AME 60614: Numerical Methods
Fall 2022, Problem Set 1

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Due: Thursday, 15 September 2022

1. The following discrete data points are were obtained from an unknown function $f(x)$.

n	x_n	$f(x_n)$	n	x_n	$f(x_n)$
1	-4.0000	-0.1245	10	0.0625	0.4706
2	-3.0625	-0.1622	11	0.2500	1.0000
3	-2.2500	-0.2195	12	0.5625	0.7423
4	-1.5625	-0.3120	13	1.0000	0.4706
5	-1.0000	-0.4706	14	1.5625	0.3120
6	-0.5625	-0.7423	15	2.2500	0.2195
7	-0.2500	-1.0000	16	3.0625	0.1622
8	-0.0625	-0.4706	17	4.0000	0.1245
9	0.0000	0.0000			

   (a) Plot the points and draw your best guess of an interpolation of the function.

   (b) Show a Lagrange interpolation of the function using all of the points.

   (c) Show a Lagrange interpolation of the function using only the even points then only the odd points.

   (d) Show piecewise Lagrange interpolations, first linear then on three subsets of the points. (i.e. $n =$ 1-7, 7-11, 11-17)

   (e) Show a cubic spline interpolation of the function. Show two different values of tension on the cubic spline.

   Submit one labeled graph for each part and comment on each.

2. Interpolate the ellipse given by the equation $x^2 + 4y^2 = 4$ using a parametric cubic spline with $N = 4$, 8, and 16, unique and approximately evenly spaced points. How many points do you have to use to apply appropriate boundary conditions? Now, see how rapidly the error converges to zero as you increase N. Plot the maximum error of the interpolation against the number of points on a log-log scale. Be sure to include enough data points to verify this trend up to a very large value of N. How small an error can you attain?

3. Find the most accurate finite difference formula and the corresponding leading error term for the following derivatives:

   (a) $f''(x)$

   (b) $f^{(iv)}(x)$

   (c) $f'''(x) - 3f'(x)$

   to the highest order of accuracy possible using only the following function values.

   i f_{i-2}, f_{i-1}, f_i, f_{i+1}, and f_{i+2}

   ii f_i, f_{i+1}, f_{i+2}, f_{i+3}, and f_{i+4}

   iii f_{i-4}, f_{i-3}, f_{i-2}, f_{i-1}, and f_i

   iv f_{i-1}, f_i, f_{i+1}, f'_{i-1}, f'_i, and f'_{i+1}

4. Use the second order accurate second derivative central difference formula

$$f''(x) \simeq \frac{f(x+h) - 2f(x) + f(x-h)}{h^2},$$

along with Richardson extrapolation to obtain a fourth and then sixth order accurate second derivative finite difference formula with leading order error. How does this result compare to the results from question 3(a)i?

5. Show the following:

   (a) $\Delta f(x) = \nabla E f(x)$
   (b) $\nabla f(x) = \Delta E^{-1} f(x)$
   (c) $\nabla \Delta f(x) = \Delta \nabla f(x) = \delta^2 f(x)$
   (d) $\Delta^n f(x) = \nabla^n f(x + nh) = \delta^n f\left(x + \frac{nh}{2}\right)$
   (e) $\Delta\left(f(x)g(x)\right) = f(x)\Delta g(x) + g(x+h)\Delta f(x)$
   (f) $\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

6. Integrate following functions of x

   (a) $\exp(x - \frac{\pi}{4})$,
   (b) $1 - \left|x - \frac{\pi}{4}\right|^{\frac{1}{3}}$,
   (c) $(x - \pi/4)^2 + \frac{3}{4} + \frac{x}{500\left(x-\frac{\pi}{4}\right)^2 + \pi}$,

   on the domain $x \in [0, 1]$. Calculating the actual integrals for each function. Then estimate the integrals using the following quadrature methods:

   i Trapezoid Rule,
   ii Simpson's Rule,
   iii 3-Point Gauss Quadrature,
   iv Adaptive Quadrature.

   Start with a few points (high target error for the adaptive quadrature) and increase the number of points (decrease target error for the adaptive quadrature) in order to generate a log-log plot for each function plotting the error against the number of points. See how small you can get your error; using a compiled language such as `Fortran` or `C` will speed this intensive calculation. Discuss your results.