selected values of x and y in Table 8.2. Use of Tan^{-1} effectively captures the correct quadrant of the complex plane corresponding to different

Table 8.2: Comparison of the action of arg, Tan^{-1}, and arctan.

x	y	$\arg(x+iy)$	$\text{Tan}^{-1}(x,y)$	$\arctan(y/x)$
1	1	$\pi/4$	$\pi/4$	$\pi/4$
-1	1	$3\pi/4$	$3\pi/4$	$-\pi/4$
-1	-1	$-3\pi/4$	$-3\pi/4$	$\pi/4$
1	-1	$-\pi/4$	$-\pi/4$	$-\pi/4$

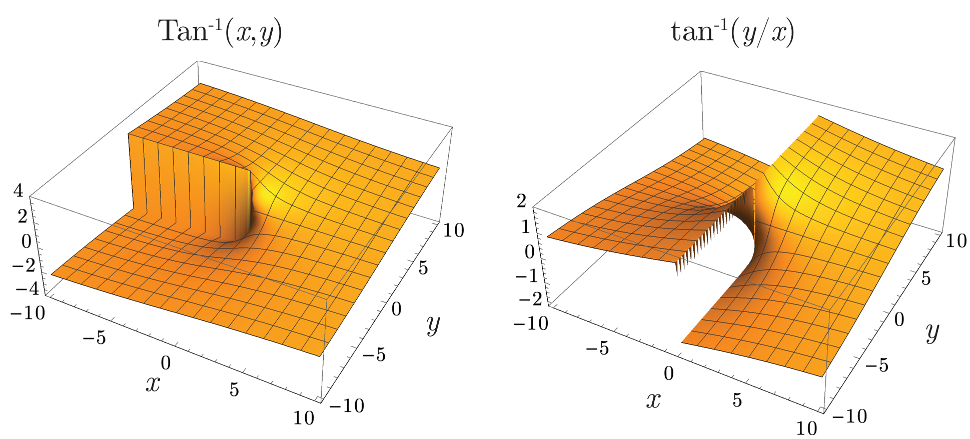


Figure 8.3: Comparison of $\text{Tan}^{-1}(x,y)$ and $\tan^{-1}(y/x)$.

positive and negative values of x and y. The function is sometimes known as Arctan or atan2. A comparison of $\mathrm{Tan}^{-1}(x,y)$ and $\tan^{-1}(y/x)$ is given in Fig. 8.3.

Now we can define the *complex conjugate* $\overline{z}$ as

$$\begin{aligned}
\overline{z} &= x - iy, && (8.51)\\
&= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} - i\frac{y}{\sqrt{x^2+y^2}}\right), && (8.52)\\
&= r\left(\cos\theta - i\sin\theta\right), && (8.53)\\
&= r\left(\cos(-\theta) + i\sin(-\theta)\right), && (8.54)\\
&= re^{-i\theta}. && (8.55)
\end{aligned}$$

Note now that

$$\begin{aligned}
z\overline{z} &= (x+iy)(x-iy) = x^2+y^2 = |z|^2, && (8.56)\\
&= re^{i\theta}re^{-i\theta}, && (8.57)\\
&= r^2, && (8.58)\\
&= |z|^2. && (8.59)
\end{aligned}$$

We also have

$$\begin{aligned}
\sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i}, && (8.60)\\
\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}. && (8.61)
\end{aligned}$$

Example 8.7

Use the polar representation of z to find all roots to the algebraic equation

$$z^4 = 1. \tag{8.62}$$

We know that $z = re^{i\theta}$. We also note that the constant 1 can be represented as

$$1 = e^{2n\pi i}, \qquad n = 0, 1, 2, \ldots \tag{8.63}$$

This will be useful in finding all roots to our equation. With this representation, Eq. (8.62) becomes

$$r^4 e^{4i\theta} = e^{2n\pi i}, \qquad n = 0, 1, 2, \ldots. \tag{8.64}$$

We have a solution when

$$r = 1, \qquad \theta = \frac{n\pi}{2}, \qquad n = 0, 1, 2, \ldots \tag{8.65}$$

There are unique solutions for $n = 0, 1, 2, 3$. For larger n, the solutions repeat. So we have four solutions

$$z = e^{0i}, \quad z = e^{i\pi/2}, \quad z = e^{i\pi}, \quad z = e^{3i\pi/2}. \tag{8.66}$$

In Cartesian form, the four solutions are

$$z = \pm 1, \quad z = \pm i. \tag{8.67}$$

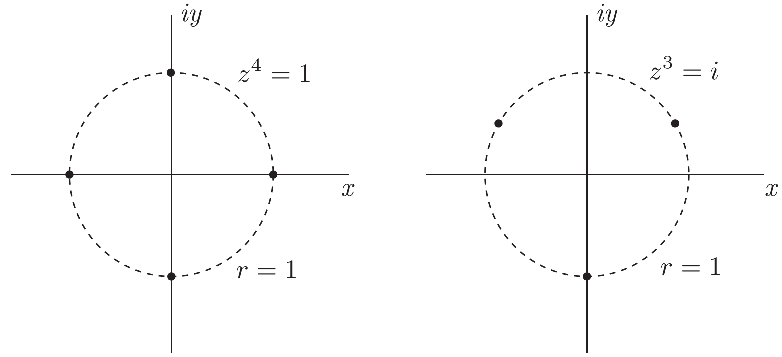


Figure 8.4: Sketches of solutions to $z^4 = 1$ and $z^3 = i$ in the complex plane.

Example 8.8

Find all roots to

$$z^3 = i. \tag{8.68}$$

We proceed in a similar fashion as for the previous example. We know that

$$i = e^{i(\pi/2+2n\pi)}, \quad n = 0, 1, 2, ... \tag{8.69}$$

Substituting this into Eq. (8.68), we get

$$r^3 e^{3i\theta} = e^{i(\pi/2+2n\pi)}, \quad n = 0, 1, 2, ... \tag{8.70}$$

Solving, we get

$$r = 1, \qquad \theta = \frac{\pi}{6} + \frac{2n\pi}{3}. \tag{8.71}$$

There are only three unique values of θ, those being $\theta = \pi/6$, $\theta = 5\pi/6$, $\theta = 3\pi/2$. So the three roots are

$$z = e^{i\pi/6}, \qquad z = e^{5i\pi/6}, \qquad z = e^{3i\pi/2}. \tag{8.72}$$

In Cartesian form these roots are

$$z = \frac{\sqrt{3}+i}{2}, \qquad z = \frac{-\sqrt{3}+i}{2}, \qquad z = -i. \tag{8.73}$$

Sketches of the solutions to this and the previous example are shown in Fig. 8.4. For both examples, the roots are uniformly distributed about the unit circle, with four roots for the quartic equation and three for the cubic.

8.3.3 Cauchy-Riemann equations

Now it is possible to define complex functions of complex variables $W(z)$. For example, take a complex function to be defined as

$$W(z) = z^2 + z, \tag{8.74}$$
$$= (x+iy)^2 + (x+iy), \tag{8.75}$$
$$= x^2 + 2xyi - y^2 + x + iy, \tag{8.76}$$
$$= \left(x^2 + x - y^2\right) + i\,(2xy + y). \tag{8.77}$$

In general, we can say

$$W(z) = \phi(x,y) + i\psi(x,y). \tag{8.78}$$

Here ϕ and ψ are *real* functions of *real* variables. We shall soon see we have chosen their symbols to usefully match the physics-based symbols of the previous section.

Now $W(z)$ is defined as *analytic* at z_0 if dW/dz exists at z_0 and is independent of the direction in which it was calculated. That is, extending Newton's definition of the derivative to apply to complex numbers, we adopt

$$\left.\frac{dW}{dz}\right|_{z=z_0} = \frac{W(z_0 + \Delta z) - W(z_0)}{\Delta z}. \tag{8.79}$$

The notation in the subscript near the vertical bar indicates that the derivative is evaluated at the point $z = z_0$. Now there are many paths that we can choose to evaluate the derivative. Let us consider two distinct paths, $y = C_1$ and $x = C_2$. We will get a result which can be shown to be valid for arbitrary paths.

For $y = C_1$, we have $\Delta z = \Delta x$, so

$$\left.\frac{dW}{dz}\right|_{z=z_0} = \frac{W(x_0 + iy_0 + \Delta x) - W(x_0 + iy_0)}{\Delta x}, \tag{8.80}$$
$$= \left.\frac{\partial W}{\partial x}\right|_{y}. \tag{8.81}$$

Here, the subscript y next to the vertical bar indicates that y is considered to be held constant.

For $x = C_2$, we have $\Delta z = i\Delta y$, so

$$\left.\frac{dW}{dz}\right|_{z=z_0} = \frac{W(x_0 + iy_0 + i\Delta y) - W(x_0 + iy_0)}{i\Delta y}, \tag{8.82}$$
$$= \frac{1}{i}\left.\frac{\partial W}{\partial y}\right|_{x}, \tag{8.83}$$
$$= -i\left.\frac{\partial W}{\partial y}\right|_{x}. \tag{8.84}$$

Here, the subscript x next to the vertical bar indicates that x is considered to be held constant. Now for an analytic function, we need the derivative to be the same for any path of integration. So certainly we must require

$$\left.\frac{\partial W}{\partial x}\right|_y = -i \left.\frac{\partial W}{\partial y}\right|_x . \tag{8.85}$$

Expanding using $W = \phi + i\psi$, and dispensing with the vertical bars, we need

$$\begin{aligned} \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} &= -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right), && (8.86)\\ &= \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}. && (8.87) \end{aligned}$$

Thus, for equality, and thus path independence of the derivative, we require

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y}, && (8.88)\\ \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x}. && (8.89) \end{aligned}$$

These are the well known *Cauchy-Riemann* equations for analytic functions of complex variables. *They are identical to our equations for incompressible irrotational fluid mechanics.* Moreover, they are identical to any of the other physical analogs from heat and mass transfer, etc., presented in Table 8.1. Consequently, *any analytic complex function is guaranteed to be a physical solution.* There are an infinite number of functions to choose from.

We define the *complex potential* as

$$W(z) = \phi(x, y) + i\psi(x, y), \tag{8.90}$$

and taking a derivative of the analytic potential, we have, using Eqs. (8.6,8.15), that

$$\begin{aligned} \frac{dW}{dz} &= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}, && (8.91)\\ &= u - iv. && (8.92) \end{aligned}$$

We can equivalently say using Eqs. (8.7, 8.14) that

$$\begin{aligned} \frac{dW}{dz} &= -i\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right), && (8.93)\\ &= \left(\frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}\right), && (8.94)\\ &= u - iv. && (8.95) \end{aligned}$$

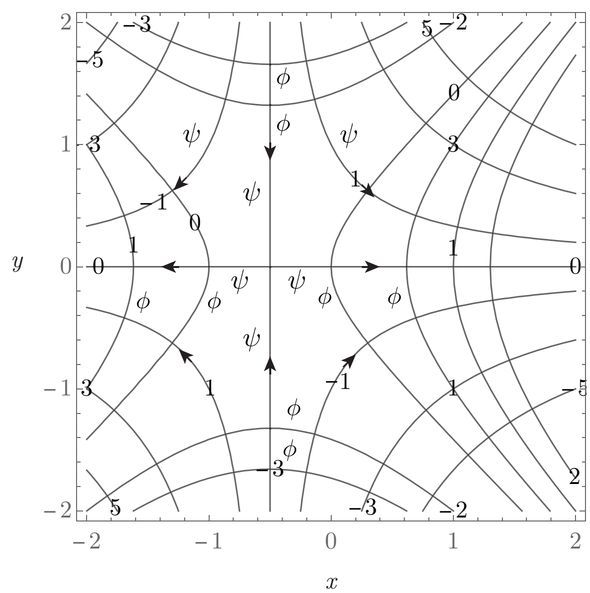


Figure 8.5: Plot of contours of $\phi(x, y) = x^2 + x - y^2$ and $\psi = 2xy + y$.

Now most common functions are easily shown to be analytic. For example, for the function $W(z) = z^2 + z$, we are tempted to apply the ordinary rules of differentiation to get $dW/dz = 2z + 1$. Let us check more carefully. We first expand to express $W(z)$ as

$$W(z) = \underbrace{(x^2 + x - y^2)}_{\phi(x,y)} + i\underbrace{(2xy + y)}_{\psi(x,y)}. \tag{8.96}$$

We see then that we have

$$\phi(x, y) = x^2 + x - y^2, \qquad \psi(x, y) = 2xy + y, \tag{8.97}$$

$$\frac{\partial \phi}{\partial x} = 2x + 1, \qquad \frac{\partial \psi}{\partial x} = 2y, \tag{8.98}$$

$$\frac{\partial \phi}{\partial y} = -2y, \qquad \frac{\partial \psi}{\partial y} = 2x + 1. \tag{8.99}$$

Note that the Cauchy-Riemann equations are satisfied because $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = -\partial\psi/\partial x$. So the derivative is independent of direction, and we can say

$$\frac{dW}{dz} = \left.\frac{\partial W}{\partial x}\right|_y = (2x + 1) + i(2y) = 2(x + iy) + 1 = 2z + 1. \tag{8.100}$$

Thus our supposition that extending the ordinary rules of derivatives for real functions to complex functions indeed works here. We plot contours of constant ϕ and ψ in Fig. 8.5. It is seen in Fig. 8.5 that lines of constant ϕ are orthogonal to lines of constant ψ consistent with the discussion in Sec. 8.2. Note also that because

$$u = \frac{\partial \phi}{\partial x} = 2x + 1, \qquad v = \frac{\partial \phi}{\partial y} = -2y, \tag{8.101}$$

the velocity vector $(u, v)^T$ is zero when $(x, y) = (-1/2, 0)$. This point is evident in Fig. 8.5. Note also that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2 = 0, \tag{8.102}$$

so if the solution were for a fluid, it would satisfy a mass conservation equation for an incompressible fluid. And the solution is also representative of an irrotational fluid as

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 0 = 0. \tag{8.103}$$

For an example of a nonanalytic function consider $W(z) = \overline{z}$. Thus

$$W(z) = x - iy. \tag{8.104}$$

So $\phi = x$ and $\psi = -y$, $\partial\phi/\partial x = 1$, $\partial\phi/\partial y = 0$, and $\partial\psi/\partial x = 0$, $\partial\psi/\partial y = -1$. Because $\partial\phi/\partial x \neq \partial\psi/\partial y$, the Cauchy-Riemann equations are not satisfied, and the derivative depends on direction.

8.4 Elementary complex potentials

Let us examine some simple analytic functions and see examples of the physics to which they correspond.

8.4.1 Uniform field

Take

$$W(z) = Az, \qquad \text{with} \qquad A \in \mathbb{C}^1. \tag{8.105}$$

Then

$$\frac{dW}{dz} = A = u - iv. \tag{8.106}$$

Because A is complex, we can say

$$A = Ue^{-i\alpha} = U\cos\alpha - iU\sin\alpha. \tag{8.107}$$

Thus

$$\begin{aligned} W(z) &= Az = (U\cos\alpha - iU\sin\alpha)(x + iy), & (8.108)\\ &= U(x\cos\alpha + y\sin\alpha) + iU(y\cos\alpha - x\sin\alpha). & (8.109) \end{aligned}$$

So

$$\phi = U(x\cos\alpha + y\sin\alpha), \qquad \psi = U(y\cos\alpha - x\sin\alpha). \tag{8.110}$$

And we get

$$u = U\cos\alpha, \qquad v = U\sin\alpha. \tag{8.111}$$

This represents a spatially uniform velocity filed with streamlines inclined at angle α to the x axis. The field is sketched in Fig. 8.6.

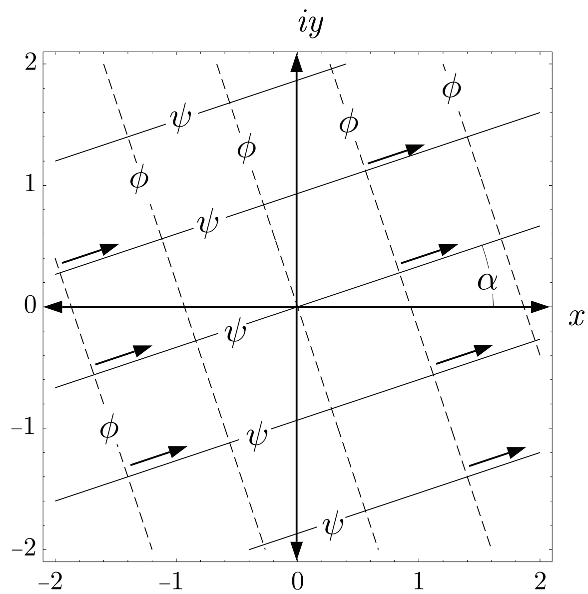


Figure 8.6: Streamlines for uniform flow.

8.4.2 Sources and sinks

Take

$$W(z) = A \ln z, \qquad A \in \mathbb{R}^1. \tag{8.112}$$

With $z = re^{i\theta}$, we have $\ln z = \ln r + i\theta$. So

$$W(z) = A \ln r + iA\theta. \tag{8.113}$$

Consequently, we have for the velocity potential and stream function

$$\phi = A \ln r, \qquad \psi = A\theta. \tag{8.114}$$

Now $\mathbf{u} = \nabla\phi$, so, after transforming to polar coordinates, omitting some details, we obtain

$$u_r = \frac{\partial \phi}{\partial r} = \frac{A}{r}, \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0. \tag{8.115}$$

So the velocity is all radial, and becomes infinite at $r = 0$. We can show that the volume flow rate is bounded, and is in fact a constant. The volume flow rate Q through a surface is

$$Q = \int_A \mathbf{u}^T \cdot \mathbf{n}\ dA = \int_0^{2\pi} u_r r\ d\theta = \int_0^{2\pi} \frac{A}{r} r\ d\theta = 2\pi A. \tag{8.116}$$

The volume flow rate is a constant. If $A > 0$, we have a source. If $A < 0$, we have a sink. The potential for a source/sink is often written as

$$W(z) = \frac{Q}{2\pi} \ln z. \tag{8.117}$$

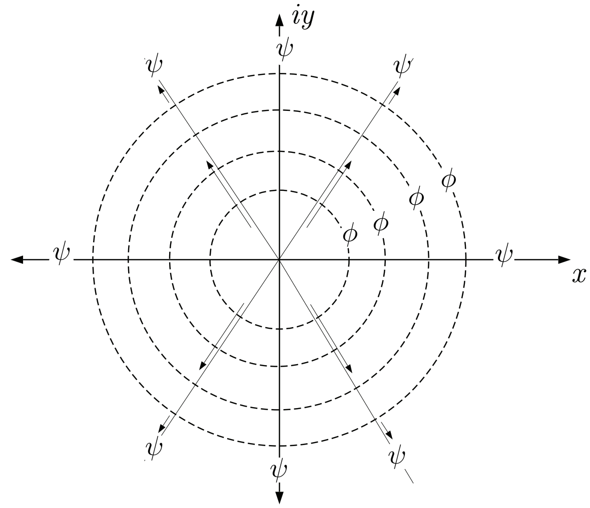


Figure 8.7: Velocity vectors and equipotential lines for source flow.

For a source located at a point z_0 which is not at the origin, we can say

$$W(z) = \frac{Q}{2\pi} \ln(z - z_0). \tag{8.118}$$

The flow is sketched in Fig. 8.7.

8.4.3 Point vortices

For an ideal point vortex, we have

$$W(z) = iB \ln z, \qquad B \in \mathbb{R}^1. \tag{8.119}$$

So

$$W(z) = iB (\ln r + i\theta) = -B\theta + iB \ln r. \tag{8.120}$$

Consequently,

$$\phi = -B\theta, \qquad \psi = B \ln r. \tag{8.121}$$

We get the velocity field from

$$u_r = \frac{\partial \phi}{\partial r} = 0, \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{B}{r}. \tag{8.122}$$

So we see that the streamlines are circles about the origin, and there is no radial component of velocity. Consider the so-called circulation of this flow

$$\Gamma = \oint_C \mathbf{u}^T \cdot d\mathbf{r} = \int_0^{2\pi} -\frac{B}{r} r \; d\theta = -2\pi B. \tag{8.123}$$

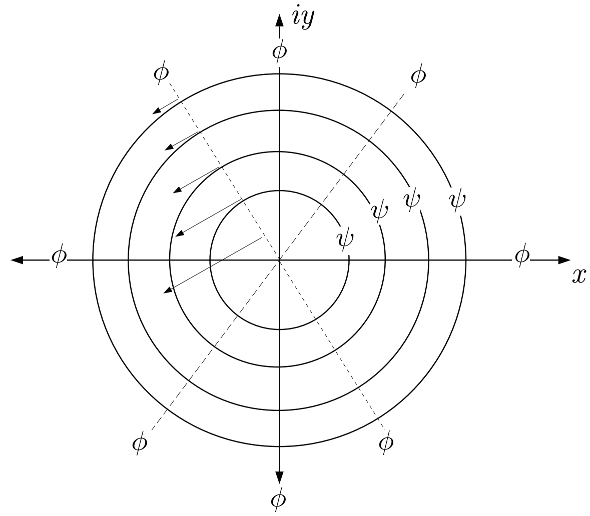


Figure 8.8: Streamlines, equipotential, and velocity vectors lines for a point vortex.

So we often write the complex potential in terms of the ideal vortex strength Γ:

$$W(z) = -\frac{i\Gamma}{2\pi} \ln z. \tag{8.124}$$

For an ideal vortex not at $z = z_0$, we say

$$W(z) = -\frac{i\Gamma}{2\pi} \ln(z - z_0). \tag{8.125}$$

The point vortex flow is sketched in Fig. 8.8.

8.4.4 Superposition of sources

Because the equation for velocity potential is linear, we can use the method of superposition to create new solutions as summations of elementary solutions. Say we want to model the effect of a wall on a source as sketched in Fig. 8.9. At the wall we want $u(0, y) = 0$. That is

$$\Re\left(\frac{dW}{dz}\right) = \Re\left(u - iv\right) = u = 0, \qquad \text{on} \qquad z = iy. \tag{8.126}$$

Now let us place a source at $z = a$ and superpose a source at $z = -a$, where a is a real number. So we have for the complex potential

$$W(z) = \underbrace{\frac{Q}{2\pi}\ln(z-a)}_{\text{original}} + \underbrace{\frac{Q}{2\pi}\ln(z+a)}_{\text{image}}, \tag{8.127}$$

$$= \frac{Q}{2\pi}\left(\ln(z-a) + \ln(z+a)\right), \tag{8.128}$$

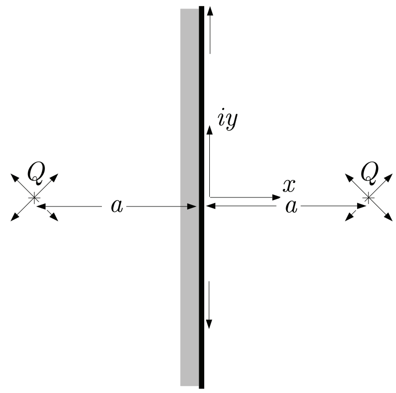


Figure 8.9: Sketch for source-wall interaction.

$$= \frac{Q}{2\pi}\left(\ln(z-a)(z+a)\right), \tag{8.129}$$

$$= \frac{Q}{2\pi}\ln(z^2-a^2), \tag{8.130}$$

$$\frac{dW}{dz} = \frac{Q}{2\pi}\frac{2z}{z^2-a^2}. \tag{8.131}$$

Now on $z = iy$, which is the location of the wall, we have

$$\frac{dW}{dz} = \frac{Q}{2\pi}\left(\frac{2iy}{-y^2-a^2}\right) = -i\underbrace{\frac{Q}{\pi}\left(\frac{y}{y^2+a^2}\right)}_{v}. \tag{8.132}$$

The term is purely imaginary; hence, the real part is zero, and we have $u = 0$ on the wall, as desired.

On the wall we have a nonzero y component of velocity:

$$v = \frac{Q}{\pi}\frac{y}{y^2+a^2}. \tag{8.133}$$

We find the location on the wall of the maximum v velocity by setting the derivative with respect to y to be zero,

$$\frac{\partial v}{\partial y} = \frac{Q}{\pi}\frac{(y^2+a^2)-y(2y)}{(y^2+a^2)^2} = 0. \tag{8.134}$$

Solving, we find a critical point at $y = \pm a$, which can be shown to be a maximum.

8.4.5 Flow in corners

Flow in or around a corner can be modeled by the complex potential

$$W(z) = Az^n, \qquad A \in \mathbb{R}^1, \tag{8.135}$$

$$
\begin{aligned}
&= A\left(re^{i\theta}\right)^n, &(8.136)\\
&= Ar^n e^{in\theta}, &(8.137)\\
&= Ar^n(\cos(n\theta) + i\sin(n\theta)). &(8.138)
\end{aligned}
$$

So we have

$$
\begin{aligned}
\phi &= Ar^n \cos n\theta, &(8.139)\\
\psi &= Ar^n \sin n\theta. &(8.140)
\end{aligned}
$$

Now recall that lines on which ψ is constant are streamlines. Examining the stream function, we obviously have streamlines when $\psi = 0$ which occurs whenever $\theta = 0$ or $\theta = \pi/n$.

For example if $n = 2$, we model a stream striking a flat wall. For this flow, we have

$$
\begin{aligned}
W(z) &= Az^2, &(8.141)\\
&= A(x + iy)^2, &(8.142)\\
&= A((x^2 - y^2) + i(2xy)). &(8.143)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\phi &= A(x^2 - y^2), &(8.144)\\
\psi &= A(2xy). &(8.145)
\end{aligned}
$$

So the streamlines are hyperbolas. For the velocity field, we take

$$
\begin{aligned}
\frac{dW}{dz} &= 2Az, &(8.146)\\
&= 2A(x + iy), &(8.147)\\
&= u - iv. &(8.148)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u &= 2Ax, &(8.149)\\
v &= -2Ay. &(8.150)
\end{aligned}
$$

This flow actually represents flow in a corner formed by a right angle or flow striking a flat plate, or the impingement of two streams. For $n = 2$, streamlines are sketched in in Fig. 8.10.

8.4.6 Doublets

We can form what is known as a doublet flow by considering the superposition of a source and sink and let the two approach each other. Consider a source and sink of equal and opposite strength straddling the y axis, each separated from the origin by a distance ϵ as sketched in Fig. 8.11. The complex velocity potential is

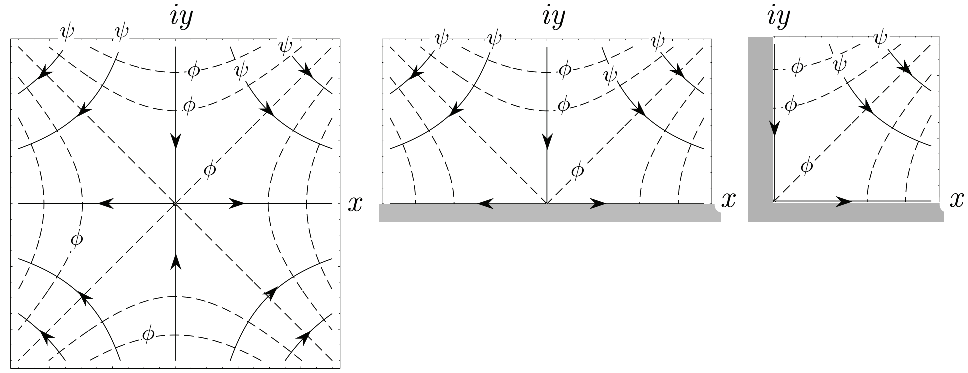


Figure 8.10: Sketch for impingement flow, stagnation flow, and flow in a corner, $n = 2$.

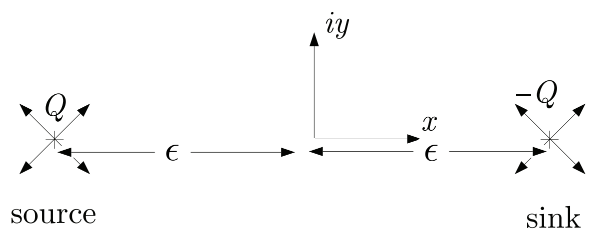


Figure 8.11: Source sink pair.

$$W(z) = \frac{Q}{2\pi}\ln(z+\epsilon) - \frac{Q}{2\pi}\ln(z-\epsilon), \tag{8.151}$$

$$= \frac{Q}{2\pi}\ln\left(\frac{z+\epsilon}{z-\epsilon}\right). \tag{8.152}$$

It can be shown by synthetic division that as $\epsilon \to 0$, that

$$\frac{z+\epsilon}{z-\epsilon} = 1 + \epsilon\frac{2}{z} + \epsilon^2\frac{2}{z^2} + \dots. \tag{8.153}$$

So the potential approaches

$$W(z) \sim \frac{Q}{2\pi}\ln\left(1 + \epsilon\frac{2}{z} + \epsilon^2\frac{2}{z^2} + \dots\right). \tag{8.154}$$

Now because $\ln(1+x) \to x$ as $x \to 0$, we get for small ϵ that

$$W(z) \sim \frac{Q}{2\pi}\epsilon\frac{2}{z} \sim \frac{Q\epsilon}{\pi z}. \tag{8.155}$$

Now if we require that

$$\lim_{\epsilon\to 0}\frac{Q\epsilon}{\pi} \to \mu, \tag{8.156}$$

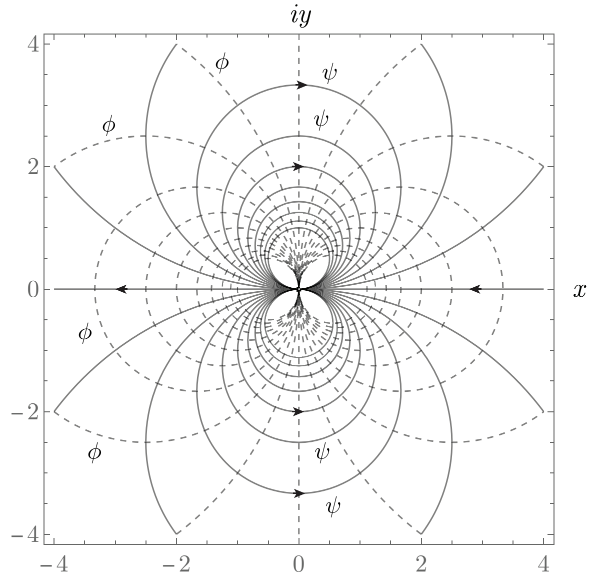


Figure 8.12: Streamlines and equipotential lines for a doublet. Notice because the sink is infinitesimally to the right of the source, there exists a directionality. This can be considered a type of *dipole moment*; in this case, the direction of the dipole is $-\mathbf{i}$.

we have

$$W(z) = \frac{\mu}{z}, \tag{8.157}$$

$$= \frac{\mu}{x+iy}\frac{x-iy}{x-iy}, \tag{8.158}$$

$$= \frac{\mu(x-iy)}{x^2+y^2}. \tag{8.159}$$

So

$$\phi(x,y) = \mu\frac{x}{x^2+y^2}, \tag{8.160}$$

$$\psi(x,y) = -\mu\frac{y}{x^2+y^2}. \tag{8.161}$$

In polar coordinates, we then say

$$\phi = \mu\frac{\cos\theta}{r}, \tag{8.162}$$

$$\psi = -\mu\frac{\sin\theta}{r}. \tag{8.163}$$

Streamlines and equipotential lines for a doublet are plotted in Fig. 8.12.

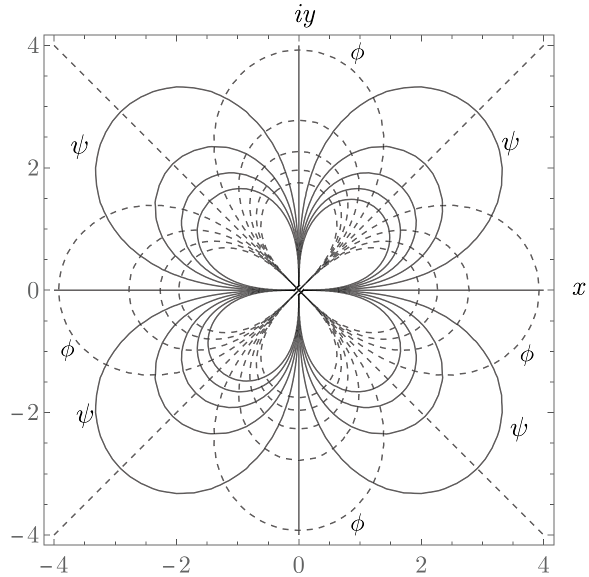


Figure 8.13: Streamlines and equipotential lines for a quadrupole, $k = 1$.

8.4.7 Quadrupoles

It is natural to examine a higher order potential, which will be called the quadrupole:

$$W(z) = \frac{k}{z^2}, \tag{8.164}$$
$$= k\frac{1}{(x+iy)(x+iy)}, \tag{8.165}$$
$$= k\frac{(x-iy)^2}{(x^2+y^2)^2}, \tag{8.166}$$
$$= k\frac{x^2-y^2-2ixy}{(x^2+y^2)^2}. \tag{8.167}$$

This gives

$$\phi(x,y) = k\frac{x^2-y^2}{(x^2+y^2)^2}, \tag{8.168}$$
$$\psi(x,y) = k\frac{-2xy}{(x^2+y^2)^2}. \tag{8.169}$$

Streamlines and equipotential lines for a quadrupole are plotted in Fig. 8.13 for $k = 1$.

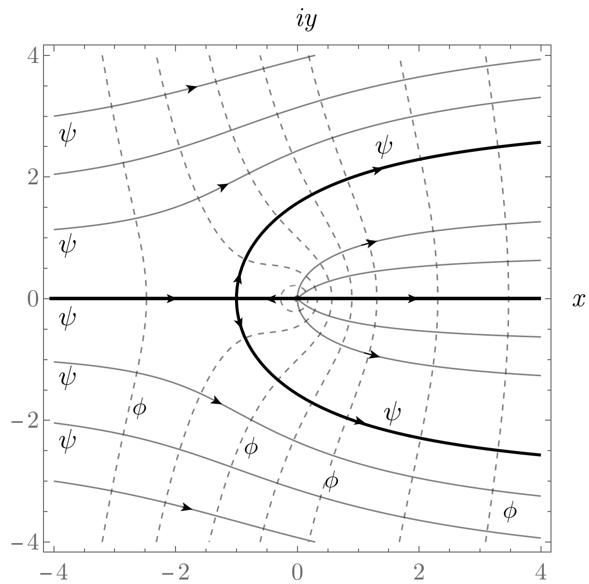


Figure 8.14: Streamlines for a Rankine half body.

8.4.8 Rankine half body

Now consider the superposition of a uniform stream and a source, which we define to be a Rankine half body:

$$
\begin{aligned}
W(z) &= Uz + \frac{Q}{2\pi}\ln z, \qquad U, Q \in \mathbb{R}^1, && (8.170)\\
&= Ure^{i\theta} + \frac{Q}{2\pi}(\ln r + i\theta), && (8.171)\\
&= Ur(\cos\theta + i\sin\theta) + \frac{Q}{2\pi}(\ln r + i\theta), && (8.172)\\
&= \left(Ur\cos\theta + \frac{Q}{2\pi}\ln r\right) + i\left(Ur\sin\theta + \frac{Q}{2\pi}\theta\right). && (8.173)
\end{aligned}
$$

So

$$
\begin{aligned}
\phi &= Ur\cos\theta + \frac{Q}{2\pi}\ln r, && (8.174)\\
\psi &= Ur\sin\theta + \frac{Q}{2\pi}\theta. && (8.175)
\end{aligned}
$$

Streamlines for a Rankine half body are plotted in Fig. 8.14. Now for the Rankine half body, it is clear that there is a point where the velocity vector $\mathbf{u} = \mathbf{0}$ somewhere on the x axis, along $\theta = \pi$. With the velocity given by

$$\frac{dW}{dz} = U + \frac{Q}{2\pi z} = u - iv, \qquad (8.176)$$

we get

$$U + \frac{Q}{2\pi}\frac{1}{r}e^{-i\theta} = u - iv, \tag{8.177}$$

$$U + \frac{Q}{2\pi}\frac{1}{r}(\cos\theta - i\sin\theta) = u - iv. \tag{8.178}$$

Thus,

$$u = U + \frac{Q}{2\pi r}\cos\theta, \tag{8.179}$$

$$v = \frac{Q}{2\pi r}\sin\theta. \tag{8.180}$$

When $\theta = \pi$, we get $u = 0$ when;

$$0 = U + \frac{Q}{2\pi r}(-1), \tag{8.181}$$

$$r = \frac{Q}{2\pi U}. \tag{8.182}$$

8.4.9 Flow over a cylinder

We can model flow past a cylinder by superposing a uniform flow with a doublet. Defining $a^2 = \mu/U$, we write

$$\begin{aligned}
W(z) &= Uz + \frac{\mu}{z} = U\left(z + \frac{a^2}{z}\right), && (8.183)\\
&= U\left(re^{i\theta} + \frac{a^2}{re^{i\theta}}\right), && (8.184)\\
&= U\left(r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)\right), && (8.185)\\
&= U\left(\left(r\cos\theta + \frac{a^2}{r}\cos\theta\right) + i\left(r\sin\theta - \frac{a^2}{r}\sin\theta\right)\right), && (8.186)\\
&= Ur\left(\cos\theta\left(1 + \frac{a^2}{r^2}\right) + i\sin\theta\left(1 - \frac{a^2}{r^2}\right)\right). && (8.187)
\end{aligned}$$

So

$$\phi = Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right), \tag{8.188}$$

$$\psi = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right). \tag{8.189}$$

Now on $r = a$, we have $\psi = 0$. Because the stream function is constant here, the curve $r = a$, a circle, must be a streamline. A sketch of the streamlines and equipotential lines is plotted in Fig. 8.15.

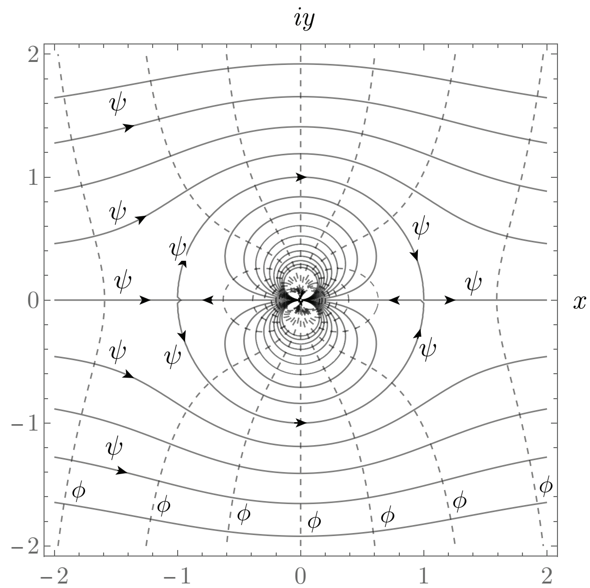


Figure 8.15: Streamlines and equipotential lines for flow over a cylinder without circulation.

For the velocities, we have

$$u_r = \frac{\partial \phi}{\partial r} = U\cos\theta\left(1+\frac{a^2}{r^2}\right) + Ur\cos\theta\left(-2\frac{a^2}{r^3}\right), \tag{8.190}$$

$$= U\cos\theta\left(1-\frac{a^2}{r^2}\right), \tag{8.191}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\sin\theta\left(1+\frac{a^2}{r^2}\right). \tag{8.192}$$

So on $r = a$, we have $u_r = 0$, and $u_\theta = -2U\sin\theta$.

There are more basic ways to describe the force on bodies using complex variables directly. We shall give those methods, but first a discussion of the motivating complex variable theory is necessary.

8.5 Contour integrals

Consider the closed contour integral of a complex function in the complex plane. For such integrals, we have a useful theory which we will not prove, but will demonstrate here. Consider contour integrals enclosing the origin with a circle in the complex plane for four functions. The contour in each is $C : z = \hat{R}e^{i\theta}$ with $\theta \in [0, 2\pi]$. For such a contour $dz = i\hat{R}e^{i\theta}\ d\theta$.

8.5.1 Simple pole

We describe a simple pole with the complex potential

$$W(z) = \frac{a}{z}, \tag{8.193}$$

and the contour integral is

$$\begin{aligned}\oint_C W(z)\,dz &= \oint_C \frac{a}{z}\,dz = \int_{\theta=0}^{\theta=2\pi} \frac{a}{\hat{R}e^{i\theta}} i\hat{R}e^{i\theta}\,d\theta, &(8.194)\\ &= ai\int_0^{2\pi} d\theta = 2\pi i a. &(8.195)\end{aligned}$$

8.5.2 Constant potential

We describe a constant with the complex potential

$$W(z) = b. \tag{8.196}$$

The contour integral is

$$\begin{aligned}\oint_C W(z)\,dz &= \oint_C b\,dz, &(8.197)\\ &= \int_{\theta=0}^{\theta=2\pi} bi\hat{R}e^{i\theta}\,d\theta, &(8.198)\\ &= \left.\frac{bi\hat{R}}{i}e^{i\theta}\right|_0^{2\pi}, &(8.199)\\ &= 0, &(8.200)\end{aligned}$$

because $e^{0i} = e^{2\pi i} = 1$.

8.5.3 Linear potential

We describe a linear field with the complex potential

$$W(z) = cz. \tag{8.201}$$

The contour integral is

$$\begin{aligned}\oint_C W(z)\,dz &= \oint_C cz\,dz = \int_{\theta=0}^{\theta=2\pi} c\hat{R}e^{i\theta} i\hat{R}e^{i\theta}\,d\theta, &(8.202)\\ &= ic\hat{R}^2\int_0^{2\pi} e^{2i\theta}\,d\theta = \left.\frac{ic\hat{R}^2}{2i}e^{2i\theta}\right|_0^{2\pi} = 0, &(8.203)\end{aligned}$$

because $e^{0i} = e^{4\pi i} = 1$.

8.5.4 Quadrupole

A quadrupole potential is described by

$$W(z) = \frac{k}{z^2}. \tag{8.204}$$

Taking the contour integral, we find

$$\oint_C \frac{k}{z^2}\,dz = k\int_0^{2\pi} \frac{i\hat{R}e^{i\theta}}{\hat{R}^2 e^{2i\theta}}\,d\theta, \tag{8.205}$$

$$= \frac{ki}{\hat{R}}\int_0^{2\pi} e^{-i\theta}\,d\theta = \frac{ki}{\hat{R}}\left(\frac{1}{-i}\right)e^{-i\theta}\Big|_0^{2\pi} = 0. \tag{8.206}$$

So the only nonzero contour integral is for functions of the form $W(z) = a/z$. If we continued, we would find all powers of z have a zero contour integral about the origin for arbitrary contours except this special one.

8.6 Laurent series

Now it can be shown that any function can be expanded, much as for a Taylor series, as a *Laurent series*:[5]

$$W(z) = \ldots + C_{-2}(z-z_0)^{-2} + C_{-1}(z-z_0)^{-1} + C_0(z-z_0)^0 + C_1(z-z_0)^1 + C_2(z-z_0)^2 + \ldots. \tag{8.207}$$

In compact summation notation, we can say

$$W(z) = \sum_{n=-\infty}^{n=\infty} C_n(z-z_0)^n. \tag{8.208}$$

Taking the contour integral of both sides we get

$$\oint_C W(z)\,dz = \oint_C \sum_{n=-\infty}^{n=\infty} C_n(z-z_0)^n\,dz, \tag{8.209}$$

$$= \sum_{n=-\infty}^{n=\infty} C_n \oint_C (z-z_0)^n\,dz. \tag{8.210}$$

From our just completed analysis, this has value $2\pi i$ only when $n = -1$, so

$$\oint_C W(z)\,dz = C_{-1}2\pi i. \tag{8.211}$$

[5]Pierre Alphonse Laurent, 1813-1854, Parisian engineer who worked on port expansion in Le Harve, submitted his work on Laurent series for a Grand Prize in 1842, with the recommendation of Cauchy, but was rejected because of a late submission.

Here C_{-1} is known as the *residue* of the Laurent series. In general we have the Cauchy integral theorem which holds that if $W(z)$ is analytic within and on a closed curve C except for a finite number of singular points, then

$$\oint_C W(z)\ dz = 2\pi i \sum \text{residues}. \tag{8.212}$$

Let us get a simple formula for C_n. We first exchange m for n in Eq. (8.208) and say

$$W(z) = \sum_{m=-\infty}^{m=\infty} C_m (z-z_0)^m. \tag{8.213}$$

Then we operate as follows:

$$\begin{aligned}
\frac{W(z)}{(z-z_0)^{n+1}} &= \sum_{m=-\infty}^{m=\infty} C_m (z-z_0)^{m-n-1}, && (8.214)\\
\oint_C \frac{W(z)}{(z-z_0)^{n+1}}\ dz &= \oint_C \sum_{m=-\infty}^{m=\infty} C_m (z-z_0)^{m-n-1}\ dz, && (8.215)\\
&= \sum_{m=-\infty}^{m=\infty} C_m \oint_C (z-z_0)^{m-n-1}\ dz. && (8.216)
\end{aligned}$$

Here C is any closed contour which has z_0 in its interior. The contour integral on the right side only has a nonzero value when $n=m$. Let us then insist that $n=m$, giving

$$\oint_C \frac{W(z)}{(z-z_0)^{n+1}}\ dz = C_n \underbrace{\oint_C (z-z_0)^{-1}\ dz}_{=2\pi i}. \tag{8.217}$$

We know from earlier analysis that the contour integral enclosing a simple pole such as found on the right side has a value of $2\pi i$. Solving, we find then that

$$C_n = \frac{1}{2\pi i}\oint_C \frac{W(z)}{(z-z_0)^{n+1}}\ dz. \tag{8.218}$$

If the closed contour C encloses no poles, then

$$\oint_C W(z)\ dz = 0. \tag{8.219}$$

We next consider some examples also described by Mei.

Example 8.9

Use complex variable methods to evaluate

$$I = \int_0^\infty \frac{dx}{1+x^2}, \qquad x \in \mathbb{R}^1. \tag{8.220}$$

We note x is real, and so the use of complex variable methods is not yet obvious. We first note the integrand is an even function with symmetry about $x=0$. This allows us to rewrite the formula with symmetric limits as

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}, \qquad x \in \mathbb{R}^1. \tag{8.221}$$

Let us replace the real variable x with a complex variable $z = x + iy$. We recognize however that the path on which I is calculated has z being purely real. So

$$I = \frac{1}{2}\int_{-\infty+0i}^{\infty+0i} \frac{dz}{1+z^2}, \qquad z \in \mathbb{C}^1. \tag{8.222}$$

On the entire path of integration $\Im(z) = y = 0$.

Now consider the integrand

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}. \tag{8.223}$$

By inspection, it has two poles, one at $z=i$ and the other at $z=-i$. We find the Laurent series expansions near both poles. Let us first consider the pole at $z=i$ in some detail. There are many ways to analyze this. Let us define $\hat{z}$ as the deviation of z from the pole: $\hat{z} = z - i$. So we have

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)} = \frac{1}{(\hat{z}+2i)\hat{z}}, \tag{8.224}$$

$$= \frac{1}{2i}\frac{1}{\hat{z}}\frac{1}{1+\frac{\hat{z}}{2i}}, \tag{8.225}$$

$$\sim \frac{1}{2i}\frac{1}{\hat{z}}\left(1 - \frac{\hat{z}}{2i} + \left(\frac{\hat{z}}{2i}\right)^2 - \left(\frac{\hat{z}}{2i}\right)^3 + \dots\right), \tag{8.226}$$

$$\sim -\frac{i}{2}\frac{1}{\hat{z}}\left(1 + \frac{i\hat{z}}{2} - \frac{\hat{z}^2}{4} - \frac{i\hat{z}^3}{8} + \dots\right), \tag{8.227}$$

$$\sim -\frac{i}{2}\frac{1}{\hat{z}} + \frac{1}{4} + \frac{\hat{z}}{8} - \frac{\hat{z}^2}{16} + \dots. \tag{8.228}$$

Returning to z, we then can easily show that the expansion near $z=i$ is

$$\frac{1}{1+z^2} \approx \underbrace{-\frac{i}{2}}_{\text{residue}}(z-i)^{-1} + \frac{1}{4}(z-i)^0 + \frac{i}{8}(z-i)^1 - \frac{1}{16}(z-i)^2 - \dots. \tag{8.229}$$

At $z=-i$, we have by the same procedure

$$\frac{1}{1+z^2} \approx \underbrace{\frac{i}{2}}_{\text{residue}}(z+i)^{-1} + \frac{1}{4}(z+i)^0 - \frac{i}{8}(z+i)^1 - \frac{1}{16}(z+i)^2 - \dots. \tag{8.230}$$

The coefficient on $(z \pm i)^{-1}$ is the residue near each pole, in this case $\pm i/2$.[6]

We now choose a closed contour C in the complex plane depicted in Fig. 8.16. Here we have

[6] We could also use partial fraction expansion to achieve the same end. It is easily verified that $1/(1+z^2) = (i/2)/(z+i) - (i/2)/(z-i)$. So the residue for the pole at $z=i$ is $-i/2$. Similarly, the residue for the pole at $z=-i$ is $i/2$.

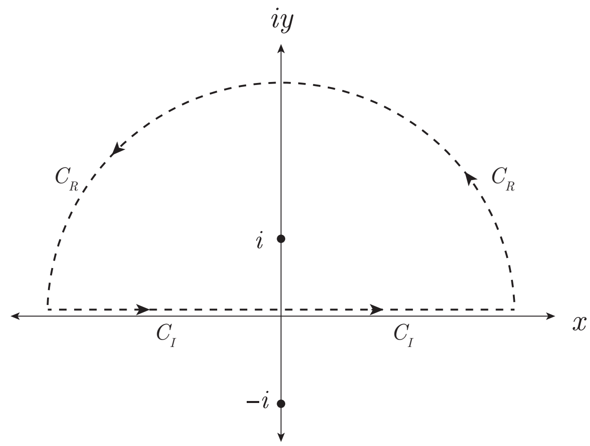


Figure 8.16: Contour integral for $\oint_C dz/(1+z^2)$.

$$C = C_R + C_I, \tag{8.231}$$

where C_R is the semicircular portion of the contour, and C_I is the portion of the contour on the real axis. We take the semicircle to have radius R and will study $R \to \infty$. When $R \to \infty$, C_I becomes the path for the original integral with which we are concerned. This contour C encloses the pole at $z = i$, but does not enclose the second pole at $z = -i$. So when we apply Eq. (8.212), we will only need to consider the residue at $z = i$. Applying Eq. (8.212), we can say

$$\oint_C \frac{dz}{z^2+1} = 2\pi i \underbrace{\left(\frac{-i}{2}\right)}_{\text{residue}} = \pi. \tag{8.232}$$

Now we are really interested in the portion of C on the real axis, namely C_I. So motivated, we rewrite Eq. (8.222) as

$$\begin{aligned} I &= \frac{1}{2}\int_{C_I} \frac{dz}{1+z^2}, && (8.233)\\ &= \frac{1}{2}\left(\underbrace{\oint_C \frac{dz}{1+z^2}}_{\pi} - \int_{C_R}\frac{dz}{1+z^2}\right), && (8.234)\\ &= \frac{1}{2}\left(\pi - \int_{C_R}\frac{dz}{1+z^2}\right). && (8.235)\end{aligned}$$

Now consider the integral applied on C_R. On C_R, we have $z = Re^{i\theta}$ with $R \to \infty$. So $dz = Rie^{i\theta}\, d\theta$. This gives

$$\begin{aligned} \int_{C_R}\frac{dz}{1+z^2} &= \int_{C_R}\frac{1}{1+R^2e^{2i\theta}} Rie^{i\theta}\, d\theta, && (8.236)\\ &= iR\int_0^{\pi}\frac{e^{i\theta}}{1+R^2e^{2i\theta}}\, d\theta. && (8.237)\end{aligned}$$

Now when we let $R \to \infty$, we can neglect the 1 in the denominator of the integrand so as to get

$$\int_{C_R}\frac{dz}{z^2+1} = iR\int_0^{\pi} R^{-2}e^{-i\theta}\, d\theta, \tag{8.238}$$

$$= \frac{i}{R}\int_0^{\pi} e^{-i\theta}\, d\theta, \tag{8.239}$$

$$= \frac{i}{R}\frac{1}{-i}\, e^{-i\theta}\Big|_0^{\pi}, \tag{8.240}$$

$$= -\frac{1}{R}(-1-1), \tag{8.241}$$

$$= \frac{2}{R}. \tag{8.242}$$

Clearly as $R \to \infty$, $\int_{C_R} \to 0$. Thus, Eq. (8.235) reduces to

$$I = \frac{\pi}{2}. \tag{8.243}$$

Note that in this case, we could have directly evaluated the integral. If we take the transformation $x = \tan\theta$, we get $dx = d\theta/\cos^2\theta$. We find

$$\int_0^{\infty} \frac{dx}{1+x^2} = \int_{\theta=0}^{\theta=\pi/2} \frac{1}{1+\tan^2\theta}\frac{d\theta}{\cos^2\theta} = \int_0^{\pi/2} \frac{d\theta}{\cos^2\theta + \sin^2\theta} = \int_0^{\pi/2} d\theta = \frac{\pi}{2}. \tag{8.244}$$

Example 8.10

Use complex variable methods to evaluate

$$I = \int_0^{\infty} \frac{\sqrt{x}}{1+x^2}\, dx, \qquad x \in \mathbb{R}^1. \tag{8.245}$$

This is similar to the previous example, except for the $\sqrt{x}$ in the numerator of the integrand. However, we will take a different approach to the contour integration. We first extend to the complex domain and say

$$I = \int_{0+0i}^{\infty+0i} \frac{\sqrt{z}}{1+z^2}\, dz, \qquad z \in \mathbb{C}^1, \tag{8.246}$$

recognizing that our integration path is confined to the real axis. Now consider this integral in the context of the closed contour depicted in Fig. 8.17. The closed contour C depicted here is

$$C = C_+ + C_R + C_-. \tag{8.247}$$

For our I, we are interested in the integral along C_+. The combination of C_+ and C_- is known as a *branch cut.* We note that the integrand can be rewritten as

$$\frac{\sqrt{z}}{1+z^2} = \frac{\sqrt{z}}{(z-i)(z+i)} = \frac{i\sqrt{z}}{2(z+i)} - \frac{i\sqrt{z}}{2(z-i)}, \tag{8.248}$$

so there are poles at $z = \pm i$, as indicated in Fig. 8.17. We find the Laurent series expansions near both poles. At $z = i$, we find, omitting details of the analysis, which is analogous to finding Taylor series coefficients,

$$\frac{\sqrt{z}}{1+z^2} \approx -\frac{i\sqrt{i}}{2}(z-i)^{-1} + \ \ldots. \tag{8.249}$$

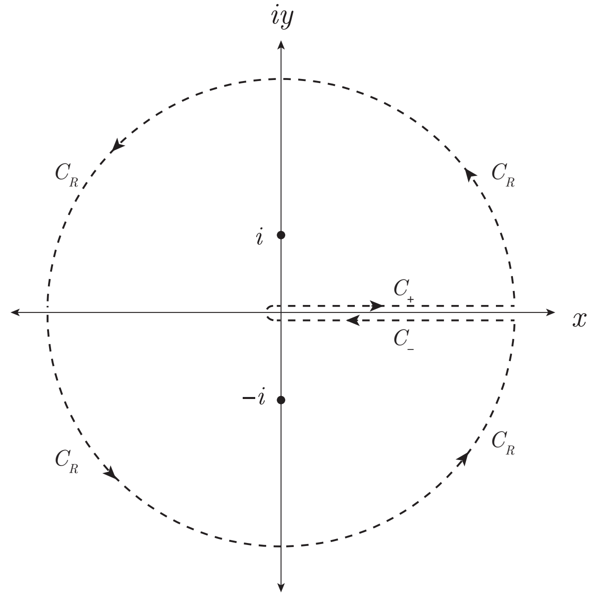


Figure 8.17: Contour integral for $\oint_C \sqrt{z}/(1+z^2)\;dz$.

At $z=-i$, we have

$$\frac{\sqrt{z}}{1+z^2} \approx \frac{i\sqrt{-i}}{2}(z+i)^{-1} + \ldots. \tag{8.250}$$

Now we need the square root to be single-valued. We can achieve this by defining

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}, \qquad \theta \in [0, 2\pi]. \tag{8.251}$$

With this, we see that $\sqrt{i} = (e^{i\pi/2})^{1/2} = e^{i\pi/4} = (1+i)/\sqrt{2}$. And we see also that $\sqrt{-i} = (e^{3\pi i/2})^{1/2} = e^{3i\pi/4} = (-1+i)/\sqrt{2}$. The sum of the residues is then

$$\sum_{residues} = -\frac{i\sqrt{i}}{2} + \frac{i\sqrt{-i}}{2}, \tag{8.252}$$

$$= \frac{\sqrt{i}}{2i} - \frac{\sqrt{-i}}{2i}, \tag{8.253}$$

$$= \frac{1}{2i}\left(\sqrt{i} - \sqrt{-i}\right), \tag{8.254}$$

$$= \frac{1}{2i}\left(\frac{1+i}{\sqrt{2}} - \frac{-1+i}{\sqrt{2}}\right), \tag{8.255}$$

$$= \frac{1}{2i}\left(\frac{2}{\sqrt{2}}\right), \tag{8.256}$$

$$= \frac{1}{\sqrt{2}i}, \tag{8.257}$$

So by Eq. (8.212), the closed contour integral is

$$\oint_C \frac{\sqrt{z}}{1+z^2}\;dz = 2\pi i\left(\frac{1}{\sqrt{2}i}\right) = \sqrt{2}\pi. \tag{8.258}$$

Now consider the portion of the integral in the far field where we are on C_R where $z = Re^{i\theta}$ and $dz = Rie^{i\theta}\, d\theta$. This portion of the integral becomes

$$\int_{C_R} \frac{\sqrt{z}}{1+z^2}\, dz = \int_0^{2\pi} \frac{\sqrt{R}e^{i\theta/2}}{1+R^2e^{2i\theta}} Rie^{i\theta}\, d\theta. \tag{8.259}$$

For large R, we can neglect the 1 in the denominator and we get

$$\int_{C_R} \frac{\sqrt{z}}{1+z^2}\, dz = \frac{i}{\sqrt{R}} \int_0^{2\pi} e^{-i\theta/2}\, d\theta. \tag{8.260}$$

While we could evaluate this integral, we see by inspection that as $R \to \infty$ that it will go to zero; thus, there is no contribution at infinity. Examining the integral then we see

$$\underbrace{\oint_C}_{\sqrt{2}\pi} = \int_{C_+} + \underbrace{\int_{C_R}}_{0} + \int_{C_-}. \tag{8.261}$$

So we have now

$$\sqrt{2}\pi = \int_{C_+} \frac{\sqrt{z}}{1+z^2}\, dz + \int_{C_-} \frac{\sqrt{z}}{1+z^2}\, dz. \tag{8.262}$$

Now on C_+, we have

$$\sqrt{z} = \sqrt{x}(e^{i0})^{1/2} = \sqrt{x}. \tag{8.263}$$

But on C_-, because of the way we defined our branch cut, we have

$$\sqrt{z} = \sqrt{x}(e^{2\pi i})^{1/2} = \sqrt{x}e^{\pi i} = -\sqrt{x}. \tag{8.264}$$

So Eq. (8.262) becomes

$$\sqrt{2}\pi = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx + \int_\infty^0 \frac{-\sqrt{x}}{1+x^2}\, dx, \tag{8.265}$$

$$= \int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx + \int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx, \tag{8.266}$$

$$= 2\int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx, \tag{8.267}$$

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx = \frac{\pi}{\sqrt{2}}. \tag{8.268}$$

8.7 Jordan's lemma

The previous examples have shown us that often the portion of the contour integral in the far field is negligible. Here we state without proof the generalization of this that is Jordan's[7] lemma, quoting liberally from Mei, pp. 246-247:

[7]Camille Jordan, 1838-1922, French engineer.

Jordan's lemma *If $f(z)$ is analytic in $\Im(z) \geq 0$ except at poles, and $|f(z)| \to 0$ on the semicircular arc C_R in the upper half plane as $R \to \infty$, then for $m > 0$*

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{imz}\, dz = 0. \tag{8.269}$$

If $m < 0$ and $f(z)$ is analytic in $\Im(z) \leq 0$ except at poles, and $|f(z)| \to 0$ as $|z| \to \infty$ in the lower half plane then Jordan's lemma holds along a semicircle C_R in the lower half plane.

Similarly, if $f(z)$ is analytic in $\Re(z) \leq 0$ except at poles, and $|f(z)| \to 0$ on the semicircle C_R in the left-half plane as $R \to \infty$, then for $m > 0$

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{mz}\, dz = 0. \tag{8.270}$$

And if $f(z)$ is analytic in $\Re(z) \geq 0$ except at poles, and $|f(z)| \to 0$ on the semicircle C_R in the right-half plane as $R \to \infty$, then for $m < 0$

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{mz}\, dz = 0. \tag{8.271}$$

8.8 Conformal mapping

Conformal mapping is a technique by which we can render results obtained for simple geometries applicable to more complicated geometries. It can apply to any physical scenario described by Laplace's equation.

8.8.1 Analog to steady two-dimensional heat transfer

Let us largely use the notation we developed in Sec. 8.2 that was motivated by incompressible, irrotational fluid mechanics, but think of it in terms of a steady two-dimensional heat transfer problem. Here we shall think of temperature T as a potential ϕ, so $T \to \phi$. Thus, Eq. (1.103) for the steady temperature field is here

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{8.272}$$

This equation employs Fourier's law, Eq. (1.99), which is recast as

$$u = -k\frac{\partial \phi}{\partial x}, \tag{8.273}$$

$$v = -k\frac{\partial \phi}{\partial y}. \tag{8.274}$$

Here we take $q_x \to u$ and $q_y \to v$. Or with $\mathbf{u} = (u, v)^T$, we could describe Fourier's law as $\mathbf{u} = -k\nabla\phi$. Loosely speaking u is the diffusion velocity of energy in the x direction and v is the diffusion velocity of energy in the y direction. In the steady-state limit, our two-dimensional energy conservation equation Eq. (1.98) is recast as

$$\nabla^T \cdot \mathbf{u} = 0, \tag{8.275}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{8.276}$$

Differentiating Eq. (8.273) with respect to y and Eq. (8.274) with respect to x, we find

$$\frac{\partial u}{\partial y} = -k\frac{\partial^2 \phi}{\partial y \partial x}, \tag{8.277}$$

$$\frac{\partial v}{\partial x} = -k\frac{\partial^2 \phi}{\partial x \partial y}. \tag{8.278}$$

Assuming sufficient continuity and differentiability of ϕ, we subtract Eq. (8.277) from Eq. (8.278) to get

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{8.279}$$

This is the analog of the irrationality condition, Eq. (8.4).

8.8.2 Mapping of one geometry to another

We look at some problems discussed also by Churchill, Brown, and Verhey. Let us imagine the notion of a complex function as a mapping from one plane to another. We shall see that this leads us to some powerful and surprising results with relevance to Laplace's equation. In short, we will see that we can solve Laplace's equation in a simple domain and use a mapping to infer a solution in a more complicated domain.

8.8.2.1 Solution in a half-plane

Let us explore this with an example. Consider the complex function

$$w(z) = \ln \frac{z-1}{z+1}. \tag{8.280}$$

This is similar to the doublet studied in Sec. 8.4.6 except rather than letting $\epsilon \to 0$, we take $\epsilon = -1$. Now $z = x + iy$, and we restrict such that $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$. Moreover w has a real and imaginary part, yielding the form

$$\xi(x,y) + i\eta(x,y) = \Re(w) + i\Im(w) = \ln \frac{z-1}{z+1}. \tag{8.281}$$

In short, we have used a complex function to define a special type of coordinate transformation in which

$$(x, y) \to (\xi, \eta), \tag{8.282}$$

via

$$\xi = \xi(x, y), \tag{8.283}$$
$$\eta = \eta(x, y), \tag{8.284}$$

where the transformation is restricted by the properties of the chosen function $w(z) = w(x + iy)$. We have also $\xi \in \mathbb{R}^1$ and $\eta \in \mathbb{R}^1$. Now we note that

$$\ln z = \ln\left(re^{i\theta}\right), \tag{8.285}$$
$$= \ln r + \ln\left(e^{i\theta}\right), \tag{8.286}$$
$$= \ln r + i\theta, \tag{8.287}$$
$$= \ln |z| + i \arg z. \tag{8.288}$$

We extend this result to express w as

$$w(z) = \ln \frac{z-1}{z+1} = \underbrace{\ln\left|\frac{z-1}{z+1}\right|}_{\xi} + i \underbrace{\arg \frac{z-1}{z+1}}_{\eta}. \tag{8.289}$$

We see then that our coordinate $\eta(x, y)$ is given by

$$\eta = \arg \frac{z-1}{z+1}, \tag{8.290}$$
$$\eta(x, y) = \arg \frac{x - 1 + iy}{x + 1 + iy}, \tag{8.291}$$
$$= \arg \frac{(x + 1 - iy)(x - 1 + iy)}{(x + 1 - iy)(x + 1 + iy)}, \tag{8.292}$$
$$= \arg \frac{x^2 + y^2 - 1 + i2y}{(x + 1)^2 + y^2}, \tag{8.293}$$
$$= \mathrm{Tan}^{-1}\left(\frac{x^2 + y^2 - 1}{(x + 1)^2 + y^2}, \frac{2y}{(x + 1)^2 + y^2}\right), \tag{8.294}$$
$$= \mathrm{Tan}^{-1}\left(x^2 + y^2 - 1, 2y\right). \tag{8.295}$$

Our other coordinate $\xi(x, y)$ is given by

$$\xi = \ln\left|\frac{z-1}{z+1}\right|, \tag{8.296}$$
$$\xi(x, y) = \ln\left|\frac{x - 1 + iy}{x + 1 + iy}\right|, \tag{8.297}$$
$$= \ln\left|\frac{(x + 1 - iy)(x - 1 + iy)}{(x + 1 - iy)(x + 1 + iy)}\right|, \tag{8.298}$$
$$= \ln\left|\frac{x^2 + y^2 - 1 + i2y}{(x + 1)^2 + y^2}\right|, \tag{8.299}$$
$$= \ln \frac{\sqrt{(x^2 + y^2 - 1)^2 + 4y^2}}{(x + 1)^2 + y^2}. \tag{8.300}$$

A plot of contours of constant ξ and η in the $x - y$ plane is given in Fig. 8.18a. Also

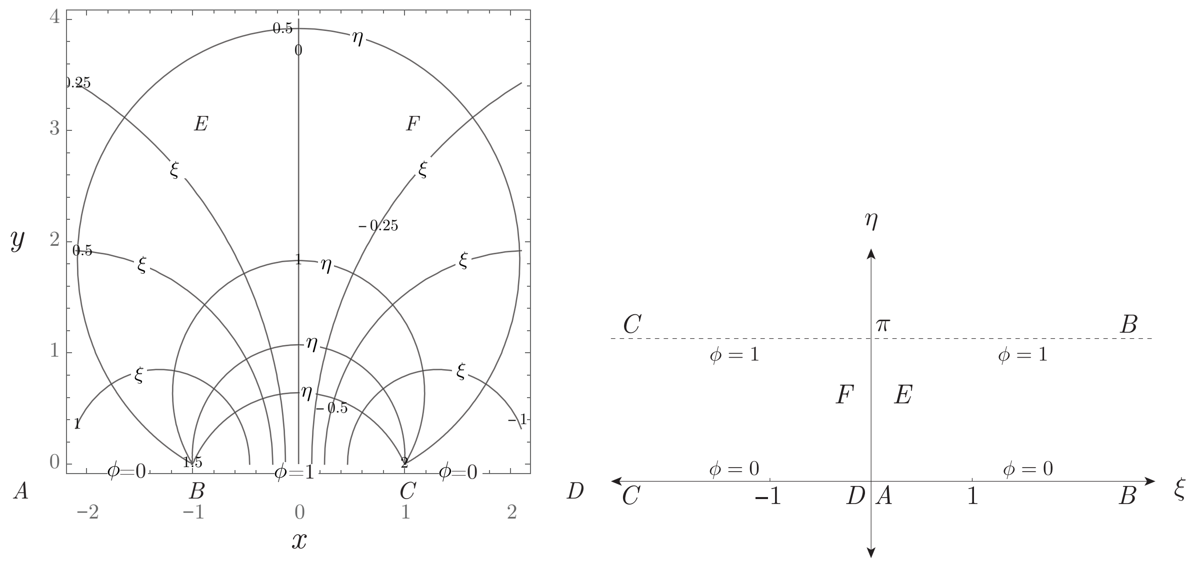


Figure 8.18: a) Contours of $\xi(x, y)$ and $\eta(x, y)$ induced by $w(z) = \ln((z-1)/(z+1))$, b) The $\xi - \eta$ plane for this mapping.

shown are four points on the x axis, A for which $x \to -\infty$, B at $x = -1$, C at $x = 1$, and D at $x \to \infty$; additionally, two points are shown in the region for $y > 0$. We have E at $(x, y) = (-1, 3)$ and F at $(x, y) = (1, 3)$.

Let us examine a special contour, namely

$$\lim_{y \to 0^+} \eta(x, y). \tag{8.301}$$

When $y = 0$, there are problems at $x = \pm 1$. Setting aside a detailed formal analysis, which is possible, we learn much by simply plotting $\eta(x, y = 0.0001)$, given in Fig. 8.19. We see that for $x \in [-1, 1]$ that η maps to π with a small error due to the finite value of y. For $x < -1$ or $x > 1$, η maps to 0. A plot of contours of $x \in (-\infty, \infty)$ for $y \approx 0$ in the $\xi - \eta$ plane is given in Fig. 8.18b. Also shown are the mappings of the points A, B, C, D, E, and F. Importantly E and F, which have $y > 0$, lie within $\eta \in [0, \pi]$.

Up to now we have simply discussed coordinate transformations. But let us now imagine that in the $x - y$ plane we have the temperature, which is the potential ϕ, defined on the boundary $y = 0$, and we are considering the domain $y > 0$. Let us further say that we have

$$\phi(x, y = 0) = \begin{cases} 0 & x < -1, \\ 1 & x \in [-1, 1], \\ 0 & x > 1. \end{cases} \tag{8.302}$$

This has been indicated on Fig. 8.18. We need to solve

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{8.303}$$

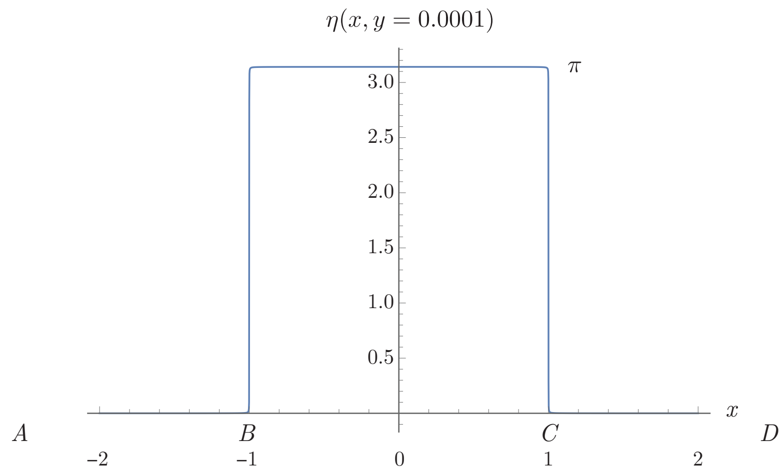


Figure 8.19: Plot of $\eta(x, y = 0.0001)$ induced by $w(z) = \ln((z-1)/(z+1))$.

with these boundary conditions. This is difficult because there is a singularity in the boundary conditions at $x = \pm 1$. Remarkably, we can achieve a solution by employing our mapping and solving

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0, \tag{8.304}$$

with the transformed boundary conditions,

$$\phi(\xi, 0) = 0, \qquad \phi(\xi, \pi) = 1. \tag{8.305}$$

We note from Fig. 8.18 that in the transformed space, the boundary conditions are particularly simple. In fact, by inspection, we note that

$$\phi(\xi, \eta) = \frac{1}{\pi}\eta, \tag{8.306}$$

satisfies Laplace's equation in the transformed space, Eq. (8.304), as well as the transformed boundary conditions at $\eta = 0$ and $\eta = \pi$. We then transform back to $x-y$ space and present our solution for the temperature field as

$$\phi(x, y) = \frac{1}{\pi}\mathrm{Tan}^{-1}\left(x^2 + y^2 - 1, 2y\right). \tag{8.307}$$

In terms of the ordinary arctan function, we could say

$$\phi(x, y) = \frac{2}{\pi}\tan^{-1}\frac{2y}{x^2 + y^2 - 1 + \sqrt{\left(x^2 + y^2 - 1\right)^2 + 4y^2}}. \tag{8.308}$$

Direct calculation in $x - y$ space for either form reveals that both Laplace's equation is satisfied as well as the boundary conditions at $y = 0$. The temperature field $\phi(x, y)$ is plotted in Fig. 8.20.

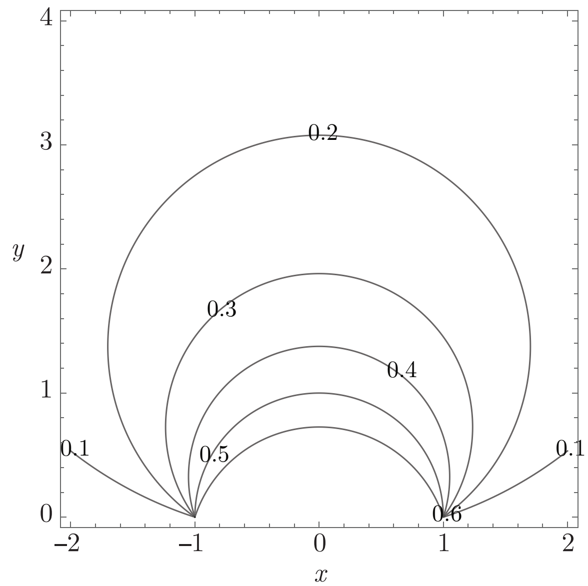


Figure 8.20: The temperature field $\phi(x, y)$ satisfying $\nabla^2\phi = 0$ with $\phi(x, 0) = 0$ for $x < -1$ and $x > 1$ and with $\phi(x, 0) = 1$ for $x \in [-1, 1]$.

8.8.2.2 Solution in a semi-infinite strip

Consider now the complex function

$$w(z) = \sin z. \tag{8.309}$$

We need to understand trigonometric functions of complex variables. We can do so by simply extending their real analogs. So, drawing upon Eq. (8.60), we say

$$\begin{aligned}
w(z) = \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, && (8.310)\\
&= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}, && (8.311)\\
&= \frac{e^{ix-y} - e^{-ix+y}}{2i}, && (8.312)\\
&= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}, && (8.313)\\
&= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right), && (8.314)\\
&= \underbrace{\sin x\cosh y}_{x_*} + i\underbrace{\cos x\sinh y}_{y_*}. && (8.315)
\end{aligned}$$

Similar to as before, the complex function has defined a coordinate transformation

$$\begin{aligned}
x_*(x, y) &= \sin x\cosh y, && (8.316)\\
y_*(x, y) &= \cos x\sinh y. && (8.317)
\end{aligned}$$

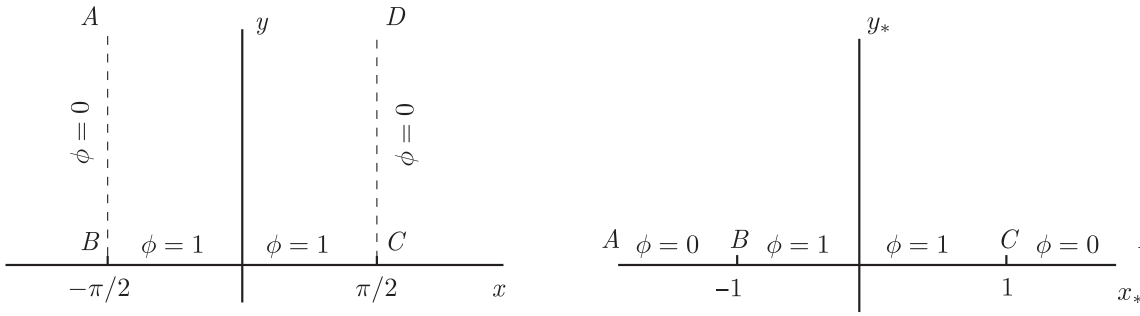


Figure 8.21: Transformation induced by $w(z) = \sin z$.

We sketch this transformation in Fig. 8.21. We note that the semi-infinite strip lying between $x = \pm\pi/2$ has been mapped to the x_* axis. And if we take our boundary conditions to be as sketched with $\phi(\pm\pi/2, y) = 0$ and $\phi(x, 0) = 1$ for $x \in [-\pi/2, \pi/2]$, we see that in the (x_*, y_*) system we have precisely the same problem we solved in the previous section. This suggests that we can simply adopt the solution of the previous section, in a transformed coordinate system. Take then Eq. (8.307) as the solution in the (x_*, y_*) system:

$$\phi(x_*, y_*) = \frac{1}{\pi}\mathrm{Tan}^{-1}\left(x_*^2 + y_*^2 - 1, 2y_*\right). \tag{8.318}$$

Then we use Eqs. (8.316,8.317) to get $\phi(x, y)$:

$$\phi(x, y) = \frac{1}{\pi}\mathrm{Tan}^{-1}\left((\sin x \cosh y)^2 + (\cos x \sinh y)^2 - 1, 2\cos x \sinh y\right). \tag{8.319}$$

Detailed use of trigonometric identities reveals that Eq. (8.319) can be rewritten as

$$\phi(x, y) = \frac{2}{\pi}\arctan\left(\frac{\cos x}{\sinh y}\right). \tag{8.320}$$

Direct calculation reveals that $\nabla^2\phi = 0$, and that $\phi(\pm\pi/2, y) = 0$ and $\phi(x, 0) = 1$. The temperature field $\phi(x, y)$ is plotted in Fig. 8.22.

8.8.2.3 Solution in a quarter-plane

Consider now the transformation

$$w(z) = \sin^{-1} z. \tag{8.321}$$

This can be rewritten as

$$z = \sin w, \tag{8.322}$$

which we take here to directly induce the transformation

$$\begin{aligned} x(\xi, \eta) &= \sin\xi \cosh\eta, & (8.323)\\ y(\xi, \eta) &= \cos\xi \sinh\eta. & (8.324) \end{aligned}$$

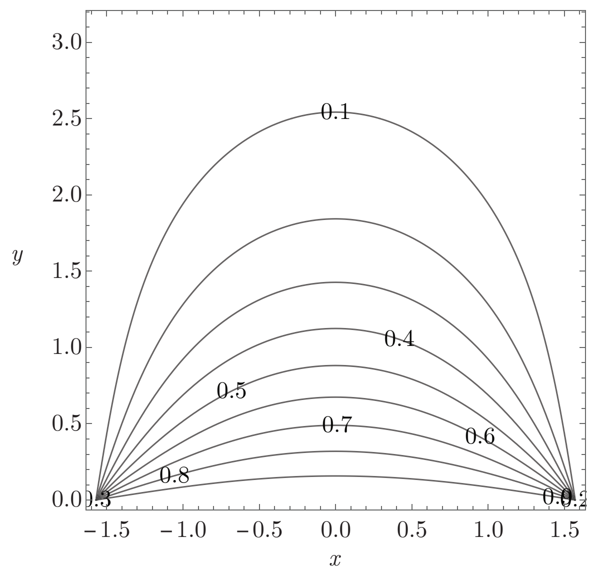


Figure 8.22: For $x \in [-\pi/2, \pi/2]$, $y \in [0, \infty]$, the temperature field $\phi(x, y)$ satisfying $\nabla^2\phi = 0$ with $\phi(\pm\pi/2, y) = 0$, $\phi(x, 0) = 1$.

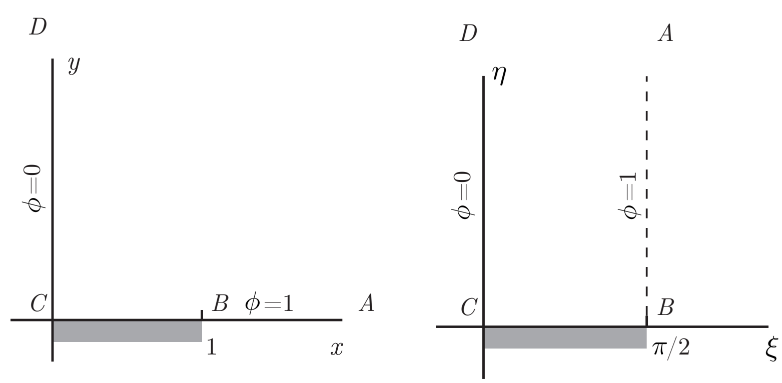


Figure 8.23: Transformation induced by $w(z) = \sin^{-1} z$ for the quarter-plane.

Here, we consider the quarter-plane in the $x-y$ system and its image under transformation to the $\xi-\eta$ system as sketched in Fig. 8.23. In the $x-y$ quarter-plane, we consider $\phi(0, y) = 0$. For $y = 0$ and $x \in [0, 1]$, we take the Neumann condition $\partial\phi/\partial y = 0$. For $y = 0$, $x > 1$, we take the Dirichlet condition $\phi(x > 1, 0) = 1$. Under the mapping, the quarter-plane transforms to a semi-infinite strip confined to $\xi \in [0, \pi/2]$ and $\eta > 0$.

The solution

$$\phi(\xi, \eta) = \frac{2}{\pi}\xi, \tag{8.325}$$

satisfies $\partial^2\phi/\partial\xi^2 + \partial^2\phi/\partial\eta^2 = 0$ and all boundary conditions. To return to the $x - y$ plane we must find the inverse transformation. Squaring both Eqs. (8.323) and (8.324) and scaling gives

$$\frac{x^2}{\sin^2\xi} = \cosh^2\eta, \tag{8.326}$$

$$\frac{y^2}{\cos^2\xi} = \sinh^2\eta. \tag{8.327}$$

Now because $\cosh^2 u - \sinh^2 u = 1$, we have

$$\frac{x^2}{\sin^2\xi} - \frac{y^2}{\cos^2\xi} = 1. \tag{8.328}$$

We can rewrite this as

$$\frac{x^2}{\sin^2\xi} - \frac{y^2}{1 - \sin^2\xi} = 1, \tag{8.329}$$

and solve for first $\sin\xi$ and then ξ to get

$$\xi = \arcsin\frac{1}{2}\left(\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}\right). \tag{8.330}$$

Then we see that

$$\phi(x, y) = \frac{2}{\pi}\arcsin\frac{1}{2}\left(\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}\right). \tag{8.331}$$

Direct substitution reveals that $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$ and that the boundary conditions are satisfied. The temperature field $\phi(x, y)$ is plotted in Fig. 8.24.

Problems

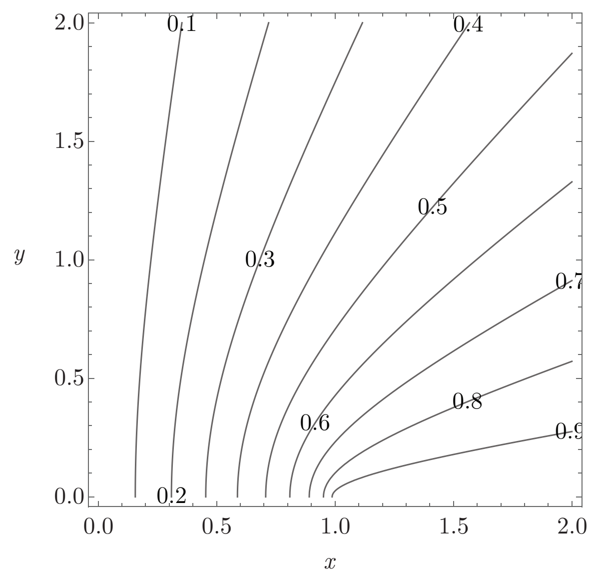


Figure 8.24: For $x \in [0, \infty)$, $y \in [0, \infty)$, the temperature field $\phi(x, y)$ satisfying $\nabla^2\phi = 0$ with $\partial\phi/\partial y = 0$ for $y = 0$, $x \in [0, 1]$, $\phi(x, 0) = 1$ for $x > 1$, and $\phi(0, y) = 0$.

Chapter 9

Integral transformation methods

see Mei, Chapters 7, 10.

Here we consider integral transformation methods.

9.1 Fourier transformations

We have familiarity with the Fourier series representation of functions, often formed using a set of orthogonal basis functions with discrete values of wave number. The Fourier transformation may be considered a limit in which the wave number varies continuously. To see how to arrive at this limit, let us begin with a more general consideration of a Fourier series representation of a function. Let us seek to expand $f(x)$ in a Fourier series expansion in terms of orthogonal basis functions $u_n(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n u_n(x). \tag{9.1}$$

Taking the basis functions to be $u_n(x) = e^{in\pi x/L}$, we express our expansion as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{9.2}$$

We recognize that via Euler's formula, Eq. (8.39), $e^{in\pi x/L} = \cos(n\pi x/L) + i\sin(n\pi x/L)$, that this can be thought of as an expansion in trigonometric functions. Now for convenience, we have chosen basis functions, $e^{in\pi x/L}$ for $n = 0, \pm 1, \pm 2, \ldots$, that are orthogonal. We could also have made the less restrictive assumption that the $u_n(x)$ were at most linearly independent, at the expense of added complication. We also could have scaled our orthogonal basis functions to render them orthonormal, though this useful practice is not commonly done. We recall that for complex functions, taking the inner product on the domain $x \in [-L, L]$

requires a complex conjugation, so

$$\langle u_m(x), u_n(x)\rangle = \int_{-L}^{L} \overline{u}_m(x) u_n(x)\; dx. \tag{9.3}$$

So for us

$$\begin{aligned}
\langle e^{im\pi x/L}, e^{in\pi x/L}\rangle &= \int_{-L}^{L} \overline{e^{im\pi x/L}} e^{in\pi x/L}\; dx, &(9.4)\\
&= \int_{-L}^{L} e^{i(n-m)\pi x/L}\; dx, &(9.5)\\
&= \left.\frac{L}{i(n-m)\pi} e^{i(n-m)\pi x/L}\right|_{-L}^{L}, \quad n \neq m, &(9.6)\\
&= \frac{2L}{(n-m)}\left(\frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{2i}\right), \quad n \neq m, &(9.7)\\
&= \frac{2L}{(n-m)} \sin(n-m)\pi, \quad n \neq m, &(9.8)\\
&= 0, \quad n \neq m. &(9.9)
\end{aligned}$$

If $n = m$, Eq. (9.5) reduces to

$$\begin{aligned}
\langle e^{im\pi x/L}, e^{in\pi x/L}\rangle &= \int_{-L}^{L} dx, \qquad n = m, &(9.10)\\
&= x|_{-L}^{L}, \qquad n = m, &(9.11)\\
&= 2L, \qquad n = m. &(9.12)
\end{aligned}$$

In summary,

$$\langle e^{im\pi x/L}, e^{in\pi x/L}\rangle = 2L\delta_{mn} = \begin{cases} 0, & m \neq n, \\ 2L, & m = n. \end{cases} \tag{9.13}$$

We next find the Fourier coefficients c_n. Operating on Eq. (9.2), we find

$$\begin{aligned}
\langle e^{im\pi x/L}, f(x)\rangle &= \langle e^{im\pi x/L}, \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}\rangle, &(9.14)\\
\int_{-L}^{L} e^{-im\pi x/L} f(x)\; dx &= \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-L}^{L} e^{i(n-m)\pi x/L}\; dx}_{=2L\delta_{mn}}, &(9.15)\\
&= 2Lc_m &(9.16)
\end{aligned}$$

Exchanging m for n and x for ξ, we can say

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(\xi) e^{-in\pi\xi/L}\; d\xi, \qquad n = 0, \pm 1 \pm 2, \ldots, \tag{9.17}$$

gives the expression for the Fourier coefficients c_n. Using this in Eq. (9.2), we can say

$$f(x) = \sum_{n=-\infty}^{\infty} \underbrace{\left(\frac{1}{2L}\int_{-L}^{L} f(\xi)e^{-in\pi\xi/L}\,d\xi\right)}_{c_n} e^{in\pi x/L}, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{9.18}$$

$$= \frac{1}{2L}\sum_{n=-\infty}^{\infty}\int_{-L}^{L} f(\xi)e^{in\pi(x-\xi)/L}\,d\xi, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{9.19}$$

Now let us allow $L \to \infty$. Following Mei's analysis on his pp. 132-133, we define α_n such that

$$\alpha_n = \frac{n\pi}{L}. \tag{9.20}$$

So we might have $\alpha_5 = 5\pi/L$ and $\alpha_4 = 4\pi/L$; thus, $\alpha_5 - \alpha_4 = \pi/L$. Generalizing, we can say

$$\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}. \tag{9.21}$$

For convenience, we now define

$$\widehat{f}(\alpha_n, x) = \frac{1}{2\pi}\int_{-L}^{L} f(\xi)e^{i\alpha_n(x-\xi)}\,d\xi. \tag{9.22}$$

Using the definition of Eq. (9.22) in Eq. (9.19), we get

$$f(x) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} 2\pi\widehat{f}(\alpha_n, x), \tag{9.23}$$

$$= \sum_{n=-\infty}^{\infty} \widehat{f}(\alpha_n, x)\Delta\alpha, \tag{9.24}$$

This appears much as the rectangular rule for discrete approximation of integrals of continuous functions. Now as we let $L \to \infty$, we see that $\Delta\alpha \to 0$, and Eq. (9.24) passes to the limit of a Riemann integral:

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\alpha, x)\,d\alpha. \tag{9.25}$$

We also see that as $L \to \infty$ that Eq. (9.22) becomes

$$\widehat{f}(\alpha, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{i\alpha(x-\xi)}\,d\xi, \tag{9.26}$$

provided $\int_{-\infty}^{\infty} |f(x)|\ dx < \infty$. We combine Eqs. (9.25, 9.26) to form

$$f(x) = \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{i\alpha(x-\xi)}\ d\xi\right)}_{\widehat{f}(\alpha,x)}\ d\alpha, \tag{9.27}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{i\alpha(x-\xi)}\ d\xi\ d\alpha. \tag{9.28}$$

Let us define the *Fourier transformation* $\mathcal{F}$ of the function $f(\xi)$ as follows:

$$\mathcal{F}(f(\xi)) = F(\alpha) = \int_{-\infty}^{\infty} f(\xi)e^{-i\alpha\xi}\ d\xi, \qquad \text{Fourier transformation.} \tag{9.29}$$

The Fourier transformation is somewhat analogous to the discrete Eq. (9.17), though they differ by a leading constant, $1/(2L)$, which has no clear analog in the limit $L \to \infty$. So $\mathcal{F}$ is somewhat analogous to the c_n from a discrete Fourier series. Next operate on Eq. (9.28) to get the *inverse Fourier transformation*:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(\xi)e^{-i\alpha\xi}\ d\xi}_{F(\alpha)}\, e^{i\alpha x}\ d\alpha, \tag{9.30}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}\ d\alpha, \qquad \text{inverse Fourier transformation.} \tag{9.31}$$

Note that Eq. (9.31) corrects an error in Mei's Eq. (7.1.8) on his p. 133. We take $f(x)$ to describe our function in the spatial domain and its image $F(\alpha)$ to represent our function in the so-called *spectral domain*. We also note that many texts will also exchange ξ for x and rewrite Eq. (9.29) as

$$\mathcal{F}(f(x)) = F(\alpha) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\ dx, \tag{9.32}$$

We note the Fourier transformation is a linear operator, so

$$\mathcal{F}(af(x) + bg(x)) = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x)). \tag{9.33}$$

Example 9.1

Find the Fourier transformation of $f(x) = \delta(x - x_0)$, with $x_0 \in \mathbb{R}^1$.

Applying Eq. (9.32), we get for the Dirac delta function

$$F(\alpha) = \int_{-\infty}^{\infty} \delta(x - x_0)e^{-i\alpha x}\ dx, \tag{9.34}$$

$$= e^{-i\alpha x_0}, \tag{9.35}$$

$$= \cos\alpha x_0 - i\sin\alpha x_0. \tag{9.36}$$

Note

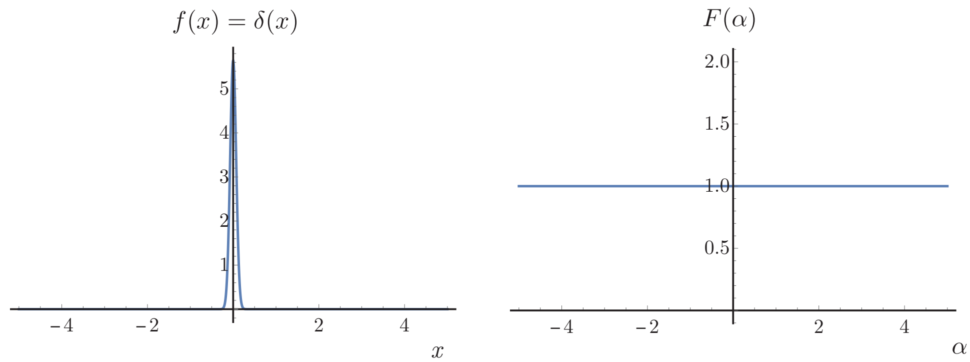


Figure 9.1: The Dirac delta function $\delta(x)$ and its Fourier transformation.

- If $f(x)$ is symmetric about $x = 0$, implying here that $x_0 = 0$, $F(\alpha)$ is purely real, and specifically here is

$$F(\alpha) = 1. \tag{9.37}$$

- Loss of symmetry of $f(x)$ induces an imaginary component of $F(\alpha)$.
- *The Dirac delta function is highly nonuniform, i.e. localized, in the x-domain; however, its image in the spectral domain is uniform throughout.* This reflects the notion that the Dirac delta function contains information at all frequencies.

For $x_0 = 0$, the function and its Fourier transformation are plotted in Fig. 9.1.

Example 9.2

Find the Fourier transformation of $f(x) = \delta(x + x_0) + \delta(x - x_0)$, with $x_0 \in \mathbb{R}^1$.

Applying Eq. (9.32), we get

$$\begin{aligned} F(\alpha) &= \int_{-\infty}^{\infty} (\delta(x + x_0) + \delta(x - x_0)) e^{-i\alpha x} \, dx, && (9.38) \\ &= e^{i\alpha x_0} + e^{-i\alpha x_0}, && (9.39) \\ &= 2 \cos \alpha x_0. && (9.40) \end{aligned}$$

Note

- Here $f(x)$ is symmetric about $x = 0$, and $F(\alpha)$ is purely real,
- Again, the function is nonuniform in the spatial domain and more uniform in the spectral domain.

For $x_0 = 1$, the function and its Fourier transformation are plotted in Fig. 9.2.

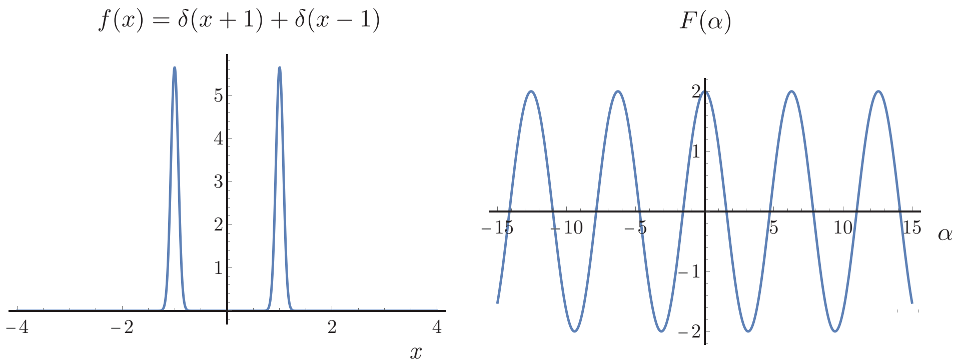


Figure 9.2: The function $f(x) = \delta(x-1) + \delta(x+1)$ and its Fourier transformation.

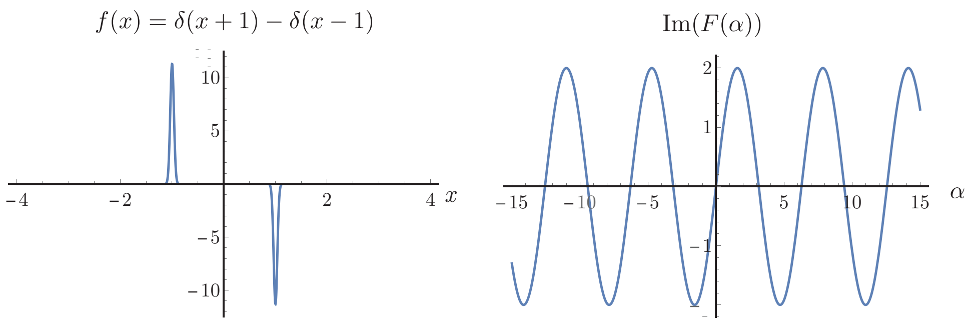


Figure 9.3: The function $f(x) = \delta(x+1) - \delta(x-1)$ and the imaginary component of its Fourier transformation.

Example 9.3

Find the Fourier transformation of $f(x) = \delta(x + x_0) - \delta(x - x_0)$, with $x_0 \in \mathbb{R}^1$.

Applying Eq. (9.32), we get

$$F(\alpha) = \int_{-\infty}^{\infty} (\delta(x + x_0) - \delta(x - x_0))e^{-i\alpha x}\, dx, \tag{9.41}$$

$$= e^{i\alpha x_0} - e^{-i\alpha x_0}, \tag{9.42}$$

$$= 2i \sin \alpha x_0. \tag{9.43}$$

Note

- Here $f(x)$ is anti-symmetric about $x = 0$, and $F(\alpha)$ is purely imaginary,
- Again the function is nonuniform in the spatial domain and more uniform in the spectral domain.

For $x_0 = 1$, the function and the imaginary component of its Fourier transformation are plotted in Fig. 9.3.

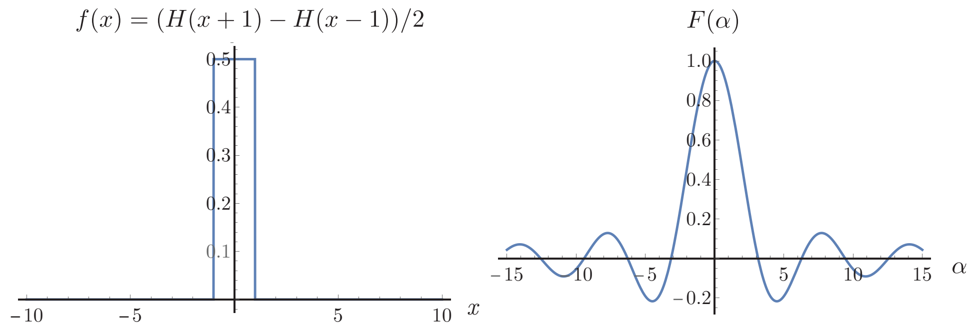


Figure 9.4: A top hat function and its Fourier transformation.

Example 9.4

Find the Fourier transformation of the top hat function

$$f(x) = \frac{1}{2a}(H(x+a) - H(x-a)). \tag{9.44}$$

Here $f(x)$ is symmetric about $x = 0$, so we expect a real-valued Fourier transformation. Note that the width of the top hat is $2a$ and the height is $1/(2a)$, so the area under the top hat is unity. Thus as $a \to 0$, our top hat approaches a Dirac delta function. Applying Eq. (9.32), we get for our top hat function

$$\begin{aligned}
F(\alpha) &= \frac{1}{2a}\int_{-\infty}^{\infty}(H(x+a) - H(x-a))e^{-i\alpha x}\, dx, && (9.45)\\
&= \frac{1}{2a}\int_{-a}^{a} e^{-i\alpha x}\, dx, && (9.46)\\
&= \frac{1}{2a}\left(\frac{1}{-i\alpha}\, e^{-i\alpha x}\Big|_{-a}^{a}\right), && (9.47)\\
&= -\frac{1}{2ia\alpha}\left(e^{-ia\alpha} - e^{ia\alpha}\right), && (9.48)\\
&= \frac{1}{a\alpha}\left(\frac{e^{ia\alpha} - e^{-ia\alpha}}{2i}\right), && (9.49)\\
&= \frac{\sin a\alpha}{a\alpha}. && (9.50)
\end{aligned}$$

The function and its Fourier transformation are plotted in Fig. 9.4 for $a = 1$. Note that in the transformed space, F is symmetric and non-singular at $\alpha = 0$. Taylor series of $F(\alpha)$ about $\alpha = 0$ verifies this as $F(\alpha) \sim 1 - a^2\alpha^2/6 + a^4\alpha^4/120 - \ldots$. If we were to broaden the top hat by increasing a, we would narrow its Fourier transformation, $F(\alpha)$. Conversely, if we were to narrow the top hat by decreasing a, we would broaden its Fourier transformation.

Example 9.5

Find the Fourier transformation of the Gaussian function $f(x) = e^{-x^2/2}$.

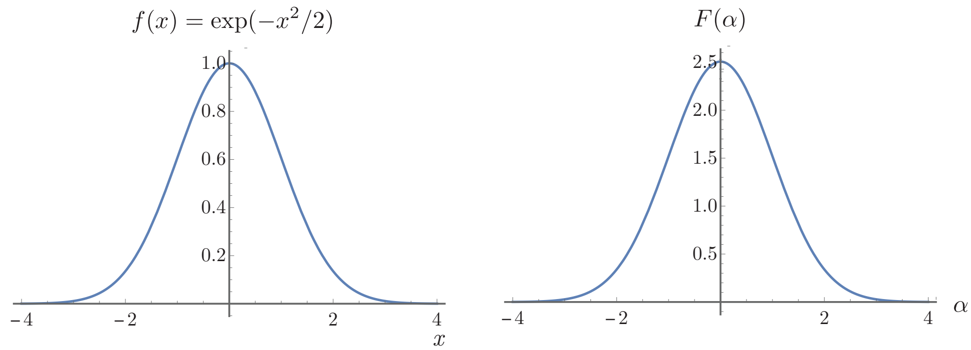


Figure 9.5: A Gaussian function and its Fourier transformation.

Applying Eq. (9.32), we get for our Gaussian function

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-i\alpha x}\, dx, \tag{9.51}$$

$$= \sqrt{2\pi} e^{-\alpha^2/2}. \tag{9.52}$$

The function and its Fourier transformation are plotted in Fig. 9.5. Remarkably, it maps into a function of the same form as its generator. So this Gaussian has identical spatial and spectral localization.

Example 9.6

Find the Fourier transformation of a cosine whose amplitude is modulated by a Gaussian function

$$f(x) = e^{-x^2/2} \cos ax. \tag{9.53}$$

Applying Eq. (9.32), we get

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-x^2/2} (\cos ax) e^{-i\alpha x}\, dx, \tag{9.54}$$

$$= \sqrt{\frac{\pi}{2}} (1 + e^{2a\alpha}) e^{-(a+\alpha)^2/2}. \tag{9.55}$$

For $a = 10$, the function and its Fourier transformation are plotted in Fig. 9.6. The function $f(x)$ appears as a pulse which oscillates. The pulse width is dictated by the exponential function. If we were to weaken the amplitude modulation, say by taking $f(x) = \exp(-x^2/20)\cos(10x)$, we would find the pulse width increased in the x domain and the spike width would decrease in the frequency domain. This suggests the function is not spatially localized but is spectrally localized. We see the single mode with wave number 10 is reflected in spectral space at $\alpha = \pm 10$. We might be surprised to see the mirror image at -10. This feature is part of all Fourier analysis, and is known as an *aliasing* effect. Only one of the peaks gives us the clue to what the wave number of the generating function was.

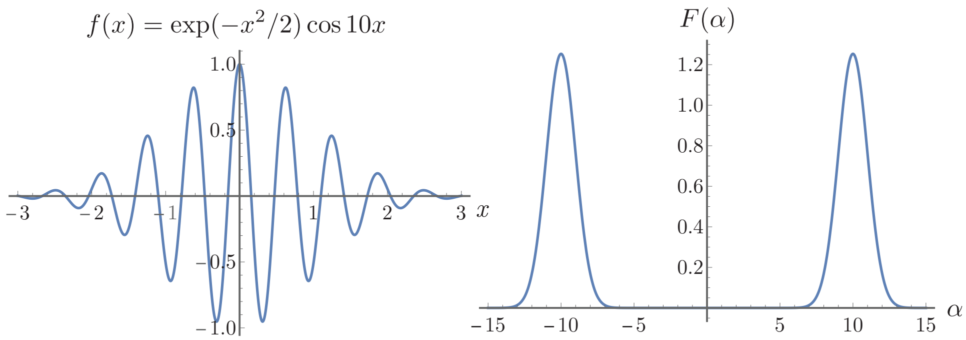


Figure 9.6: A Gaussian modulated cosine function and its Fourier transformation.

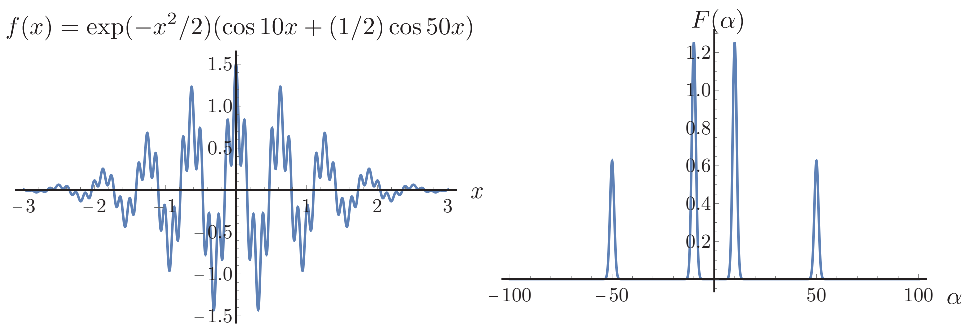


Figure 9.7: A Gaussian modulated sum of cosines and its Fourier transformation.

Let us add another frequency mode and consider the Fourier transformation of

$$f(x) = e^{-x^2/2}\left(\cos 10x + \frac{1}{2}\cos 50x\right). \tag{9.56}$$

The Fourier transformation can be shown to be

$$\sqrt{\frac{\pi}{2}} e^{-\frac{\alpha^2}{2} - 1250}\left(2e^{1200}\cosh 10\alpha + \cosh 50\alpha\right). \tag{9.57}$$

The function and its Fourier transformation are plotted in Fig. 9.7. We clearly see the two peaks at $\alpha = 10$ and $\alpha = 50$, as well as their aliases.

Example 9.7

Find the Fourier transformation of a sine whose amplitude is modulated by a Gaussian function

$$f(x) = e^{-x^2/2}\sin ax. \tag{9.58}$$

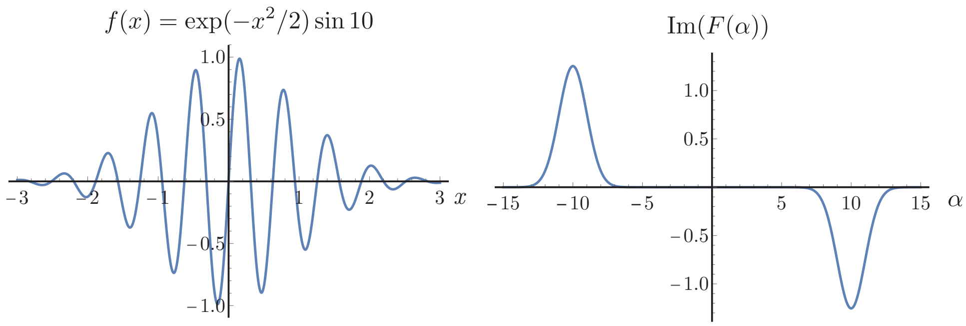


Figure 9.8: A Gaussian modulated sine function and its Fourier transformation.

Applying Eq. (9.32), we get

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-x^2/2}(\sin ax)e^{-i\alpha x}\ dx, \tag{9.59}$$

$$= -i\sqrt{\frac{\pi}{2}}(-1+e^{2a\alpha})e^{-(a+\alpha)^2/2}. \tag{9.60}$$

For $a = 10$, the function and its Fourier transformation are plotted in Fig. 9.8. In contrast to the even cosine function of of the previous example which induced a purely real $F(\alpha)$, the odd sine function induces a purely imaginary $F(\alpha)$. Other features are similar to the previous example involving cosine.

Example 9.8

If $u = u(x,t)$ and $u \to 0$ as $x \to \pm\infty$, find the Fourier transformation of $\partial u/\partial x$.

Let us start by taking

$$\mathcal{F}(u(x,t)) = U(\alpha,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\alpha x}\ dx. \tag{9.61}$$

We then have

$$\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}e^{-i\alpha x}\ dx, \tag{9.62}$$

$$= e^{-i\alpha x}u\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-i\alpha)ue^{-i\alpha x}\ dx. \tag{9.63}$$

Because we have insisted that u vanish as $x \to \pm\infty$, this reduces to

$$\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = i\alpha\int_{-\infty}^{\infty} ue^{-i\alpha x}\ dx, \tag{9.64}$$

$$= i\alpha U(\alpha,t). \tag{9.65}$$

In general, we can say that

$$\mathcal{F}\left(\frac{\partial^n u}{\partial x^n}\right) = (i\alpha)^n U(\alpha,t). \tag{9.66}$$

Example 9.9

Solve the heat equation with general initial conditions using Fourier transformations:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \qquad u(x \to \pm\infty, t) \to 0, \quad u(x,0) = f(x). \tag{9.67}$$

Let us take the Fourier transformation of the heat equation:

$$\begin{aligned}
\mathcal{F}\left(\frac{\partial u}{\partial t}\right) &= \nu \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right), && (9.68)\\
\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\alpha x}\, dx &= -\nu\alpha^2 U(\alpha,t), && (9.69)\\
\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u e^{-i\alpha x}\, dx &= -\nu\alpha^2 U(\alpha,t), && (9.70)\\
\frac{\partial U}{\partial t} &= -\nu\alpha^2 U(\alpha,t), && (9.71)\\
U(\alpha,t) &= C(\alpha)\exp\left(-\nu\alpha^2 t\right). && (9.72)
\end{aligned}$$

Now our initial condition gives us

$$\begin{aligned}
u(x,0) &= f(x), && (9.73)\\
\mathcal{F}\left(u(x,0)\right) &= \mathcal{F}\left(f(x)\right), && (9.74)\\
U(\alpha,0) &= \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx, && (9.75)\\
&= F(\alpha). && (9.76)
\end{aligned}$$

Substituting this transformed initial condition into Eq. (9.72) gives

$$\begin{aligned}
U(\alpha,0) &= F(\alpha) = C(\alpha)\exp(0), && (9.77)\\
F(\alpha) &= C(\alpha). && (9.78)
\end{aligned}$$

Therefore, our solution in transformed space is

$$U(\alpha,t) = F(\alpha)\exp(-\nu\alpha^2 t). \tag{9.79}$$

To return to the (x,t) domain, we employ the inverse Fourier transformation of Eq. (9.31) to get

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x - \nu\alpha^2 t}\, d\alpha. \tag{9.80}$$

In terms of f, we can use Eq. (9.29) to say

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} f(\xi) e^{-i\alpha\xi}\, d\xi\right)}_{F(\alpha)} e^{i\alpha x - \nu\alpha^2 t}\, d\alpha, && (9.81)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\int_{-\infty}^{\infty} e^{i\alpha(x-\xi) - \nu\alpha^2 t}\, d\alpha\, d\xi. && (9.82)
\end{aligned}$$

Symbolic calculation reveals this reduces to

$$u(x,t) = \frac{1}{2\sqrt{\pi \nu t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4\nu t}} \, d\xi. \tag{9.83}$$

This is equivalent to Eq. (2.81).

Example 9.10

We have already seen this problem posed in Eq. (2.85), and a solution was obtained. Following the example of Mei, p. 137, solve the heat equation with a Dirac delta distribution as an initial condition:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \qquad u(x \to \pm\infty, t) \to 0, \quad u(x,0) = \delta(x). \tag{9.84}$$

We know $F(\alpha) = 1$ from Eq. (9.37), and the solution of Eq. (9.80) becomes

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu\alpha^2 t} e^{i\alpha x} \, d\alpha, \tag{9.85}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu\alpha^2 t} \left(\cos \alpha x + i \sin \alpha x\right) \, d\alpha, \tag{9.86}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu\alpha^2 t} \cos \alpha x \, d\alpha + \frac{i}{2\pi} \underbrace{\int_{-\infty}^{\infty} e^{-\nu\alpha^2 t} \sin \alpha x \, d\alpha}_{=0}, \tag{9.87}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu\alpha^2 t} \cos \alpha x \, d\alpha. \tag{9.88}$$

The integral involving sin is zero because the limits are symmetric and the function is odd in α. We next break the integral into two pieces:

$$u(x,t) = \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{-\nu\alpha^2 t} \cos \alpha x \, d\alpha + \int_{0}^{\infty} e^{-\nu\alpha^2 t} \cos \alpha x \, d\alpha \right). \tag{9.89}$$

Because of symmetry about $\alpha = 0$, the $\int_{-\infty}^{0}$ is equal to the $\int_{0}^{\infty}$. Thus, we can also say

$$u(x,t) = \frac{1}{2\pi} \left(\int_{0}^{\infty} e^{-\nu\alpha^2 t} \cos \alpha x \, d\alpha + \int_{0}^{\infty} e^{-\nu\alpha^2 t} \cos \alpha x \, d\alpha \right), \tag{9.90}$$

$$= \frac{1}{\pi} \int_{0}^{\infty} e^{-\nu\alpha^2 t} \cos \alpha x \, d\alpha. \tag{9.91}$$

Let us now change variables, exchanging α for β via

$$\alpha = \frac{\beta}{\sqrt{\nu t}}. \tag{9.92}$$

Thus for fixed t, we have

$$d\alpha = \frac{d\beta}{\sqrt{\nu t}}. \tag{9.93}$$

Substituting into Eq. (9.91) to eliminate α in favor of β, we find

$$u(x,t) = \frac{1}{\pi\sqrt{\nu t}} \int_0^\infty e^{-\beta^2} \cos \frac{\beta x}{\sqrt{\nu t}} \, d\beta. \tag{9.94}$$

Now define for convenience

$$\mu = \frac{x}{\sqrt{\nu t}}, \tag{9.95}$$

$$I(\mu) = \int_0^\infty e^{-\beta^2} \cos \mu\beta \, d\beta. \tag{9.96}$$

With these definitions, Eq. (9.94) becomes

$$u(x,t) = \frac{1}{\pi\sqrt{\nu t}} I(\mu), \tag{9.97}$$

Let us consider $I(\mu)$. Differentiating Eq. (9.96), we get

$$\frac{dI}{d\mu} = -\int_0^\infty \beta e^{-\beta^2} \sin \mu\beta \, d\beta. \tag{9.98}$$

Let us integrate the right side by parts to obtain

$$\frac{dI}{d\mu} = \underbrace{\frac{1}{2} e^{-\beta^2} \sin \mu\beta \bigg|_0^\infty}_{=0} - \frac{1}{2} \underbrace{\int_0^\infty e^{-\beta^2} \mu \cos \mu\beta \, d\beta}_{=\mu I(\mu)}, \tag{9.99}$$

$$= -\frac{1}{2}\mu I \tag{9.100}$$

This atypical equation is actually a first order ordinary differential equation for the variable I, which is itself an integral. We can get a condition at $\mu = 0$ by considering the definition of Eq. (9.96) applied at $\mu = 0$:

$$I(0) = \int_0^\infty e^{-\beta^2} \cos(0) \, d\beta = \int_0^\infty e^{-\beta^2} \, d\beta. \tag{9.101}$$

This integral is well-known to have a value which can be determined by a coordinate change, as described earlier on p. 176. Using this, we find

$$I(0) = \frac{\sqrt{\pi}}{2}. \tag{9.102}$$

Solution of Eqs. (9.100, 9.102) gives by separation of variables

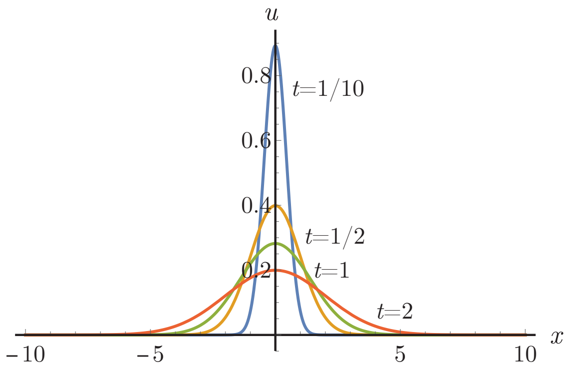
$$\frac{dI}{I} = -\frac{\mu \, d\mu}{2}, \tag{9.103}$$

$$\ln I = -\frac{\mu^2}{4} + C, \tag{9.104}$$

$$I = \hat{C} e^{-\mu^2/4}, \tag{9.105}$$

$$= \frac{\sqrt{\pi}}{2} e^{-\mu^2/4}, \tag{9.106}$$

$$= \frac{\sqrt{\pi}}{2} e^{\frac{-x^2}{4\nu t}}. \tag{9.107}$$



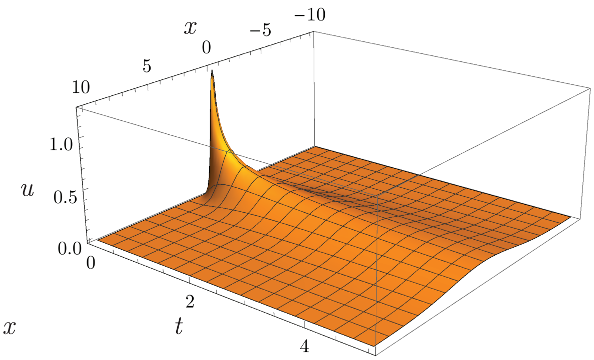


Figure 9.9: Solution to the heat equation for $\nu = 1$ for an initial pulse distribution.

Substituting into Eq. (9.97) to eliminate I, we get

$$u(x,t) = \frac{1}{2\sqrt{\pi\nu t}} \exp\left(\frac{-x^2}{4\nu t}\right). \tag{9.108}$$

Note this is fully consistent with the more general result of Eq. (9.83) for $f(\xi) = \delta(\xi)$. We plot results for $u(x,t)$ in Fig. 9.9 for $\nu = 1$.

We close with a discussion of the notion of *convolution*. Let us say we have two functions f and g and their respective Fourier transformations:

$$\mathcal{F}(f) = F(\alpha) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\, dx, \tag{9.109}$$

$$\mathcal{F}(g) = G(\alpha) = \int_{-\infty}^{\infty} g(x)e^{-i\alpha x}\, dx. \tag{9.110}$$

We also have the inverse Fourier transformations

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}\, d\alpha, \tag{9.111}$$

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)e^{i\alpha x}\, d\alpha. \tag{9.112}$$

Let us define the convolution of f and g by the operation

$$f * g \equiv \int_{-\infty}^{\infty} g(\xi)f(x-\xi)\, d\xi. \tag{9.113}$$

Now use Eq. (9.111) to eliminate $f(x-\xi)$:

$$f * g = \int_{-\infty}^{\infty} g(\xi) \underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha(x-\xi)}\, d\alpha\right)}_{f(x-\xi)} d\xi, \tag{9.114}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi)F(\alpha)e^{i\alpha(x-\xi)}\,d\alpha\,d\xi, && (9.115)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi)F(\alpha)e^{i\alpha(x-\xi)}\,d\xi\,d\alpha, && (9.116)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}\underbrace{\int_{-\infty}^{\infty} g(\xi)e^{-i\alpha\xi}\,d\xi}_{G(\alpha)}\,d\alpha, && (9.117)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)G(\alpha)e^{i\alpha x}\,d\alpha, && (9.118)\\
&= \mathcal{F}^{-1}\left(F(\alpha)G(\alpha)\right), && (9.119)\\
\mathcal{F}(f*g) &= \mathcal{F}\left(\mathcal{F}^{-1}\left(F(\alpha)G(\alpha)\right)\right), && (9.120)\\
&= F(\alpha)G(\alpha), && (9.121)\\
&= \mathcal{F}(f)\mathcal{F}(g). && (9.122)
\end{aligned}$$

9.2 Laplace transformations

The Laplace transformation is a technique often applied to linear ordinary differential equation to allow them to be transformed to algebraic equations, which are more easily solved. In a similar fashion as the Fourier transformation, the Laplace transformation can be extended to apply to partial differential equations so as to convert them to ordinary differential equations. As discussed by Mei, there are some problems for which Fourier transformation integrals are not convergent but which have no such problems under the Laplace transformation.

Let us see how the Laplace transformation can be considered as a special case of the Fourier transformation. Consider the function

$$g(x) = H(x)e^{-cx}f(x), \qquad (9.123)$$

where $H(x)$ is the Heaviside unit step function and $c \in \mathbb{R}^1 > 0$. Let us take the Fourier transformation of $g(x)$:

$$\begin{aligned}
\mathcal{F}(g(x)) = G(\lambda) &= \int_{-\infty}^{\infty} g(x)e^{-i\lambda x}\,dx, && (9.124)\\
&= \int_{-\infty}^{\infty} H(x)e^{-cx}f(x)e^{-i\lambda x}\,dx, && (9.125)\\
&= \int_{0}^{\infty} e^{-(c+i\lambda)x}f(x)\,dx, && (9.126)\\
&= \int_{0}^{\infty} e^{-sx}f(x)\,dx. && (9.127)
\end{aligned}$$

where we have defined

$$s = c + i\lambda. \qquad (9.128)$$

The inverse Fourier transformation is

$$g(x) = H(x)e^{-cx}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)e^{i\lambda x}\ d\lambda. \tag{9.129}$$

Thus, we have

$$H(x)f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)e^{(c+i\lambda)x}\ d\lambda. \tag{9.130}$$

From Eq. (9.128), we have

$$\lambda = -i(s-c), \tag{9.131}$$

$$d\lambda = -i\ ds. \tag{9.132}$$

Thus,

$$H(x)f(x) = \frac{-i}{2\pi}\int_{c-i\infty}^{c+i\infty} G(-i(s-c))e^{sx}\ ds, \tag{9.133}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(-i(s-c))e^{sx}\ ds. \tag{9.134}$$

Let us define

$$\overline{F}(s) = G(-i(s-c)), \tag{9.135}$$

and

$$F(x) = H(x)f(x). \tag{9.136}$$

Then we define the Laplace transformation $\mathcal{L}(F(x))$ as

$$\mathcal{L}(F(x)) = \overline{F}(s) = \int_0^{\infty} e^{-sx}F(x)\ dx. \tag{9.137}$$

We define the inverse Laplace transformation, $\mathcal{L}^{-1}$, as

$$\mathcal{L}^{-1}(\overline{F}(s)) = F(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{F}(s)e^{sx}\ ds. \tag{9.138}$$

Because c is positive, the path of integration is on a vertical line to the right of the origin in the complex plane.

Now, most physical problems involving the Laplace transformation involve time t rather than distance x. So following convention, we simply trade x for t in the definition of the Laplace transformation and its inverse:

$$\mathcal{L}(F(t)) = \overline{F}(s) = \int_0^{\infty} e^{-st}F(t)\ dt. \tag{9.139}$$

$$\mathcal{L}^{-1}(\overline{F}(s)) = F(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{F}(s)e^{st}\ ds. \tag{9.140}$$

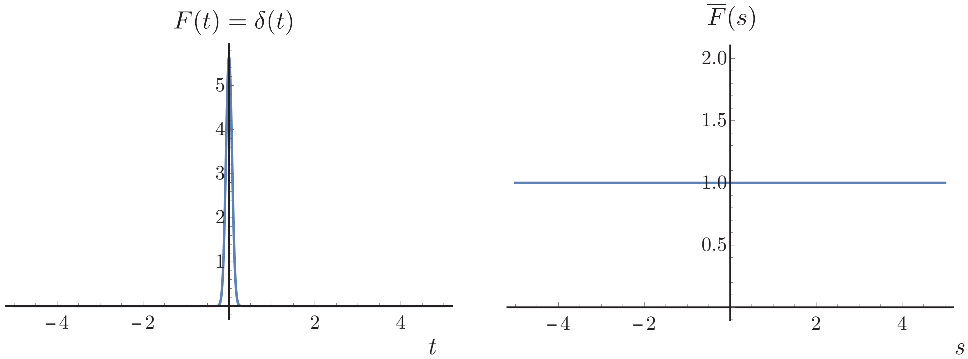


Figure 9.10: The Dirac delta function $\delta(t)$ and its Laplace transformation.

Because the Laplace transformation is only defined for $t \geq 0$, we can effectively ignore any part of $F(t)$ for $t < 0$.

Example 9.11

Find the Laplace transformation of $F(t) = \delta(t - t_0)$ with $t_0 = \mathbb{R}^1 \geq 0$.

Applying Eq. (9.139), we get

$$\mathcal{L}(\delta(t-t_0)) = \overline{F}(s) = \int_0^\infty e^{-st}\delta(t-t_0)\,dt, \tag{9.141}$$

$$= e^{-st_0}. \tag{9.142}$$

For $t_0 = 0$, the Dirac delta function and its Laplace transformation are plotted in Fig. 9.10 for $s \in \mathbb{R}^1$. Note here that $F(t) = 0$ already for $t < 0$.

Example 9.12

Find the Laplace transformation of $F(t) = H(t - t_0)$ with $t_0 = \mathbb{R}^1 \geq 0$.

Applying Eq. (9.139), we get

$$\mathcal{L}(H(t-t_0)) = \overline{F}(s) = \int_0^\infty e^{-st}H(t-t_0)\,dt, \tag{9.143}$$

$$= \int_{t_0}^\infty e^{-st}\,dt, \tag{9.144}$$

$$= -\frac{1}{s}e^{-st}\bigg|_{t_0}^{\infty}, \tag{9.145}$$

$$= \frac{e^{-st_0}}{s}. \tag{9.146}$$

We must make the additional restriction that $s > 0$ here. For $t_0 = 0$, the Heaviside function and its Laplace transformation are plotted in Fig. 9.11 for $s \in \mathbb{R}^1$. Note here that $F(t) = 0$ already for $t < 0$.

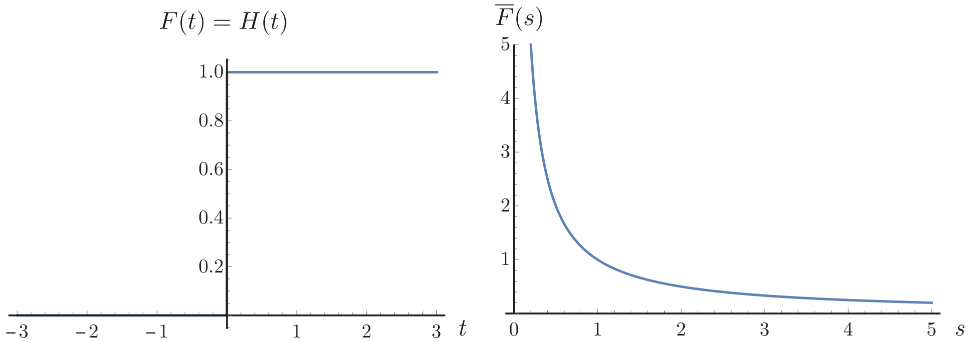


Figure 9.11: The Heaviside function $H(t)$ and its Laplace transformation.

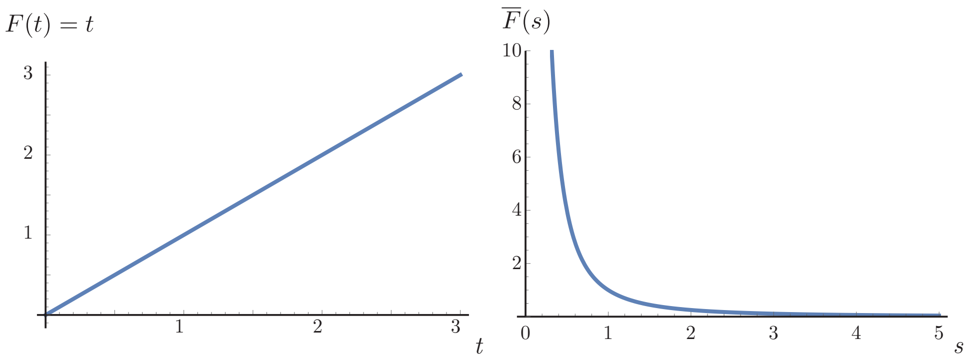


Figure 9.12: The function $F(t) = t$ and its Laplace transformation.

Example 9.13

Find the Laplace transformation of $F(t) = t$.

Applying Eq. (9.139), we get

$$\mathcal{L}(t) = \overline{F}(s) = \int_0^\infty e^{-st} t \, dt, \tag{9.147}$$

$$= -\left.\frac{e^{-st}(1+st)}{s^2}\right|_0^\infty, \tag{9.148}$$

$$= \frac{1}{s^2}. \tag{9.149}$$

We must make the additional restoration that $s > 0$ here. The function $F(t) = t$ and its Laplace transformation are plotted in Fig. 9.12 for $s \in \mathbb{R}^1$. We only plot $F(t)$ for $t > 0$ because it is on that domain that $\overline{F}$ is defined. For the more general $F(t) = t^n$, it is easily shown that $\overline{F}(s) = \Gamma(n+1)/s^{n+1}$.

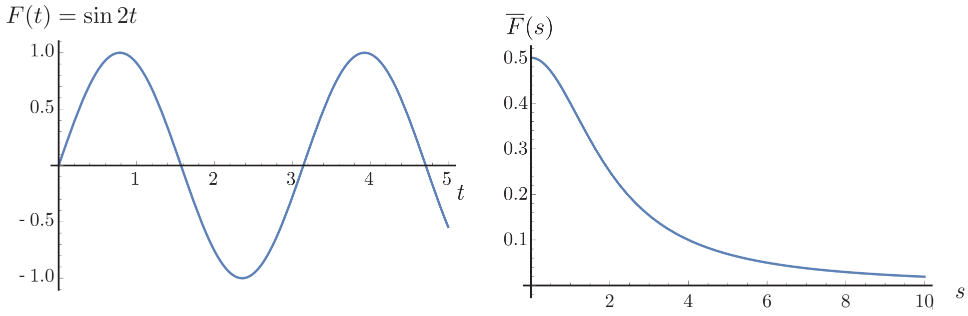


Figure 9.13: The function $F(t) = \sin 2t$ and its Laplace transformation.

Example 9.14

Find the Laplace transformation of $F(t) = b \sin at$, for $a, b \in \mathbb{R}^1 > 0$.

Applying Eq. (9.139), we get

$$\mathcal{L}(b \sin at) = \overline{F}(s) = \int_0^\infty b e^{-st} \sin at \; dt, \tag{9.150}$$

$$= -\left. \frac{b e^{-st}(s \sin at + a \cos ax)}{a^2 + s^2} \right|_0^\infty, \tag{9.151}$$

$$= \frac{ab}{a^2 + s^2}. \tag{9.152}$$

For $a = 2$, $b = 1$, the function $F(t) = \sin 2t$ and its Laplace transformation are plotted in Fig. 9.13 for $s \in \mathbb{R}^1$.

Example 9.15

Find the Laplace transformation of $F(t) = \exp(-t^2/2)$

Applying Eq. (9.139), we get

$$\mathcal{L}(\exp(-t^2/2)) = \overline{F}(s) = \int_0^\infty e^{-st} e^{-t^2/2} \; dt. \tag{9.153}$$

Omitting details, we find

$$\overline{F}(s) = \sqrt{\frac{\pi}{2}} e^{\frac{s^2}{2}} \operatorname{erfc}\left(\frac{s}{\sqrt{2}} \right). \tag{9.154}$$

The function $F(t) = \exp(-t^2/2)$ and its Laplace transformation are plotted in Fig. 9.14.

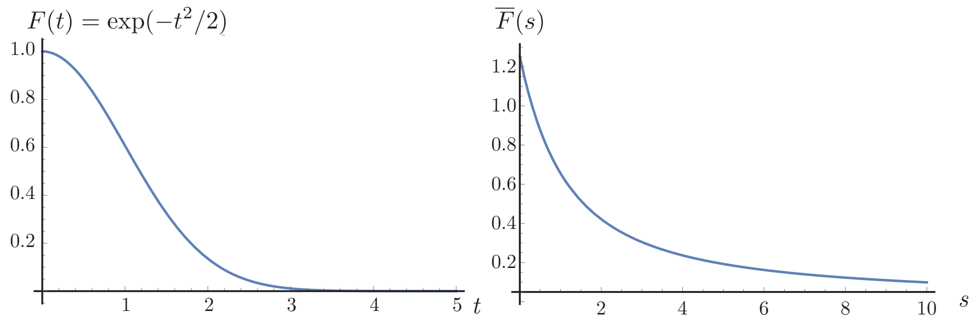


Figure 9.14: The function $F(t) = \exp(-t^2/2)$ and its Laplace transformation.

Example 9.16

If $u = u(t)$, find the Laplace transformation of du/dt.

Let us take

$$\mathcal{L}(u(t)) = \overline{U}(s) = \int_0^\infty e^{-st} u(t)\ dt. \tag{9.155}$$

We then have

$$\begin{aligned} \mathcal{L}\left(\frac{du}{dt}\right) &= \int_0^\infty \frac{du}{dt} e^{-st}\ dt, && (9.156)\\ &= e^{-st}u\Big|_0^\infty - \int_0^\infty (-s)ue^{-st}\ dt, && (9.157)\\ &= s\overline{U}(s) - u(0). && (9.158) \end{aligned}$$

In general, one can show that

$$\mathcal{L}\left(\frac{d^n u}{dt^n}\right) = s^n\overline{U}(s) - s^{n-1}u(0) - \ldots - s^0 \left.\frac{d^{n-1}u}{dt^{n-1}}\right|_{x=0}. \tag{9.159}$$

Example 9.17

Use Laplace transformations and their inverses to solve

$$\frac{d^2u}{dt^2} + 9u = 0, \quad u(0) = 0,\ \dot{u}(0) = 2. \tag{9.160}$$

Take the Laplace transformation of the system, Eq. (9.160), to get

$$\begin{aligned} \mathcal{L}\left(\frac{d^2u}{dt^2} + 9u\right) &= \mathcal{L}(0), && (9.161)\\ \mathcal{L}\left(\frac{d^2u}{dt^2}\right) + \mathcal{L}(9u) &= \mathcal{L}(0), && (9.162)\\ \left(s^2\overline{U} - s\cancel{u(0)} - \dot{u}(0)\right) + 9\overline{U} &= 0, && (9.163) \end{aligned}$$

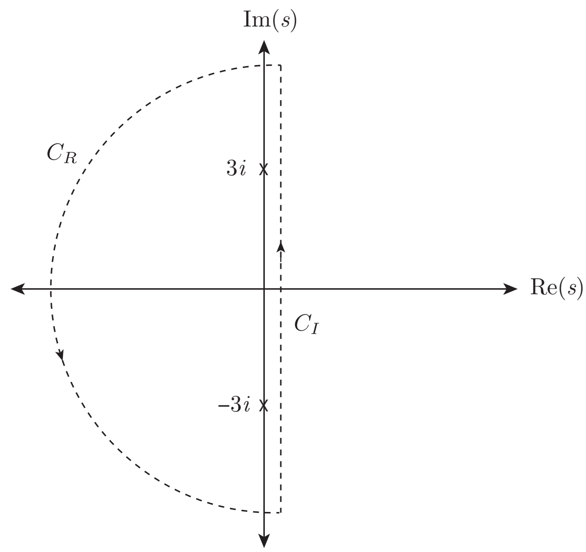


Figure 9.15: Contour integration path for inverse Laplace transformation integral associated with $d^2u/dt^2 + 9u = 0$.

$$
\begin{aligned}
s^2\overline{U} - 2 + 9\overline{U} &= 0, && (9.164)\\
\overline{U}(s^2 + 3^2) &= 2, && (9.165)\\
\overline{U}(s) &= \frac{2}{s^2 + 3^2}. && (9.166)
\end{aligned}
$$

Comparing to Eq. (9.152), we induce that

$$u(t) = \frac{2}{3}\sin 3t. \tag{9.167}$$

It is easy to verify that the differential equation and conditions at $t = 0$ are satisfied.

Let us see if we can use the more formal machinery of the inverse Laplace transformation to deduce Eq. (9.167). Substituting Eq. (9.166) into Eq. (9.140), we get

$$
\begin{aligned}
\mathcal{L}^{-1}(\overline{U}(s)) = u(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2e^{st}}{s^2+9}\,ds, && (9.168)\\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2e^{st}}{(s+3i)(s-3i)}\,ds, && (9.169)\\
& && (9.170)
\end{aligned}
$$

The integrand has two poles on the imaginary axis at

$$s = 0 \pm 3i. \tag{9.171}$$

Consider now the contour integral depicted in Fig. 9.15. We have the closed contour C as the sum of two portions of the contour:

$$C = C_I + C_R. \tag{9.172}$$

We can use Eq. (8.212) to give us $\oint_C$. First we need the residues of the integrand. Finding a Laurent series in the neighborhood of both poles gives us

$$\frac{2e^{st}}{(s+3i)(s-3i)} = \frac{\mp(i/3)e^{\pm 3it}}{s \mp 3i} + \ldots, \qquad s = \pm 3i. \tag{9.173}$$

The residues are thus $\mp(i/3)e^{\pm 3it}$. So we get

$$\begin{aligned}
\sum \text{residues} &= \frac{i}{3}(e^{-3it} - e^{3it}), && (9.174)\\
\oint_C = 2\pi i \sum \text{residues} &= -\frac{2\pi}{3}(e^{-3it} - e^{3it}), && (9.175)\\
&= \frac{4\pi i}{3}\frac{e^{3it} - e^{-3it}}{2i}, && (9.176)\\
&= \frac{4\pi i}{3}\sin 3t. && (9.177)
\end{aligned}$$

Now

$$\begin{aligned}
u(t) &= \frac{1}{2\pi i}\int_{C_I}, && (9.178)\\
&= \frac{1}{2\pi i}\left(\oint_C - \int_{C_R}\right), && (9.179)\\
&= \frac{1}{2\pi i}\left(\frac{4\pi i}{3}\sin 3t - \int_{C_R}\right), && (9.180)\\
&= \frac{2}{3}\sin 3t - \frac{1}{2\pi i}\int_{C_R}\frac{2e^{st}}{s^2+9}\,ds. && (9.181)
\end{aligned}$$

We now apply Jordan's lemma, Eq. (8.270), to $\int_{C_R}$. We note that C_R lies in the region where $\Re(s) \leq 0$. And for us $f(s) = 2/(s^2+9)$. Clearly, on C_R with $s = Re^{i\theta}$, we see as $R \to \infty$ that $|f(s)| \to 0$. So as long as $t > 0$, we have $\int_{C_R} = 0$. Thus, we get

$$u(t) = \frac{2}{3}\sin 3t. \tag{9.182}$$

Let us now discuss convolution in the context of the Laplace transformation. As with the convolution for the Fourier transformation, let us assume we have two functions F and G, and their respective Laplace and inverse Laplace transformations:

$$\begin{aligned}
\mathcal{L}(F) = \overline{F}(s) &= \int_0^\infty e^{-st}F(t)\,dt, && (9.183)\\
\mathcal{L}(G) = \overline{G}(s) &= \int_0^\infty e^{-st}G(t)\,dt. && (9.184)
\end{aligned}$$

Let us define the convolution as

$$F * G = \int_0^t G(\tau)F(t-\tau)\,d\tau. \tag{9.185}$$

Then we operate as follows

$$\begin{aligned}
\mathcal{L}(F*G) &= \int_0^\infty e^{-st}\left(\int_0^t G(\tau)F(t-\tau)\,d\tau\right)\,dt, && (9.186)\\
&= \int_0^\infty \int_0^t e^{-st}G(\tau)F(t-\tau)\,d\tau\,dt. && (9.187)
\end{aligned}$$

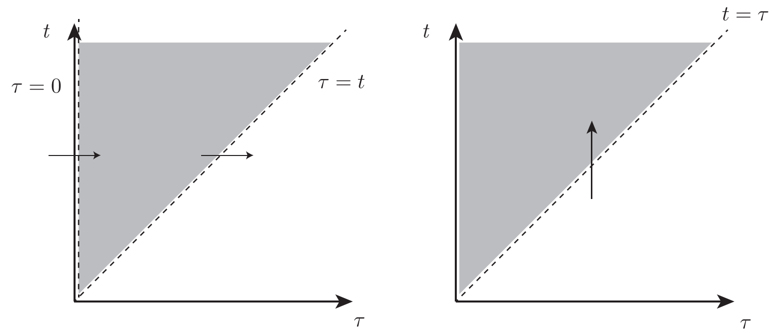


Figure 9.16: Sketch of area of integration and limits, depending on order of integration.

The domain of integration is sketched in Fig. 9.16. The graph on the left is bounded by the curves $\tau = 0$ and $\tau = t$ and lies between the curves $t = 0$ and $t \to \infty$. When we switch the order of integration, we have to carefully change the limits. When we first integrate on t, we must enter the domain at $t = \tau$ and exit at $t \to \infty$. Then we must bound this area by $\tau = 0$ and $\tau = \infty$. So the integral becomes

$$\mathcal{L}(F * G) = \int_0^\infty \int_\tau^\infty e^{-st} G(\tau) F(t-\tau)\, dt\, d\tau. \tag{9.188}$$

Let now $\hat{t} = t - \tau$. Then $d\hat{t} = dt$ and

$$\begin{aligned}
\mathcal{L}(F * G) &= \int_0^\infty \int_0^\infty e^{-s(\hat{t}+\tau)} G(\tau) F(\hat{t})\, d\hat{t}\, d\tau, && (9.189)\\
&= \int_0^\infty \int_0^\infty e^{-s(\hat{t}+\tau)} G(\tau) F(\hat{t})\, d\tau\, d\hat{t}, && (9.190)\\
&= \int_0^\infty e^{-s\hat{t}} F(\hat{t}) \int_0^\infty e^{-s\tau} G(\tau)\, d\tau\, d\hat{x}, && (9.191)\\
&= \left(\int_0^\infty e^{-s\hat{t}} F(\hat{t})\, d\hat{t}\right) \left(\int_0^\infty e^{-s\tau} G(\tau)\, d\tau\right), && (9.192)\\
&= \overline{F}(s)\overline{G}(s). && (9.193)
\end{aligned}$$

So

$$\mathcal{L}(F * G) = \mathcal{L}(F)\mathcal{L}(G). \tag{9.194}$$

Example 9.18

Following the analysis of Mei, pp. 272-275, apply the Laplace transformation method to solve Stokes' first problem, considered in Sec. 6.1:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \quad u(y,0) = 0, \quad u(0,t) = (U)H(t), \quad u(\infty,t) = 0. \tag{9.195}$$

We consider the Laplace transformation to operate on t and thus it does not impact y. So $\mathcal{L}(u(y,t)) = \overline{u}(y,s)$. And here $H(t)$ is the Heaviside function in t, which is a more complete way to formulate Stokes' first problem than done previously. Taking the Laplace transformation of the governing equation, we get

$$s\overline{u}(y,s) - \underbrace{u(y,0)}_{=0} = \nu \frac{\partial^2 \overline{u}}{\partial y^2}, \tag{9.196}$$

$$\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{s}{\nu}\overline{u} = 0, \tag{9.197}$$

$$\overline{u}(y,s) = C_1(s) \exp\left(\sqrt{\frac{s}{\nu}}y\right) + C_2(s) \exp\left(-\sqrt{\frac{s}{\nu}}y\right). \tag{9.198}$$

We need a bounded solution as $y \to \infty$, so we take $C_1(s) = 0$, giving

$$\overline{u}(y,s) = C_2(s) \exp\left(-\sqrt{\frac{s}{\nu}}y\right). \tag{9.199}$$

Now, we evaluate $C_2(s)$ by employing the boundary condition. We thus need to take the Laplace transformation of the boundary condition at $y = 0$, which is

$$\mathcal{L}(u(0,t)) = \mathcal{L}((U)H(t)), \tag{9.200}$$

$$= \frac{U}{s}. \tag{9.201}$$

Thus $C_2(s) = U/s$, and we get

$$\overline{u}(y,s) = \frac{U}{s} \exp\left(-\sqrt{\frac{s}{\nu}}y\right). \tag{9.202}$$

Now we need to take the inverse Laplace transformation to find $u(y,t)$:

$$u(y,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{u}(y,s) e^{st}\, ds, \tag{9.203}$$

$$= \frac{U}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{st-\sqrt{s/\nu}y}\, ds, \tag{9.204}$$

$$= \frac{U}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{st} e^{-\sqrt{s/\nu}y}\, ds. \tag{9.205}$$

Obviously there is a pole at $s = 0$. Let us employ a special contour which avoids this pole at the expense of introducing a branch cut as we take the contour integral sketched in Fig. 9.17. We have the closed contour C as

$$C = C_I + C_{R1} + C_+ + C_\epsilon + C_- + C_{R2}. \tag{9.206}$$

For C_{R1} and C_{R2}, we will let $R \to \infty$, and for C_ϵ, we will let $\epsilon \to 0$. Our contour integral will take the form

$$\oint_C = \int_{C_I} + \int_{C_{R1}} + \int_{C_+} + \int_{C_\epsilon} + \int_{C_-} + \int_{C_{R2}} = 0. \tag{9.207}$$

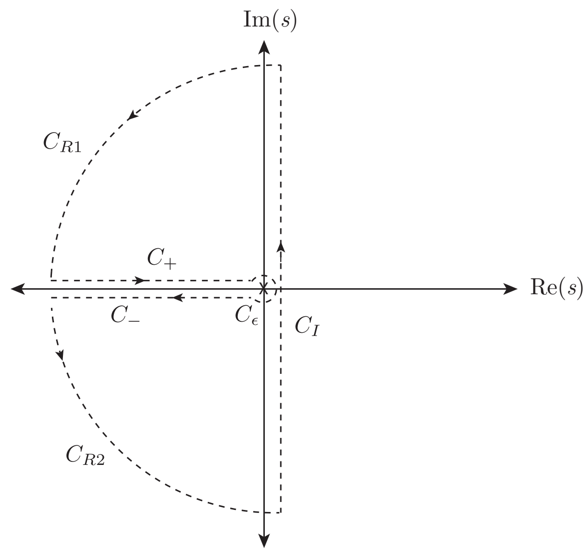


Figure 9.17: Contour integration path for inverse Laplace transformation integral associated with $\partial u/\partial t = \nu \partial^2 u/\partial y^2$.

There are no residues to consider by the nature of our contour, which has avoided the singularity at $s = 0$. And we are interested in $\int_{C_I}$ for as needed by our inverse Laplace transformation. By Jordan's lemma, Sec. 8.7, both $\int_{C_{R1}}$ and $\int_{C_{R2}}$ vanish as $R \to \infty$, for $t > 0$. On C_ϵ, we let

$$s = \epsilon e^{i\theta}, \qquad ds = \epsilon i e^{i\theta}\, d\theta. \tag{9.208}$$

and consider

$$\begin{aligned}
\int_{C_\epsilon} \frac{1}{s} e^{st} e^{-\sqrt{s/\nu}y}\, ds &= \lim_{\epsilon \to 0} \int_{\pi}^{-\pi} \frac{1}{\epsilon e^{i\theta}} \underbrace{e^{O(\epsilon)} e^{O(\sqrt{\epsilon})}}_{\to 1} (\epsilon i e^{i\theta}\, d\theta), && (9.209)\\
&= \int_{\pi}^{-\pi} i\, d\theta, && (9.210)\\
&= -2\pi i. && (9.211)
\end{aligned}$$

Note that this corrects a small error in Mei's analysis on p. 274. Now along $C_\pm$, we introduce the positive real variable v so as to have

$$s = v e^{\pm i\pi} = -v, \qquad v \in \mathbb{R}^1 > 0, \qquad ds = -dv. \tag{9.212}$$

Also on $C_\pm$ we have

$$\sqrt{s} = \sqrt{v} e^{\pm i\pi/2} = \pm i \sqrt{v}. \tag{9.213}$$

We get then on C_+

$$\begin{aligned}
\int_{C_+} \frac{1}{s} e^{st} e^{-\sqrt{s/\nu}y}\, ds &= \int_{\infty}^{0} \frac{1}{-v} e^{-vt} e^{-i\sqrt{v/\nu}y}\, (-dv), && (9.214)\\
&= -\int_0^\infty \frac{1}{v} e^{-vt} e^{-i\sqrt{v/\nu}y}\, dv. && (9.215)
\end{aligned}$$

We get on C_-

$$\int_{C_-} \frac{1}{s} e^{st} e^{-\sqrt{s/\nu}y}\, ds = \int_0^\infty \frac{1}{-v} e^{-vt} e^{i\sqrt{v/\nu}y}\, (-dv), \tag{9.216}$$

$$= \int_0^\infty \frac{1}{v} e^{-vt} e^{i\sqrt{v/\nu}y}\, dv. \tag{9.217}$$

Adding, we get

$$\int_{C_-} + \int_{C_+} = \int_0^\infty \frac{1}{v} e^{-vt} \left(e^{i\sqrt{v/\nu}y} - e^{-i\sqrt{v/\nu}y} \right) dv, \tag{9.218}$$

$$= 2i \int_0^\infty \frac{1}{v} e^{-vt} \sin(\sqrt{v/\nu}y)\, dv, \tag{9.219}$$

$$= 2\pi i \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}} \right), \tag{9.220}$$

where the last integral was obtained with the aid of symbolic software. So Eq. (9.207) tells us

$$\int_{C_I} = -\int_{C_\epsilon} - \int_{C_-} - \int_{C_+}, \tag{9.221}$$

$$\int_{C_I} \frac{1}{s} e^{st} e^{-\sqrt{s/\nu}y}\, ds = 2\pi i - 2\pi i \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}} \right), \tag{9.222}$$

$$\underbrace{\frac{U}{2\pi i} \int_{C_I} \frac{1}{s} e^{st} e^{-\sqrt{s/\nu}y}\, ds}_{u(y,t)} = U \left(1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}} \right) \right), \tag{9.223}$$

$$u(y,t) = U \left(1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}} \right) \right), \tag{9.224}$$

$$= U \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}} \right). \tag{9.225}$$

This is fully equivalent to our earlier result found in in Eq. (6.70) using slightly different notation.

Problems

Chapter 10

Linear integral equations

see Powers and Sen, Chapter 8.

In this chapter, adopted largely from Powers and Sen[1] we introduce an important, though often less emphasized, topic: *integral equations*. Integral equations, and their cousins the integro-differential equations, often arise naturally in engineering problems where nonlocal effects are significant, i.e. when what is happening at a given point in space-time is affected by the past or by points at a distance, or by both. They may arise in such areas as radiation heat transfer and statistical physics. They also arise in problems involving the Green's functions of linear operators, which may originate from a wide variety of problems in engineering such as heat transfer, elasticity, or electromagnetics. Our focus will be on linear integral equations, though one could extend to the nonlinear theory if desired. More common studies of linear equations of the sort $\mathbf{L}y = f$ typically address cases where $\mathbf{L}$ is either a linear differential operator or a matrix. Here we take it to be an integral. We will then be able to apply standard notions from eigenvalue analysis to aid in the interpretation of the solutions to such equations. When the integral operator is discretized, integral equations can be approximated as linear algebra problems.

10.1 Definitions

We consider integral equations that take the form

$$h(x)y(x) = f(x) + \lambda \int_a^b K(x,s)y(s)\, ds. \tag{10.1}$$

Such an equation is linear in $y(x)$, the unknown dependent variable for which we seek a solution. Here $K(x,s)$, the so-called *kernel*, is known, $h(x)$ and $f(x)$ are known functions,

[1] J. M. Powers and M. Sen, 2015, *Mathematical Methods in Engineering*, Cambridge University Press, New York.

and λ is a constant parameter. We could rewrite Eq. (10.1) as

$$\underbrace{\left(h(x)\,(\cdot)|_{s=x} - \lambda \int_a^b K(x,s)\,(\cdot)\;ds\right)}_{\mathbf{L}} y(s) = f(x), \tag{10.2}$$

so that it takes the explicit form $\mathbf{L}y = f$. Here $(\cdot)$ is a placeholder for the operand. If $f(x) = 0$, our integral equation is *homogeneous*. When a and b are fixed constants, Eq. (10.1) is called a *Fredholm*[2] *equation*. If the upper limit is instead the variable x, we have a *Volterra*[3] *equation*:

$$h(x)y(x) = f(x) + \lambda \int_a^x K(x,s)y(s)\;ds. \tag{10.3}$$

A Fredholm equation whose kernel has the property $K(x,s) = 0$ for $s > x$ is in fact a Volterra equation. If one or both of the limits is infinite, the equation is known as *singular integral equation*, e.g.

$$h(x)y(x) = f(x) + \lambda \int_a^\infty K(x,s)y(s)\;ds. \tag{10.4}$$

If $h(x) = 0$, we have what is known as a *Fredholm equation of the first kind*:

$$0 = f(x) + \lambda \int_a^b K(x,s)y(s)\;ds. \tag{10.5}$$

Here, we can expect difficulties in solving for $y(s)$ if for a given x, $K(x,s)$ takes on a value of zero or near zero for $s \in [a,b]$. That is because when $K(x,s) = 0$, it maps all $y(s)$ into zero, rendering the solution nonunique. The closer $K(x,s)$ is to zero, the more challenging it is to estimate $y(s)$.

If $h(x) = 1$, we have a *Fredholm equation of the second kind*:

$$y(x) = f(x) + \lambda \int_a^b K(x,s)y(s)\;ds. \tag{10.6}$$

Equations of this kind have a more straightforward solution than those of the first kind.

10.2 Homogeneous Fredholm equations

Let us here consider homogeneous Fredholm equations, i.e. those with $f(x) = 0$.

[2] Erik Ivar Fredholm, 1866-1927, Swedish mathematician.
[3] Vito Volterra, 1860-1940, Italian mathematician.

10.2.1 First kind

A homogeneous Fredholm equation of the first kind takes the form

$$0 = \int_a^b K(x,s)y(s)\ ds. \tag{10.7}$$

Solutions to Eq. (10.7) are functions $y(s)$ which lie in the null space of the linear integral operator. Certainly, $y(s) = 0$ satisfies, but there may be other nontrivial solutions, based on the nature of the kernel $K(x,s)$. Certainly, for a given x, if there are points or regions where $K(x,s) = 0$ in $s \in [a,b]$, one would expect nontrivial and nonunique $y(s)$ to exist which would still satisfy Eq. (10.7). Also, if $K(x,s)$ oscillates appropriately about zero for $s \in [a,b]$, one may find nontrivial and nonunique $y(s)$.

Example 10.1

Find solutions $y(x)$ to the homogeneous Fredholm equation of the first kind

$$0 = \int_0^1 xsy(s)\ ds. \tag{10.8}$$

Assuming $x \neq 0$, we can factor to say

$$0 = \int_0^1 sy(s)\ ds. \tag{10.9}$$

Certainly solutions for y are nonunique. For example, any function which is odd and symmetric about $s = 1/2$ and scaled by s will satisfy, e.g.

$$y(x) = C\frac{\sin(2n\pi x)}{x}, \qquad C \in \mathbb{R}^1, \quad n \in \mathbb{Q}^1. \tag{10.10}$$

The piecewise function

$$y(x) = \begin{cases} C, & x = 0, \\ 0, & x \in (0,1], \end{cases} \tag{10.11}$$

also satisfies, where $C \in \mathbb{R}^1$.

10.2.2 Second kind

A homogeneous Fredholm equation of the second kind takes the form

$$y(x) = \lambda \int_a^b K(x,s)y(s)\ ds. \tag{10.12}$$

Obviously, when $y(s) = 0$, Eq. (10.12) is satisfied. But we might expect that there exist nontrivial eigenfunctions and corresponding eigenvalues which also satisfy Eq. (10.12). This is because Eq. (10.12) takes the form of $(1/\lambda)y = \mathbf{L}y$, where $\mathbf{L}$ is the linear integral operator. In the theory of integral equations, it is more traditional to have the eigenvalue λ play the role of the reciprocal of the usual eigenvalue.

10.2.2.1 Separable kernel

In the special case in which the kernel is what is known as a *separable kernel* or *degenerate kernel* with the form:

$$K(x,s) = \sum_{i=1}^{N} \phi_i(x)\psi_i(s), \tag{10.13}$$

significant simplification arises. We then substitute into Eq. (10.12) to get

$$\begin{aligned} y(x) &= \lambda \int_a^b \left(\sum_{i=1}^{N} \phi_i(x)\psi_i(s) \right) y(s)\ ds, & (10.14) \\ &= \lambda \sum_{i=1}^{N} \phi_i(x) \underbrace{\int_a^b \psi_i(s) y(s)\ ds}_{c_i}. & (10.15) \end{aligned}$$

Then we define the constants c_i, $i = 1, \ldots, N$, as

$$c_i = \int_a^b \psi_i(s) y(s)\ ds, \qquad i = 1, \ldots, N, \tag{10.16}$$

and find

$$y(x) = \lambda \sum_{i=1}^{N} c_i \phi_i(x). \tag{10.17}$$

We get the constants c_i by substituting Eq. (10.17) into Eq. (10.16):

$$\begin{aligned} c_i &= \int_a^b \psi_i(s) \lambda \sum_{j=1}^{N} c_j \phi_j(s)\ ds, & (10.18) \\ &= \lambda \sum_{j=1}^{N} c_j \underbrace{\int_a^b \psi_i(s)\phi_j(s)\ ds}_{B_{ij}}. & (10.19) \end{aligned}$$

Defining the constant matrix B_{ij} as $B_{ij} = \int_a^b \psi_i(s)\phi_j(s)\ ds$, we then have

$$c_i = \lambda \sum_{j=1}^{N} B_{ij} c_j. \tag{10.20}$$

In Gibbs notation, we would say

$$\begin{aligned} \mathbf{c} &= \lambda \mathbf{B} \cdot \mathbf{c}, & (10.21) \\ \mathbf{0} &= (\lambda \mathbf{B} - \mathbf{I}) \cdot \mathbf{c}. & (10.22) \end{aligned}$$

This is an eigenvalue problem for $\mathbf{c}$. Here the reciprocal of the traditional eigenvalues of $\mathbf{B}$ give the values of λ, and the eigenvectors are the associated values of $\mathbf{c}$.

Example 10.2

Find the eigenvalues and eigenfunctions for the homogeneous Fredholm equation of the second kind with the degenerate kernel, $K(x,s) = xs$ on the domain $x \in [0,1]$:

$$y(x) = \lambda \int_0^1 xsy(s)\ ds. \tag{10.23}$$

The equation simplifies to

$$y(x) = \lambda x \int_0^1 sy(s)\ ds. \tag{10.24}$$

Take then

$$c = \int_0^1 sy(s)\ ds, \tag{10.25}$$

so that

$$y(x) = \lambda xc. \tag{10.26}$$

Thus,

$$c = \int_0^1 s\lambda sc\ ds, \tag{10.27}$$

$$1 = \lambda \int_0^1 s^2\ ds, \tag{10.28}$$

$$= \lambda \left. \frac{s^3}{3} \right|_0^1, \tag{10.29}$$

$$= \lambda \left(\frac{1}{3}\right), \tag{10.30}$$

$$\lambda = 3. \tag{10.31}$$

Thus, there is a single eigenfunction, $y = x$ associated with a single eigenvalue, $\lambda = 3$. Any constant multiplied by the eigenfunction is also an eigenfunction.

10.2.2.2 Non-separable kernel

For many problems, the kernel is not separable, and we must resort to numerical methods. Let us consider Eq. (10.12) with $a = 0$, $b = 1$:

$$y(x) = \lambda \int_0^1 K(x,s)y(s)\ ds. \tag{10.32}$$

Now, while there are many sophisticated numerical methods to evaluate the integral in Eq. (10.32), it is easiest to convey our ideas via the simplest method: the rectangular rule with evenly spaced intervals. Let us distribute N points uniformly in $x \in [0, 1]$ so that $x_i = (i-1)/(N-1)$, $i = 1, \ldots, N$. We form the same distribution for $s \in [0, 1]$ with $s_j = (j-1)/(N-1)$, $j = 1, \ldots, N$. For a given $x = x_i$, this distribution defines $N-1$ rectangles of width $\Delta s = 1/(N-1)$ and of height $K(x_i, s_j) \equiv K_{ij}$. We can think of K_{ij} as a matrix of dimension $(N-1) \times (N-1)$. We can estimate the integral by adding the areas of all of the individual rectangles. By the nature of the rectangular rule, this method has a small asymmetry which ignores the influence of the function values at $i = j = N$. In the limit of large N, this is not a problem. Next, let $y(x_i) \equiv y_i$, $i = 1, \ldots, N-1$, and $y(s_j) \equiv y_j$, $j = 1, \ldots, N-1$, and write Eq. (10.32) in a discrete approximation as

$$y_i = \lambda \sum_{j=1}^{N-1} K_{ij} y_j \Delta s. \tag{10.33}$$

In vector form, we could say

$$\begin{aligned} \mathbf{y} &= \lambda \mathbf{K} \cdot \mathbf{y} \Delta s, & (10.34)\\ \mathbf{0} &= \left(\mathbf{K} - \frac{1}{\lambda \Delta s} \mathbf{I} \right) \cdot \mathbf{y}, & (10.35)\\ &= (\mathbf{K} - \sigma \mathbf{I}) \cdot \mathbf{y}. & (10.36) \end{aligned}$$

Obviously, this is a eigenvalue problem in linear algebra. The eigenvalues of $\mathbf{K}$, $\sigma_i = 1/(\lambda_i \Delta s)$, $i = 1, \ldots, N-1$, approximate the eigenvalues of $\mathbf{L} = \int_0^1 K(x,s)(\cdot)\, ds$, and the eigenvectors are approximations to the eigenfunctions of $\mathbf{L}$.

Example 10.3

Find numerical approximations of the first nine eigenvalues and eigenfunctions of the homogeneous Fredholm equation of the second kind

$$y(x) = \lambda \int_0^1 \sin(10xs) y(s)\, ds. \tag{10.37}$$

Discretization leads to a matrix equation in the form of Eq. (10.34). For display purposes only, we examine a coarse discretization of $N = 6$. In this case, our discrete equation is

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}}_{\mathbf{y}} = \lambda \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.389 & 0.717 & 0.932 & 1.000 \\ 0 & 0.717 & 1.000 & 0.675 & -0.058 \\ 0 & 0.932 & 0.675 & -0.443 & -0.996 \\ 0 & 1.000 & -0.058 & -0.996 & 0.117 \end{pmatrix}}_{\mathbf{K}} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}}_{\mathbf{y}} \underbrace{\left(\frac{1}{5}\right)}_{\Delta s}. \tag{10.38}$$

Obviously, $\mathbf{K}$ is not full rank because of the row and column of zeros. In fact it has a rank of 4. The zeros exist because $K(x,s) = 0$ for both $x = 0$ and $s = 0$. This however, poses no issues for computing the eigenvalues and eigenvectors. However $N = 6$ is too small to resolve either the eigenvalues or

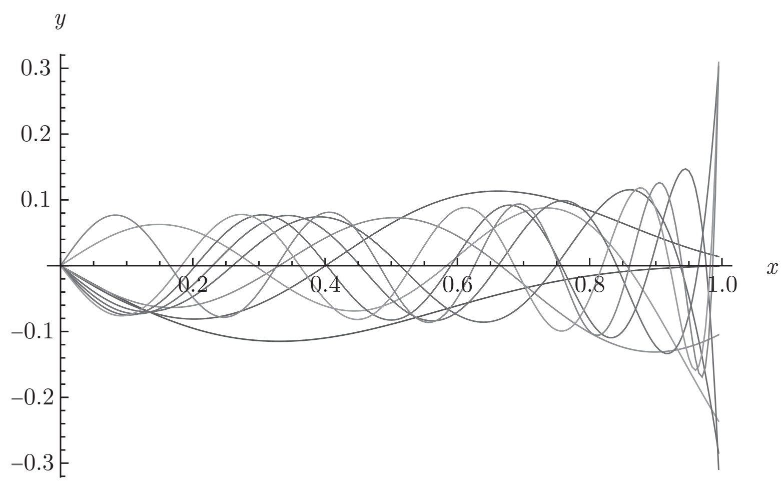


Figure 10.1: First nine eigenfunctions for $y(x) = \lambda \int_0^1 \sin(10xs)y(s)\ ds$.

eigenfunctions of the underlying continuous operator. Choosing $N = 201$ points gives acceptable resolution for the first nine eigenvalues, which are

$$\lambda_1 = 2.523, \qquad \lambda_2 = -2.526, \qquad \lambda_3 = 2.792, \tag{10.39}$$

$$\lambda_4 = -7.749, \qquad \lambda_5 = 72.867, \qquad \lambda_6 = -1225.2, \tag{10.40}$$

$$\lambda_7 = 3.014 \times 10^4, \qquad \lambda_8 = -1.011 \times 10^6, \qquad \lambda_9 = 4.417 \times 10^7. \tag{10.41}$$

The corresponding eigenfunctions are plotted in Fig. 10.1.

Example 10.4

Find numerical approximations of the first six eigenvalues and eigenfunctions of the homogeneous Fredholm equation of the second kind

$$y(x) = \lambda \int_0^1 g(x,s)y(s)\ ds, \tag{10.42}$$

where

$$g(x,s) = \begin{cases} x(s-1), & x \le s, \\ s(x-1), & x \ge s. \end{cases} \tag{10.43}$$

This kernel is the Green's function for the problem $d^2y/dx^2 = f(x)$ with $y(0) = y(1) = 0$. The Green's function solution is $y(x) = \int_0^1 g(x,s)f(s)\ ds$. For our example problem, we have $f(s) = \lambda y(s)$; thus, we are also solving the eigenvalue problem $d^2y/dx^2 = \lambda y$.

Choosing $N = 201$ points gives acceptable resolution for the first six eigenvalues, which are

$$\lambda_1 = -9.869, \qquad \lambda_2 = -39.48, \qquad \lambda_3 = -88.81, \tag{10.44}$$

$$\lambda_4 = -157.9, \qquad \lambda_5 = -246.6, \qquad \lambda_6 = -355.0. \tag{10.45}$$

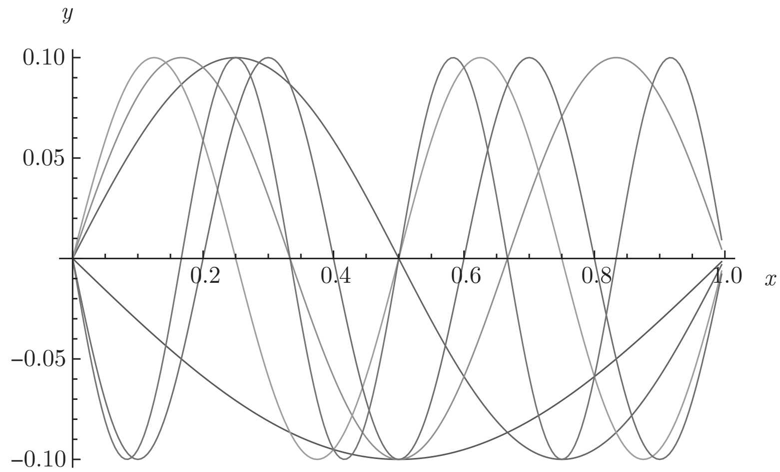


Figure 10.2: First six eigenfunctions for $y(x) = \lambda \int_0^1 g(x,s)y(s)\ ds$, where $g(x,s)$ is the Green's function for $d^2y/dx^2 = f(x)$, $y(0) = y(1) = 0$.

These compare well with the known eigenvalues of $\lambda = -n^2\pi^2$, $n = 1, 2, \ldots$:

$$\lambda_1 = -9.870, \qquad \lambda_2 = -39.48, \qquad \lambda_3 = -88.83, \tag{10.46}$$

$$\lambda_4 = -157.9, \qquad \lambda_5 = -246.7, \qquad \lambda_6 = -355.3. \tag{10.47}$$

The corresponding eigenfunctions are plotted in Fig. 10.2. The eigenfunctions appear to approximate well the known eigenfunctions $\sin(n\pi x)$, $n = 1, 2, \ldots$.

We can gain some understanding of the accuracy of our method by studying how its error converges as the number of terms in the approximation increases. There are many choices as to how to evaluate the error. Here let us choose one of the eigenvalues, say λ_4; others could have been chosen. We know the exact value is $\lambda_4 = -16\pi^2$. Let us take the relative error to be

$$e_4 = \frac{|\lambda_{4N} + 16\pi^2|}{16\pi^2}, \tag{10.48}$$

where λ_{4N} here is understood to be the numerical approximation to λ_4, which is a function of N. Fig. 10.3 shows the convergence, which is well approximated by the curve fit $e_4 \approx 15.99N^{-2.04}$.

10.3 Inhomogeneous Fredholm equations

Inhomogeneous integral equations can also be studied, and we do so here.

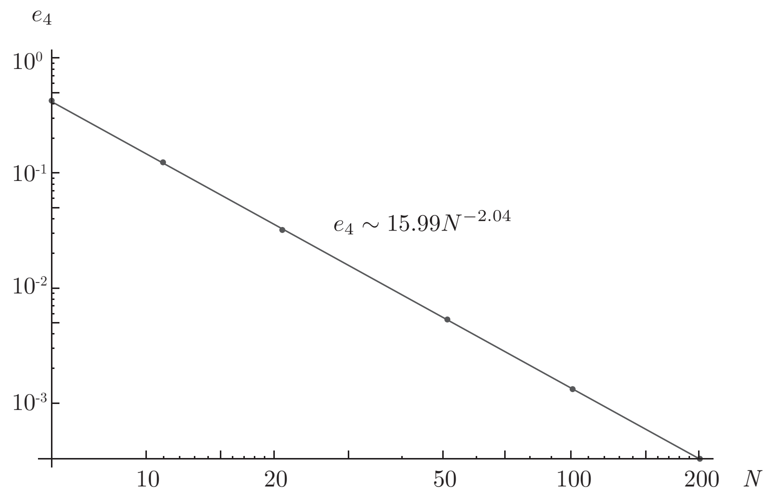


Figure 10.3: Convergence of the relative error in approximation of $\lambda_4 = -16\pi^2$ for $y(x) = \lambda \int_0^1 g(x,s)y(s)\,ds$, where $g(x,s)$ is the Green's function for $d^2y/dx^2 = f(x)$, $y(0) = y(1) = 0$.

10.3.1 First kind

Example 10.5

Consider solutions $y(x)$ to the inhomogeneous Fredholm equation of the first kind

$$0 = x + \int_0^1 \sin(10xs)y(s)\,ds. \tag{10.49}$$

Here we have $f(x) = x$, $\lambda = 1$, and $K(x,s) = \sin(10xs)$. For a given value of x, we have $K(x,s) = 0$, when $s = 0$, and so we expect a nonunique solution for y.

Let us once again solve this by discretization techniques identical to previous examples. In short,

$$0 = f(x) + \int_0^1 K(x,s)y(s)\,ds, \tag{10.50}$$

leads to the matrix equation

$$\mathbf{0} = \mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s, \tag{10.51}$$

where $\mathbf{f}$ is a vector of length $N-1$ containing the values of $f(x_i), i = 1, \ldots, N-1$, $\mathbf{K}$ is a matrix of dimension $(N-1) \times (N-1)$ populated by values of $K(x_i, s_j), i = 1, \ldots, N-1$, $j = 1, \ldots, N-1$, and $\mathbf{y}$ is a vector of length $N-1$ containing the unknown values of $y(x_j), j = 1, \ldots, N-1$.

When we evaluate the rank of $\mathbf{K}$, we find for $K(x,s) = \sin(xs)$ that the rank of the discrete $\mathbf{K}$ is $r = N-2$. This is because $K(x,s)$ evaluates to zero at $x = 0$ and $s = 0$. Thus, the right null space is of dimension unity. Now, we have no guarantee that $\mathbf{f}$ lies in the column space of $\mathbf{K}$, so the best we can imagine is that there exists a unique solution $\mathbf{y}$ that minimizes $||\mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s||_2$ which itself has no components in the null space of $\mathbf{K}$, so that $\mathbf{y}$ itself is of minimum "length." So, we say our best $\mathbf{y}$ is

$$\mathbf{y} = -\frac{1}{\Delta s}\mathbf{K}^+ \cdot \mathbf{f}, \tag{10.52}$$

where $\mathbf{K}^+$ is the Moore-Penrose pseudoinverse of $\mathbf{K}$.

Letting $N = 6$ gives rise to the matrix equation

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{0}} = \underbrace{\begin{pmatrix} 0 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.389 & 0.717 & 0.932 & 1.000 \\ 0 & 0.717 & 1.000 & 0.675 & -0.058 \\ 0 & 0.932 & 0.675 & -0.443 & 0.996 \\ 0 & 1.000 & -0.058 & 0.996 & 0.117 \end{pmatrix}}_{\mathbf{K}} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_2 \\ y_4 \\ y_5 \end{pmatrix}}_{\mathbf{y}} \underbrace{\left(\frac{1}{5}\right)}_{\Delta s}. \tag{10.53}$$

Solving for the $\mathbf{y}$ of minimum length which minimizes $||\mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s||_2$, we find

$$\mathbf{y} = \begin{pmatrix} 0 \\ -4.29 \\ 1.32 \\ -0.361 \\ 0.057 \end{pmatrix}. \tag{10.54}$$

We see by inspection that the vector $(1, 0, 0, 0, 0)^T$ lies in the right null space of $\mathbf{K}$. So $\mathbf{K}$ operating on any scalar multiple, α, of this null space vector maps into zero, and does not contribute to the error. So the following set of solution vectors $\mathbf{y}$ all have the same error in approximation:

$$\mathbf{y} = \begin{pmatrix} \alpha \\ -4.29 \\ 1.32 \\ -0.361 \\ 0.057 \end{pmatrix}, \qquad \alpha \in \mathbb{R}^1. \tag{10.55}$$

We also find the error to be, for $N = 6$

$$||\mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s||_2 = 0. \tag{10.56}$$

Because the error is zero, we have selected a function $f(x) = x$, whose discrete approximation lies in the column space of $\mathbf{K}$; for more general functions, this will not be the case. This is a consequence of our selected function, $f(x) = x$, evaluating to zero at $x = 0$. For example, for $f(x) = x + 2$, we would find $||\mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s||_2 = f(0) = 2$, with all of the error at $x = 0$, and none at other points in the domain.

This seems to be a rational way to approximate the best continuous $y(x)$ to satisfy the continuous integral equation. However, as N increases, we find the approximation $\mathbf{y}$ does not converge to a finite well-behaved function, as displayed for $N = 6, 51, 101$ in Fig. 10.4. This lack of convergence is likely related to the ill-conditioned nature of $\mathbf{K}$. For $N = 6$ the condition number c, that is the ratio of the largest and smallest singular values is $c = 45$; for $N = 51$, we find $c = 10^{137}$; for $N = 101$, $c = 10^{232}$. This ill-conditioned behavior is typical for Fredholm equations of the first kind. While the function itself does not converge with increasing N, the error $||\mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s||_2$ remains zero for all N for $f(x) = x$ (or any other f which has $f(0) = 0$).

10.3.2 Second kind

Example 10.6

Identify the solution $y(x)$ to the inhomogeneous Fredholm equation of the second kind

$$y(x) = x + \int_0^1 \sin(10xs)y(s)\, ds. \tag{10.57}$$

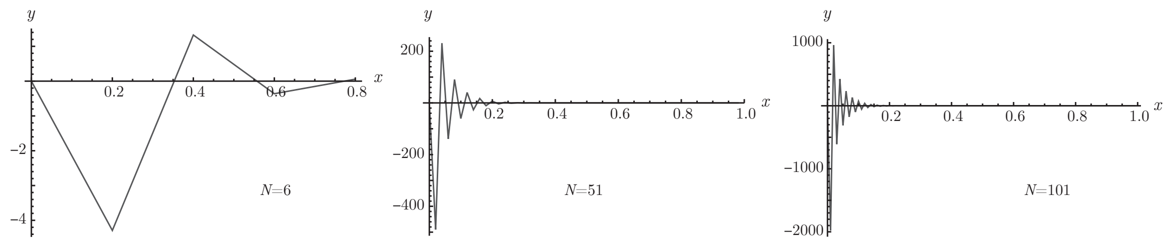


Figure 10.4: Approximations $\mathbf{y} \approx y(x)$ which have minimum norm while best satisfying the discrete Fredholm equation of the first kind $\mathbf{0} = \mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s$, modeling the continuous $0 = x + \int_0^1 \sin(xs)y(s)\ ds$.

Again, we have $f(x) = x$, $\lambda = 1$, and $K(x, s) = \sin(10xs)$. Let us once again solve this by discretization techniques identical to previous examples. In short,

$$y(x) = f(x) + \int_0^1 K(x, s)y(s)\ ds, \tag{10.58}$$

leads to the matrix equation

$$\mathbf{y} = \mathbf{f} + \mathbf{K} \cdot \mathbf{y}\Delta s, \tag{10.59}$$

where $\mathbf{f}$ is a vector of length $N - 1$ containing the values of $f(x_i), i = 1, \ldots, N - 1$, $\mathbf{K}$ is a matrix of dimension $(N - 1) \times (N - 1)$ populated by values of $K(x_i, s_j), i = 1, \ldots, N - 1$, $j = 1, \ldots, N - 1$, and $\mathbf{y}$ is a vector of length $N - 1$ containing the unknown values of $y(x_j), j = 1, \ldots, N - 1$.

Solving for $\mathbf{y}$, we find

$$\mathbf{y} = (\mathbf{I} - \mathbf{K}\Delta s)^{-1} \cdot \mathbf{f}. \tag{10.60}$$

The matrix $\mathbf{I} - \mathbf{K}\Delta s$ is not singular, and thus we find a unique solution. The only error in this solution is that associated with the discrete nature of the approximation. This discretization error approaches zero as N becomes large. The converged solution is plotted in Fig. 10.5. In contrast to Fredholm equations of the first kind, those of the second kind generally have unambiguous solution.

10.4 Fredholm alternative

The Fredholm alternative applies to integral equations, as well as many other types of equations. Consider, respectively, the inhomogeneous and homogeneous Fredholm equations of the second kind,

$$y(x) = f(x) + \lambda \int_a^b K(x, s)y(s)\ ds, \tag{10.61}$$

$$y(x) = \lambda \int_a^b K(x, s)y(s)\ ds. \tag{10.62}$$

For such systems, given $K(x, s)$, $f(x)$, and nonzero $\lambda \in \mathbb{C}^1$ either

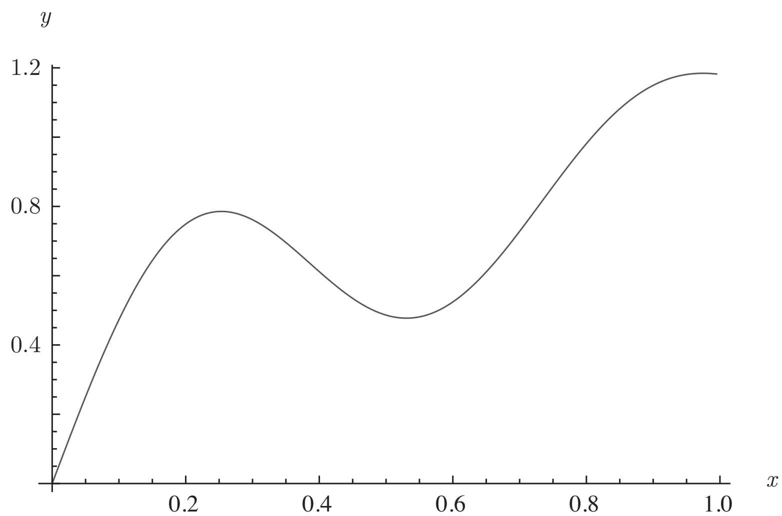


Figure 10.5: The function $y(x)$ which solves the Fredholm equation of the second kind $y(x) = x + \int_0^1 \sin(10xs)y(s)\, ds$.

- Eq. (10.61) can be uniquely solved for all $f(x)$, or
- Eq. (10.62) has a nontrivial nonunique solution.

10.5 Fourier series projection

We can use the eigenfunctions of the linear integral operator as a basis on which to project a general function. This then yields a Fourier series approximation of the general function.

First let us take the inner product to be defined in a typical fashion for functions $u, v \in \mathbb{L}_2[0, 1]$:

$$\langle u, v \rangle = \int_0^1 u(s)v(s)\, ds. \tag{10.63}$$

If $u(s)$ and $v(s)$ are sampled at N uniformly spaced points in the domain $s \in [0, 1]$, with $s_1 = 0$, $s_N = 1$, $\Delta s = 1/(N-1)$, the inner product can be approximated by what amounts to the rectangular method of numerical integration:

$$\langle u, v \rangle \approx \sum_{n=1}^{N-1} u_n v_n \Delta s. \tag{10.64}$$

Then if we consider u_n, v_n, to be the components of vectors $\mathbf{u}$ and $\mathbf{v}$, each of length $N-1$, we can cast the inner product as

$$\begin{aligned} \langle u, v \rangle &\approx (\mathbf{u}^T \cdot \mathbf{v})\Delta s, && (10.65)\\ &\approx \left(\mathbf{u}\sqrt{\Delta s}\right)^T \cdot \left(\mathbf{v}\sqrt{\Delta s}\right). && (10.66) \end{aligned}$$

The functions u and v are orthogonal if $\langle u, v\rangle = 0$ when $u \neq v$. The norm of a function is, as usual,

$$||u||_2 = \sqrt{\langle u, u\rangle} = \sqrt{\int_0^1 u^2(s)\ ds}. \tag{10.67}$$

In the discrete approximation, we have

$$\begin{aligned} ||u||_2 &\approx \sqrt{(\mathbf{u}^T\cdot\mathbf{u})\Delta s}, & (10.68)\\ &\approx \sqrt{\left(\mathbf{u}\sqrt{\Delta s}\right)^T\cdot\left(\mathbf{u}\sqrt{\Delta s}\right)} & (10.69)\end{aligned}$$

Now consider the integral equation defining our eigenfunctions $y(x)$, Eq. (10.32):

$$y(x) = \lambda\int_0^1 K(x,s)y(s)\ ds. \tag{10.70}$$

We restrict attention to problems where $K(x,s) = K(s,x)$. With this, the integral operator is self-adjoint, and the eigenfunctions are thus guaranteed to be orthogonal. Consequently, we are dealing with a problem from Hilbert-Schmidt theory. Discretization, as before, leads to Eq. (10.36):

$$\mathbf{0} = (\mathbf{K}-\sigma\mathbf{I})\cdot\mathbf{y}. \tag{10.71}$$

Because $K(x,s) = K(s,x)$, its discrete form gives $K_{ij} = K_{ji}$. Thus, $\mathbf{K} = \mathbf{K}^T$, and the discrete operator is self-adjoint. We find a set of $N-1$ eigenvectors, each of length $N-1$, $\mathbf{y}_i$, $i = 1,\ldots,N-1$. The eigenvalues are $\sigma_i = 1/(\lambda_i\Delta s)$, $i = 1,\ldots,N-1$.

Now if $y_i(x)$ is the eigenfunction, we can define a corresponding orthonormal eigenfunction $\varphi_i(x)$, by scaling $y_i(x)$ by its norm:

$$\varphi_i(x) = \frac{y_i(x)}{||y_i||_2} = \frac{y_i(x)}{\sqrt{\int_0^1 y_i^2(s)\ ds}}. \tag{10.72}$$

The discrete analog, properly scaled to render $\boldsymbol{\phi}_i$ to be of unit magnitude, is

$$\begin{aligned} \boldsymbol{\phi}_i &= \frac{\mathbf{y}_i\sqrt{\Delta s}}{\sqrt{(\mathbf{y}_i^T\cdot\mathbf{y}_i)\Delta s}}, & (10.73)\\ &= \frac{\mathbf{y}_i}{||\mathbf{y}_i||_2}. & (10.74)\end{aligned}$$

Now for an M-term Fourier series, we approximate $f(x)$ by

$$f(x) \approx f(x_j) = \mathbf{f}_p^T = \sum_{i=1}^{M}\alpha_i\varphi_i(x_j) = \boldsymbol{\alpha}^T\cdot\boldsymbol{\Phi}. \tag{10.75}$$

Here $\mathbf{f}_p$ is an $(N-1)\times 1$ vector containing the projection of f, $\mathbf{\Phi}$ is a matrix of dimension $M \times (N-1)$ with each row populated by a normalized eigenvector $\boldsymbol{\phi}_i$:

$$\mathbf{\Phi} = \begin{pmatrix} \dots & \boldsymbol{\phi}_1 & \dots \\ \dots & \boldsymbol{\phi}_2 & \dots \\ & \vdots & \\ \dots & \boldsymbol{\phi}_M & \dots \end{pmatrix}. \tag{10.76}$$

So if $M = 4$, we would have the approximation

$$f(x) = f(x_j) = \mathbf{f}_p = \alpha_1 \boldsymbol{\phi}_1 + \alpha_2 \boldsymbol{\phi}_2 + \alpha_3 \boldsymbol{\phi}_3 + \alpha_4 \boldsymbol{\phi}_4. \tag{10.77}$$

However, we need an expression for the Fourier coefficients $\boldsymbol{\alpha}$. Now in the continuous limit, $\alpha_i = \int_0^1 f(s)\varphi_i(s)\, ds$. The discrete analog of this is

$$\boldsymbol{\alpha}^T = (\mathbf{f}^T \cdot \mathbf{\Phi}^T), \tag{10.78}$$

$$\boldsymbol{\alpha} = \mathbf{\Phi} \cdot \mathbf{f}. \tag{10.79}$$

The vector $\mathbf{f}$ is of length $N-1$ and contains the values of $f(x)$ evaluated at each x_i. When $M = N-1$, the matrix $\mathbf{\Phi}$ is square, and moreover, orthogonal. Thus, its norm is unity, and its transpose is its inverse. When square, it can always be constructed such that its determinant is unity, thus rendering it to be a rotation. In this case, $\mathbf{f}$ is rotated by $\mathbf{\Phi}$ to form $\boldsymbol{\alpha}$.

We could also represent Eq. (10.75) as

$$\mathbf{f}_p = \mathbf{\Phi}^T \cdot \boldsymbol{\alpha}. \tag{10.80}$$

Using Eq. (10.79) to eliminate $\boldsymbol{\alpha}$ in Eq. (10.80), we can say

$$\mathbf{f}_p = \underbrace{\mathbf{\Phi}^T \cdot \mathbf{\Phi}}_{\mathbf{P}} \cdot \mathbf{f}. \tag{10.81}$$

The matrix $\mathbf{\Phi}^T \cdot \mathbf{\Phi}$ is a projection matrix $\mathbf{P}$:

$$\mathbf{P} = \mathbf{\Phi}^T \cdot \mathbf{\Phi}. \tag{10.82}$$

The matrix $\mathbf{P}$ has dimension $(N-1)\times(N-1)$ and is of rank M. It has M eigenvalues of unity and $N-1-M$ eigenvalues which are zero. If $\mathbf{\Phi}$ is square, $\mathbf{P}$ becomes the identity matrix $\mathbf{I}$, rendering $\mathbf{f}_p = \mathbf{f}$, and no information is lost. That is, the approximation at each of the $N-1$ points is exact. Still, if the underlying function $f(x)$ has fine scale structures, one must take $N-1$ to be sufficiently large to capture those structures.

Example 10.7

Find a Fourier series approximation for the function $f(x) = 1 - x^2$, $x \in [0,1]$, where the basis functions are the orthonormalized eigenfunctions of the integral equation

$$y(x) = \lambda \int_0^1 \sin(10xs)y(s)\, ds. \tag{10.83}$$

We have found the unnormalized eigenfunction approximation $\mathbf{y}$ in an earlier example by solving the discrete equation

$$\mathbf{0} = (\mathbf{K} - \sigma \mathbf{I}) \cdot \mathbf{y}. \tag{10.84}$$

Here $\mathbf{K}$ is of dimension $(N-1) \times (N-1)$, is populated by $\sin(10 x_i s_j)$, $i, j = 1, \ldots, N-1$, and is obviously symmetric.

Let us first select a coarse approximation with $N = 6$. Thus, $\Delta s = 1/(N-1) = 1/5$. This yields the same $\mathbf{K}$ we saw earlier in Eq. (10.38):

$$\mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.389 & 0.717 & 0.932 & 1.000 \\ 0 & 0.717 & 1.000 & 0.675 & -0.058 \\ 0 & 0.932 & 0.675 & -0.443 & -0.996 \\ 0 & 1.000 & -0.058 & -0.996 & 0.117 \end{pmatrix}. \tag{10.85}$$

We then find the eigenvectors of $\mathbf{K}$ and use them to construct the matrix $\mathbf{\Phi}$. For completeness, we present $\mathbf{\Phi}$ for the case where $M = N - 1 = 5$:

$$\mathbf{\Phi}_{5\times 5} = \begin{pmatrix} \ldots & \boldsymbol{\phi}_1 & \ldots \\ \ldots & \boldsymbol{\phi}_2 & \ldots \\ \ldots & \boldsymbol{\phi}_3 & \ldots \\ \ldots & \boldsymbol{\phi}_4 & \ldots \\ \ldots & \boldsymbol{\phi}_5 & \ldots \end{pmatrix} = \begin{pmatrix} 0 & 0.60 & 0.70 & 0.39 & 0.092 \\ 0 & 0.49 & 0.023 & -0.67 & -0.55 \\ 0 & 0.39 & -0.23 & -0.38 & 0.80 \\ 0 & -0.50 & 0.68 & -0.50 & 0.20 \\ 1.0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.86}$$

Now, let us consider an $M = 3$-term Fourier series approximation. Then we will restrict attention to the first three eigenfunctions and consider $\mathbf{\Phi}$ to be a matrix of dimension $M \times (N-1) = 3 \times 5$:

$$\mathbf{\Phi} = \begin{pmatrix} \ldots & \boldsymbol{\phi}_1 & \ldots \\ \ldots & \boldsymbol{\phi}_2 & \ldots \\ \ldots & \boldsymbol{\phi}_3 & \ldots \end{pmatrix} = \begin{pmatrix} 0 & 0.60 & 0.70 & 0.39 & 0.092 \\ 0 & 0.49 & 0.023 & -0.67 & -0.55 \\ 0 & 0.39 & -0.23 & -0.38 & 0.80 \end{pmatrix}. \tag{10.87}$$

Now we consider the value of $f(x)$ at each of the $N-1$ sample points given in the vector $\mathbf{x}$:

$$\mathbf{x} = \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{2}{5} \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \end{pmatrix}. \tag{10.88}$$

At each point $f(x)$ gives us the vector $\mathbf{f}$, of length $N-1$:

$$\mathbf{f} = \begin{pmatrix} 1 \\ \frac{24}{25} \\ \frac{21}{25} \\ \frac{16}{25} \\ \frac{9}{25} \end{pmatrix} = \begin{pmatrix} 1.00 \\ 0.96 \\ 0.84 \\ 0.64 \\ 0.36 \end{pmatrix} \tag{10.89}$$

We can find the projected value $\mathbf{f}_p$ by direct application of Eq. (10.81):

$$\mathbf{f}_p = \underbrace{\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0.60 & 0.49 & 0.39 \\ 0.70 & 0.023 & -0.23 \\ 0.39 & -0.67 & -0.38 \\ 0.092 & -0.55 & 0.80 \end{pmatrix}}_{\mathbf{\Phi}^T} \underbrace{\begin{pmatrix} 0 & 0.60 & 0.70 & 0.39 & 0.092 \\ 0 & 0.49 & 0.023 & -0.67 & -0.55 \\ 0 & 0.39 & -0.23 & -0.38 & 0.80 \end{pmatrix}}_{\mathbf{\Phi}}}_{\mathbf{P}} \underbrace{\begin{pmatrix} 1.00 \\ 0.96 \\ 0.84 \\ 0.64 \\ 0.36 \end{pmatrix}}_{\mathbf{f}},$$

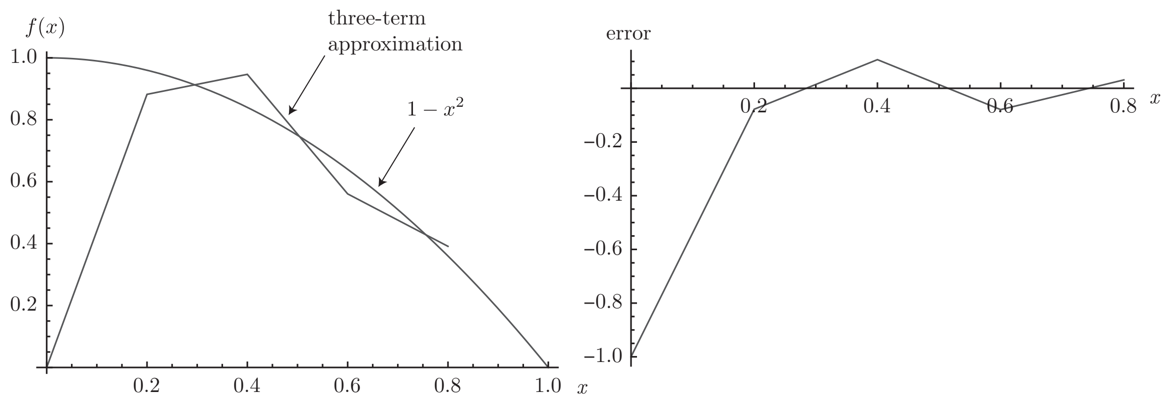


Figure 10.6: $M = 3$-term Fourier approximation of $f(x) = 1 - x^2$, $x \in [0, 1]$ where the basis functions are eigenfunctions of the $N - 1 = 5$-term discretization of the integral operator with the symmetric kernel $K(x, s) = \sin(10xs)$, along with the error distribution.

$$= \begin{pmatrix} 0 \\ 0.88 \\ 0.95 \\ 0.56 \\ 0.39 \end{pmatrix}. \tag{10.90}$$

A plot of the $M = 3$-term approximation for $N - 1 = 5$ superposed onto the exact solution, and in a separate plot, the error distribution, is shown in Fig. 10.6. We see the approximation is generally a good one even with only three terms. At $x = 0$, the approximation is bad because all the selected basis functions evaluate to zero there, while the function evaluates to unity.

The Fourier coefficients $\boldsymbol{\alpha}$ are found from Eq. (10.79) and are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 0.60 & 0.70 & 0.39 & 0.092 \\ 0 & 0.49 & 0.023 & -0.67 & -0.55 \\ 0 & 0.39 & -0.23 & -0.38 & 0.80 \end{pmatrix} \begin{pmatrix} 1.00 \\ 0.96 \\ 0.84 \\ 0.64 \\ 0.36 \end{pmatrix} = \begin{pmatrix} 1.4 \\ -0.14 \\ 0.23 \end{pmatrix}. \tag{10.91}$$

So the Fourier series is

$$\mathbf{f}_p = \alpha_1 \boldsymbol{\phi}_1 + \alpha_2 \boldsymbol{\phi}_2 + \alpha_3 \boldsymbol{\phi}_3 = 1.4 \begin{pmatrix} 0 \\ 0.60 \\ 0.70 \\ 0.39 \\ 0.092 \end{pmatrix} - 0.14 \begin{pmatrix} 0 \\ 0.49 \\ 0.023 \\ -0.67 \\ -0.55 \end{pmatrix} + 0.23 \begin{pmatrix} 0 \\ 0.39 \\ -0.23 \\ -0.38 \\ 0.80 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.88 \\ 0.95 \\ 0.56 \\ 0.39 \end{pmatrix}. \tag{10.92}$$

If we increase $N - 1$, while holding M fixed, our basis functions become smoother, but the error remains roughly the same. If we increase M while holding $N - 1$ fixed, we can reduce the error; we achieve no error when $M = N - 1$. Let us examine a case where $N - 1 = 100$, so the basis functions are much smoother, and $M = 20$, so the error is reduced. A plot of the $M = 20$-term approximation for $N - 1 = 100$ superposed onto the exact solution, and in a separate plot, the error distribution, is shown in Fig. 10.7.

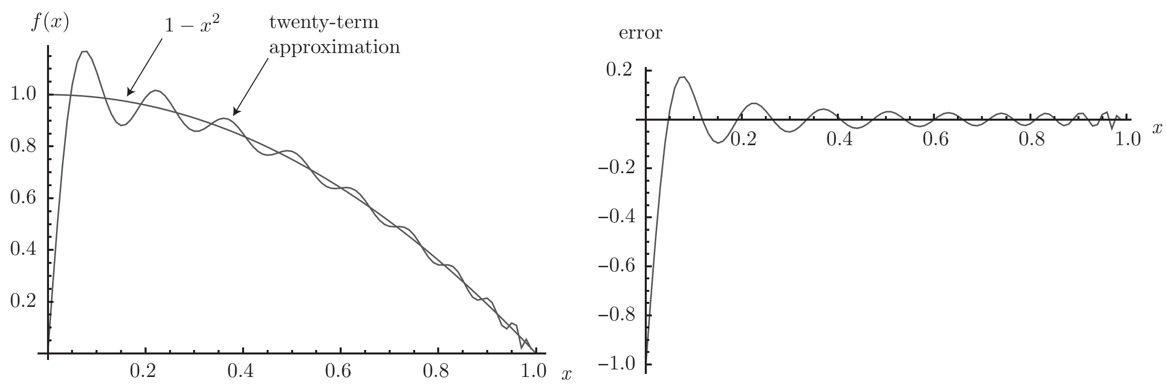


Figure 10.7: $M = 20$-term Fourier approximation of $f(x) = 1 - x^2$, $x \in [0, 1]$ where the basis functions are eigenfunctions of the $N - 1 = 100$-term discretization of the integral operator with the symmetric kernel $K(x, s) = \sin(10xs)$, along with the error distribution.

Problems

1. Solve the Volterra equation
$$a + \int_0^t e^{bs} u(s)\, ds = ae^{bt}.$$
Hint: Differentiate.

2. Find any and all eigenvalues λ and associated eigenfunctions y which satisfy
$$y(x) = \lambda \int_0^1 \frac{x}{s} y(s)\, ds.$$

3. Find a numerical approximation to the first six eigenvalues and eigenfunctions of
$$y(x) = \lambda \int_0^1 \cos(10xs) y(s)\, ds.$$
Use sufficient resolution to resolve the eigenvalues to three digits of accuracy. Plot on a single graph the first six eigenfunctions.

4. Find numerical approximations to $y(x)$ via a process of discretization and, where appropriate, Moore-Penrose pseudoinverse to the equations
   (a)
   $$0 = x + \int_0^1 \cos(10xs) y(s)\, ds.$$
   (b)
   $$y(x) = x + \int_0^1 \cos(10xs) y(s)\, ds.$$
   In each, demonstrate whether or not the solution converges as the discretization is made finer.

5. Find any and all solutions, $y(x)$, which satisfy
   (a) $y(x) = \int_0^1 y(s)\, ds$,
   (b) $y(x) = x + \int_0^1 y(s)\, ds$,
   (c) $y(x) = \int_0^1 x^2 s^2 y(s)\, ds$,
   (d) $y(x) = x^2 + \int_0^1 x^2 s^2 y(s)\, ds$.

6. Using the eigenfunctions $y_i(x)$ of the equation
$$y(x) = \lambda \int_0^1 e^{xs} y(s)\, ds,$$
approximate the following functions $f(x)$ for $x \in [0, 1]$ in ten-term expansions of the form
$$f(x) = \sum_{i=1}^{10} \alpha_i y_i(x),$$
   (a) $f(x) = x$,
   (b) $f(x) = \sin(\pi x)$.
   The eigenfunctions will need to be estimated by numerical approximation.

Bibliography

R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Springer, New York, 1988.

M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, Dover Publications, New York, 1964.

A. A. Andronov, *Qualitative Theory of Second Order Dynamical Systems*, John Wiley & Sons, New York, 1973.

P. J. Antsaklis, and A. N. Michel, *Linear Systems*, Birkhäuser, Boston, 1997.

P. J. Antsaklis, and A. N. Michel, *A Linear Systems Primer*, Birkhäuser, Boston, 2007.

T. M. Apostol, *Calculus: One-Variable Calculus, with an Introduction to Linear Algebra*, Vol. 1, Second Edition, John Wiley & Sons, New York, 1991.

T. M. Apostol, *Calculus: Multi-Variable Calculus and Linear Algebra with Applications to Differential Equations and Probability*, Vol. 2, Second Edition, John Wiley & Sons, New York, 1991.

G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists*, Seventh Edition, Academic Press, Waltham, MA, 2012.

R. Aris, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Dover Publications, New York, 1962.

V. I. Arnold, *Ordinary Differential Equations*, MIT Press, Cambridge, MA, 1973.

V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, New York, 1983.

D. Arrowsmith and C. M. Place, *Dynamical Systems: Differential Equations, Maps, and Chaotic Behaviour*, Chapman Hall/CRC, Boca Raton, FL, 1992.

N. H. Asmar, *Applied Complex Analysis with Partial Differential Equations*, Prentice-Hall, Upper Saddle River, NJ, 2002.

G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press, Cambridge, UK, 1996.

R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.

C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, Springer-Verlag, New York, 1999.

M. L. Boas, *Mathematical Methods in the Physical Sciences*, Third Edition, John Wiley & Sons, New York, 2005.

A. I. Borisenko and I. E. Tarapov, *Vector and Tensor Analysis with Applications*, Dover Publications, New York, 1968.

W. E. Boyce, R. C. DiPrima, and D. B. Meade,*Elementary Differential Equations and Boundary Value Problems*, Twelfth Edition, John Wiley & Sons, Hoboken, NJ, 2022.

K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, SIAM, Philadelphia, 1996.

M. Braun, *Differential Equations and Their Applications*, Springer-Verlag, New York, 1983.

I. N. Bronshtein and K. A. Semendyayev, *Handbook of Mathematics*, Springer, Berlin, 1998.

B. J. Cantwell, *Introduction to Symmetry Analysis*, Cambridge University Press, Cambridge, UK, 2002.

C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, 1988.

J. Carr, *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.

G. F. Carrier and C. E. Pearson, *Ordinary Differential Equations*, SIAM, Philadelphia, 1991.

R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill, New York, 1941.

R. V. Churchill, J. W. Brown, and R. F. Verhey, *Complex Variables and Applications*, Third Edition, McGraw-Hill, New York, 1976.

P. G. Ciarlet, *Introduction to Numerical Linear Algebra and Optimisation*, Cambridge University Press, Cambridge, UK, 1989.

T. B. Co, *Methods of Applied Mathematics for Engineers and Scientists*, Cambridge University Press, Cambridge, UK, 2013.

J. A. Cochran, H. C. Wiser and B. J. Rice, *Advanced Engineering Mathematics*, Second Edition, Brooks/Cole, Monterey, CA, 1987.

E. A. Coddington, *An Introduction to Ordinary Differential Equations*, Dover Publications, New York, 1989.

E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Krieger, Malabar, FL, 1987.

R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vols. 1 and 2, John Wiley & Sons, New York, 1989.

R. Courant, *Differential and Integral Calculus*, Vols. 1 and 2, John Wiley & Sons, New York, 1988.

I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

L. Debnath and P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, Third Edition, Elsevier, Amsterdam, 2005.

J. W. Dettman, *Mathematical Methods in Physics and Engineering*, McGraw-Hill, New York, 1962.

P. G. Drazin, *Nonlinear Systems*, Cambridge University Press, Cambridge, UK, 1992.

R. D. Driver, *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, 1977.

J. Feder, *Fractals*, Plenum Press, New York, 1988.

B. A. Finlayson, *The Method of Weighted Residuals and Variational Principles*, Academic Press, New York, 1972.

C. A. J. Fletcher, *Computational Techniques for Fluid Dynamics*, Second Edition, Springer, Berlin, 1991.

B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, UK, 1998.

B. Friedman, *Principles and Techniques of Applied Mathematics*, Dover Publications, New York, 1956.

I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Dover Publications, New York, 2000.

J. Gleick, *Chaos*, Viking, New York, 1987.

G. H. Golub and C. F. Van Loan, *Matrix Computations*, Fourth Edition, The Johns Hopkins University Press, Baltimore, MD, 2013.

S. W. Goode, *An Introduction to Differential Equations and Linear Algebra*, Prentice-Hall, Englewood Cliffs, NJ, 1991.

B. Goodwine, *Engineering Differential Equations: Theory and Applications*, Springer, New York, 2011.

D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM, Philadelphia, 1977.

M. D. Greenberg, *Foundations of Applied Mathematics*, Prentice-Hall, Englewood Cliffs, NJ, 1978.

M. D. Greenberg, *Advanced Engineering Mathematics*, Second Edition, Pearson, Upper Saddle River, NJ, 1998.

D. H. Griffel, *Applied Functional Analysis*, Dover Publications, New York, 2002.

J. Guckenheimer and P. H. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 2002.

J. Hale and H. Koçak, *Dynamics and Bifurcations*, Springer-Verlag, New York, 1991.

M. T. Heath, *Scientific Computing*, Revised Second Edition, SIAM, Philadelphia, PA, 2018.

F. B. Hildebrand, *Advanced Calculus for Applications*, Second Edition, Prentice-Hall, Englewood Cliffs, NJ, 1976.

E. J. Hinch, *Perturbation Methods*, Cambridge University Press, Cambridge, UK, 1991.

M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, Boston, 1974.

M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, Third Edition, Academic Press, Waltham, MA, 2013.

M. H. Holmes, *Introduction to Perturbation Methods*, Springer-Verlag, New York, 1995.

M. H. Holmes, *Introduction to the Foundations of Applied Mathematics*, Springer-Verlag, New York, 2009.

R. A. Howland, *Intermediate Dynamics: a Linear Algebraic Approach*, Springer, New York, 2006.

J. H. Hubbard and B. B. Hubbard, *Vector Calculus, Linear Algebra, and Differential Forms: a Unified Approach*, Fifth Edition, Matrix Editions, Ithaca, NY, 2015.

M. Humi and W. Miller, *Second Course in Ordinary Differential Equations for Scientists and Engineers*, Springer-Verlag, New York, 1988.

A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Second Edition, Cambridge University Press, Cambridge, UK, 2009.

E. T. Jaynes, *Probability Theory: the Logic of Science*, Cambridge University Press, Cambridge, UK, 2003.

H. Jeffreys and B. Jeffreys, *Methods of Mathematical Physics*, Third Edition, Cambridge University Press, Cambridge, UK, 1972.

D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: an Introduction for Scientists and Engineers*, Fourth Edition, Oxford University Press, Oxford, UK, 2007.

P. B. Kahn, *Mathematical Methods for Engineers and Scientists*, Dover Publications, New York, 2004.

W. Kaplan, *Advanced Calculus*, Fifth Edition, Addison-Wesley, Boston, 2003.

D. C. Kay, *Tensor Calculus*, Schaum's Outline Series, McGraw-Hill, New York, 1988.

J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics*, Springer-Verlag, New York, 1981.

J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods*, Springer-Verlag, New York, 1996.

A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover Publications, New York, 1999.

L. D. Kovach, *Advanced Engineering Mathematics*, Addison-Wesley, Reading, MA, 1982.

E. Kreyszig, *Advanced Engineering Mathematics*, Tenth Edition, John Wiley & Sons, New York, 2011.

E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, 1978.

C. Lanczos, *The Variational Principles of Mechanics*, Fourth Edition, Dover Publications, New York, 2000.

P. D. Lax, *Functional Analysis*, John Wiley & Sons, New York, 2002.

P. D. Lax, *Linear Algebra and its Applications*, Second Edition, John Wiley & Sons, Hoboken, NJ, 2007.

J. R. Lee, *Advanced Calculus with Linear Analysis*, Academic Press, New York, 1972.

R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, Cambridge, UK, 2002.

R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations*, SIAM, Philadelphia, 2007.

A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, Second Edition, Springer, Berlin, 1992.

C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences*, SIAM, Philadelphia, 1988.

J. D. Logan, *Applied Mathematics*, Fourth Edition, John Wiley & Sons, Hoboken, NJ, 2013.

R. J. Lopez, *Advanced Engineering Mathematics*, Addison Wesley Longman, Boston, 2001.

D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles*, Dover Publications, New York, 1989.

J. E. Marsden and A. Tromba, *Vector Calculus*, Sixth Edition, W. H. Freeman, San Francisco, 2011.

J. Mathews and R. L. Walker, *Mathematical Methods of Physics*, Addison-Wesley, Redwood City, CA, 1970.

A. J. McConnell, *Applications of Tensor Analysis*, Dover Publications, New York, 1957.

C. C. Mei, *Mathematical Analysis in Engineering*, Cambridge University Press, Cambridge, UK, 1997.

A. N. Michel and C. J. Herget, *Applied Algebra and Functional Analysis*, Dover Publications, New York, 1981.

R. K. Miller and A. N. Michel, *Ordinary Differential Equations*, Dover Publications, New York, 2007.

P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vols. 1 and 2, McGraw-Hill, New York, 1953.

J. A. Murdock, *Perturbations, Theory and Methods*, SIAM, Philadelphia, 1987.

G. M. Murphy, *Ordinary Differential Equations and Their Solutions*, Dover Publications, New York, 2011.

T. Myint-U and L. Debnath, *Linear Partial Differential Equations*, Birkhäuser, Boston, MA, 2007.

W. L. Oberkampf and C. J. Roy, *Verification, Validation and Uncertainty Quantification in Scientific Computing*, Second Edition, Cambridge, Cambridge, UK.

J. T. Oden and L. F. Demkowicz, *Applied Functional Analysis*, Second Edition, CRC, Boca Raton, FL, 2010.

P. V. O'Neil, *Advanced Engineering Mathematics*, Eighth Edition, Cengage, Boston, CT, 2018.

L. Perko, *Differential Equations and Dynamical Systems*, Third Edition, Springer, Berlin, 2006.

J. M. Powers and M. Sen, *Mathematical Methods in Engineering*, Cambridge University Press, New York, 2015.

item J. M. Powers, *Combustion Thermodynamics and Dynamics*, Cambridge University Press, New York, NY, 2016.

J. M. Powers, *Mechanics of Fluids*, Cambridge University Press, Cambridge, UK, 2024.

W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in Fortran 77*, Cambridge University Press, Cambridge, UK, 1986.

A. Prosperetti, *Advanced Mathematics for Applications*, Cambridge University Press, Cambridge, UK, 2011.

J. N. Reddy, *Applied Functional Analysis and Variational Methods in Engineering*, McGraw-Hill, New York, 1986.

J. N. Reddy and M. L. Rasmussen, *Advanced Engineering Analysis*, John Wiley & Sons, New York, 1982.

R. M. Redheffer, *Differential Equations: Theory and Applications*, Jones and Bartlett, Boston, 1991.

L. F. Richardson, *Weather Prediction by Numerical Process*, Cambridge University Press, Cambridge, UK, 1922.

R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial-Value Problems*, Second Edition, Krieger, Malabar, FL, 1994.

F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Dover Publications, New York, 1990.

K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering: a Comprehensive Guide*, Third Edition, Cambridge University Press, Cambridge, UK, 2006.

P. D. Ritger and N. J. Rose, *Differential Equations with Applications*, Dover Publications, New York, 2010.

P. J. Roache, *Verification and Validation in Computational Science and Engineering*, Hermosa, Albuquerque, NM, 1998.

J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, Cambridge, UK, 2001.

M. Rosenlicht, *Introduction to Analysis*, Dover Publications, New York, 1968.

T. L. Saaty and J. Bram, *Nonlinear Mathematics*, Dover Publications, New York, 2010.

H. Sagan, *Boundary and Eigenvalue Problems in Mathematical Physics*, Dover Publications, New York, 1989.

D. A. Sanchez, R. C. Allen, and W. T. Kyner, *Differential Equations*, Addison-Wesley, Boston,1988.

H. M. Schey, *Div, Grad, Curl, and All That*, Fourth Edition, W.W. Norton, London, 2005.

M. J. Schramm, *Introduction to Real Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1996.

L. A. Segel, *Mathematics Applied to Continuum Mechanics*, Dover Publications, New York, 1987.

D. S. Sivia and J. Skilling, *Data Analysis: a Bayesian Tutorial*, Second Edition, Oxford University Press, Oxford, UK, 2006.

I. S. Sokolnikoff and R. M. Redheffer, *Mathematics of Physics and Modern Engineering*, Second Edition, McGraw-Hill, New York, 1966.

G. Stephenson and P. M. Radmore, *Advanced Mathematical Methods for Engineering and Science Students*, Cambridge University Press, Cambridge, UK, 1990.

G. Strang, *Linear Algebra and its Applications*, Fourth Edition, Cengage, Boston, 2006.

G. Strang, *Introduction to Applied Mathematics*, Wellesley-Cambridge, Wellesley, MA, 1986.

G. Strang, *Computational Science and Engineering*, Wellesley-Cambridge, Wellesley, MA, 2007.

S. H. Strogatz, *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering*, Second Edition, Westview, Boulder, CO, 2015.

R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Second Edition, Springer, New York, 1997.

G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, Ninth Edition, Addison-Wesley, Boston, 1995.

L. N. Trefethen and M. Embree, *Spectra and Pseudospectra: the Behavior of Nonnormal Matrices and Operators*, Princeton University Press, Princeton, NJ, 2005.

L. N. Trefethen and D. Bau, *Numerical Linear Algebra*, SIAM, Philadelphia, 1997.

M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford, CA, 1975.

A. Varma and M. Morbinelli, *Mathematical Methods in Chemical Engineering*, Oxford University Press, Oxford, UK, 1997.

G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1999.

S. Wiggins, *Global Bifurcations and Chaos: Analytical Methods*, Springer-Verlag, New York, 1988.

S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Second Edition Springer-Verlag, New York, 2003.

H. J. Wilcox and L. W. Lamm, *An Introduction to Lebesgue Integration and Fourier Series*, Dover Publications, New York, 2012.

C. R. Wylie and L. C. Barrett, *Advanced Engineering Mathematics*, Sixth Edition, McGraw-Hill, New York, 1995.

D. Xiu, *Numerical Methods for Stochastic Computations*, Princeton University Press, Princeton, NJ, 2010.

E. Zauderer, *Partial Differential Equations of Applied Mathematics*, Third Edition, Wiley, Hoboken, NJ, 2006.

E. Zeidler, *Applied Functional Analysis: Main Principles and Their Applications*, Springer-Verlag, New York, 1995.

E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Springer-Verlag, New York, 1999.

D. G. Zill, *Advanced Engineering Mathematics*, Seventh Edition, Jones and Bartlett, Boston, 2022.