AME 60614: Numerical Methods
Fall 2022, Problem Set 3
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Due: Thursday, November 10, 2022

1. Consider the following nonlinear boundary value problem

$$f'' = f^3 - ff', \tag{1}$$

with $1 \leq x \leq 2$, $f(1) = 1/2$, and $f(2) = 1/3$.

   (a) Find the actual solution $f(x)$.

   (b) Use the direct method to approximate the solution to (1). Plot the error norm $||\cdot||_\infty$ versus $h = \{10^{-1}, 10^{-2}, \ldots, 10^{-5}\}$ on a log-log scale for each fixed number of iterations $N = \{10^1, 10^2, \ldots 10^5\}$.

2. Consider the following nonlinear boundary value problem

$$f'' = -(f')^2 - f + \ln x, \tag{2}$$

with $x \in [1, 2]$, $f(1) = 0$, and $f(2) = \ln 2$.

   (a) Find the actual solution $f(x)$.

   (b) Assume an initial value $f'(1) = 4$. Use the nonlinear shooting method and Newton's method to approximate the solution to (2). For each fixed number of iterations $N = \{1, \ldots, 6\}$, plot the error norm $||\cdot||_1$ at $t = 2$ versus $h = \{10^{-2}, 10^{-3}, \ldots, 10^{-9}\}$ on a log-log scale.

   (c) For each step size $h = \{10^{-2}, 10^{-3}, \ldots, 10^{-9}\}$, determine the iteration step $k > 2$ such that $||\cdot||_1$ at $t = 2$ for $k - 1$ iterations is less than $||\cdot||_1$ at $t = 2$ for k iterations.

3. Consider the following simple system of two linear differential equations:

$$\begin{aligned} x' &= \alpha x + \beta y, \\ y' &= \beta x + \alpha y, \end{aligned} \tag{3}$$

with initial conditions $x(0) = 2$ and $y(0) = 0$.

   (a) Find the solution $x(t)$ and $y(t)$ to the system of differential equations.

   (b) Derive the conditions on α and β such that $x(t)$ and $y(t)$ exponentially decay to 0.

   (c) Derive the conditions on α and β such that (3) is stiff.

   (d) Write out the numerical algorithm for solving (3) using the explicit Euler method.

(e) Find the solution x_n and y_n to the difference equation in terms of α, β, h, and n.

(f) Derive the condition on h such that the numerical solution will mimic the behavior of the exact solution.

(g) Repeat steps (d)-(f) using the implicit trapezoidal method.

4. Consider the following two-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{4}$$

Let us assume a uniform grid in x, y and a constant time step.

(a) Derive the numerical finite difference method for (4) using a forward difference in time and the formula for the Laplacian based on a centered difference in both x and y.

(b) Derive the stability condition (ignoring the boundary condition) for the numerical finite difference method determined in the previous step.

5. Einstein's most cited paper does not deal with relativity but rather Brownian motion (*Ann. Phys., Lpz* (17): 549–560, 1905). It is the founding document in the theory of stochastic processes and ultimately provided a link between deterministic and statistical models. The rough idea of this link can easily be seen by working through the following problem. Suppose that in a three-dimensional random walk, at each time step h it is equally likely to move right δx or left or up δz or down or forward δy or back.

(a) Derive the difference equation for this problem.

(b) Derive the partial differential equation governing this process if δx, δy, $\delta z \to 0$ and $h \to 0$ such that

$$\begin{aligned}
\lim_{\delta x, h \to 0} \frac{\delta x^2}{h} &= \kappa, \\
\lim_{\delta y, h \to 0} \frac{\delta y^2}{h} &= \kappa, \\
\lim_{\delta z, h \to 0} \frac{\delta z^2}{h} &= \kappa.
\end{aligned}$$

(c) Assume that the probability of staying in place is zero and the initial position of the random walker is known with certainty. Compute four different random walker trajectories over 10^6 time steps and compare them graphically.

6. Similar to the previous problem, suppose that in a one-dimensional random walk, at each time step h the probability of moving to the right k is a, and the probability of moving to the left k is also a. The probability of staying in place is b.

(a) Derive the difference equation for this problem.

(b) Derive the partial differential equation governing this process if $k \to 0$ and $h \to 0$ such that

$$\lim_{k,h \to 0} \frac{k^2}{h} = \kappa.$$

(c) Assume that the probability of staying in place is zero and the initial position of the random walker is known with certainty. Determine the most likely location of the random walker and the associated probability at the following three time steps:

i. $N_1 = 1 \times 10^5$

ii. $N_2 = 5 \times 10^5$

iii. $N_3 = 1 \times 10^6$

(d) Using your numerical intuition, what is the most likely location of the random walker after an arbitrary number of time steps.

7. Consider Bessel's differential equation of order zero given by

$$x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + \mu^2 x^2 f = 0, \tag{5}$$

with $f'(0) = 0$ and $f(1) = 0$.

(a) Use the direct method to solve for the eigenvalues μ.

(b) Numerically approximate ten successive eigenfunctions on a single plot.