AME 60614: Numerical Methods
Fall 2022, Problem Set 2

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Due: Thursday, October 6, 2022

1. The population growth of many kinds of insects is sometimes modeled by the following ordinary differential equation:

$$\frac{dN}{dt} = \alpha N - \beta N^2, \qquad N(0) = N_o,$$

where N is the population, α is the birth rate coefficient, and β is the death rate coefficient. If $N_o = 10^5$ *insect*, $\alpha = 0.1$ day^{-1}, and $\beta = 8 \times 10^{-7}$ $insect^{-1} day^{-1}$, what is the population of this kind after 20 years?

2. The following ordinary differential equation model a simple one-step chemical reaction:

$$\frac{d\lambda}{dt} = 2500(1-\lambda)\exp\left[\frac{-E}{T}\right],$$

where T is the temperature, E is the activation energy, and λ is the reaction progress variable, $0 \le \lambda \le 1$. For a specific case $E = 50$ and $T = 12 + 1.9\sqrt{1-\lambda} - 9(1-\lambda)$,

   (a) using any explicit scheme, plot the temperature distribution, for $\lambda(0) = 0, \;\; t \in [0, 1.5]$.

   (b) perform a convergence study by plotting the norm of the relative error L_2 vs. the grid size h.

3. Solve the following ODE:

$$y' = t^2 \; y \;\; \cos(y+t)^3, \qquad y(0) = 1, \qquad t \in [0, 3],$$

   (a) using the following schemes:

      i. Leapfrog.

      ii. Trapezoidal.

      iii. Second order Runge-Kutta.

      iv. Fourth order Runge-Kutta.

   (b) Investigate experimentally the order of accuracy of each scheme by performing a convergence study. Plot the norm of the relative error L_∞ vs. the grid size h on a log-log scale and estimate the order of accuracy. (At least show five solutions.)

   (c) Discuss your results.

4. The following numerical scheme:

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}\right],$$

is proposed. By utilizing the model problem, $y' = \lambda y$,

(a) classify this scheme in terms of consistency and order of accuracy.

(b) Derive the characteristic equation and find its roots. Find the order of accuracy of this scheme by expanding the roots of the characteristic equation in powers of h.

(c) Investigate the stability of this scheme for:

i. λ purely real.

ii. λ purely imaginary.

(d) Discuss your results.

5. For the following set of ODEs:

$$\begin{aligned} y' &= -4z - 0.1y, \quad y(0) = 1, \\ z' &= -2 \times 10^4 z, \quad z(0) = 1, \end{aligned}$$

where $t \in [0, 1]$,

(a) Find the eigenvalues and the stiffness ratio for this system.

(b) Solve this system using any explicit Runge-Kutta scheme. In order to yield a solution, what is the largest grid size?

(c) Discuss your results.

6. For the following multistep numerical scheme:

$$\begin{aligned} y^*_{n+1} &= y_n + hf_n, \\ y^{**}_{n+1} &= y_n + \frac{h}{2}\left[f^*_{n+1} - f_n\right], \\ y_{n+1} &= y_n + \frac{h}{2}\left[(1-\alpha)f^{**}_{n+1} + \alpha f^*_{n+1} + f_n\right]. \end{aligned}$$

(a) By using the model equation, $y' = \lambda y$, what is the maximum order of accuracy of this scheme?

(b) Plot the stability region for $\alpha = [0, 0.5, 0.75, \text{and } 1]$.