AME 60614: Numerical Methods
Fall 2022, Problem Set 4

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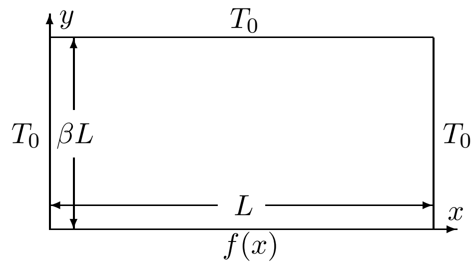
Due: Thursday, December 8, 2022

1. The cross section of a rectangular bar with aspect ratio β has temperature distribution $T(x, y, t)$. The bar is sufficiently long, and has uniformity along its z-dimension, so it can be idealized as a two-dimensional problem. Three sides of the bar are kept at a uniform temperature T_0. The fourth side of the bar is subject to a temperature profile $f(x)$.



This temperature profile is governed by the partial differential equation

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right),$$

where α is the coefficient of thermal diffusivity. This equation has the analytical solution for $T^*(x, y, t) = T(x, y, t) - T_0$ of

$$T^*(x,y,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \left[\sinh\left(n\pi\left(\beta - \frac{y}{L}\right)\right) - \sum_{m=1}^{\infty} B_{n,m} \sin\left(\frac{m\pi y}{\beta L}\right) \exp\left(\frac{-\alpha\pi^2(n^2\beta^2 + m^2)t}{\beta^2 L^2}\right)\right],$$

where the coefficients for $f(x) = 1 + T_0$ are $A_n = \frac{2(1-(-1)^n)}{n\pi \sinh(n\pi\beta)}$ and $B_{n,m} = \frac{m(\exp(n\pi\beta) - \exp(-n\pi\beta))}{\pi(n^2\beta^2 + m^2)}$. Set up the finite difference equation using the second order central difference method in the spatial dimensions and the explicit Euler method in the temporal dimension. Solve the equation with $\Delta x = \frac{1}{100}$ m, $\Delta y = \frac{1}{400}$ m, $L = 1$ m, $\beta = \frac{1}{4}$, $\alpha = 294 \frac{\text{m}^2}{\text{s}}$, and the initial temperature profile of $T(x, y) = T_0$. Compare the numerical solution to the analytical solution at $t = 1.79 \times 10^{-6}$ s, 3.06×10^{-6} s, 1.770×10^{-5} s, 2.2520×10^{-4} s, and 4.5845×10^{-4} s.

2. Take the finite difference equation from problem 1 and set $\Delta x = \Delta y = h$. Show that by substituting in the formula for maximum time step allowed for stability, the difference equation is equivalent to the Jacobi method. Investigate the convergence of the Jacobi method; using the PDE, calculate what the residual of the Jacobi method is equal to in terms of α, h, and $\frac{\partial T}{\partial t}$. Now, estimate the number of iterations, n_{itr}, necessary to converge to a residual with a norm of 10^{-10} for each mode (i.e. $f(x) = \sin(n\pi x)$) from $n = 1$ up to the largest allowable n by Nyquist sampling. Use the analytic solution, $A_n = (\sinh(n\pi\beta))^{-1}$, $t = n_{\text{itr}}\Delta t$, and $h = \frac{1}{100}$, then check your estimations numerically. What norm are you estimating? Plot your estimated and actual iterations against the mode n on one graph. Describe how well your estimation matches and why. Now, use the weighted Jacobi method with $\omega = \frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ and plot the number of iterations that each of these methods takes to converge to the same norm.

3. Solve for the steady state temperature profile of the bar from problems 1 and 2, now with $f(x) = \sin(\pi x)$, using the Jacobi method (using the optimum ω found in problem 2), Gauss-Seidel method, and a direct method (i.e. using matrix inversion) on varying grid sizes. Start with a course grid and work your way up to a fine grid (I recommend using $h = \frac{1}{2^n}$). Obtain the maximum convergence possible for each method on each grid (the direct method may not be feasible for the finer grids) and record the norm of the residual. Compare the results to the analytical steady state temperature, which is

$$T(x,y) = \frac{\sin(\pi x)\sinh\left(\pi\left(\frac{1}{4} - y\right)\right)}{\sinh\left(\frac{\pi}{4}\right)}.$$

Plot the norms of the difference between the numerical results and the analytical result and the norms of the residual against the number of grid points. Contrast the two types of convergence.

4. Consider the following hyperbolic PDE:

$$\frac{\partial u}{\partial t} - \frac{\partial xu}{\partial x} = 0, \qquad x \in [0,5], \quad t \in [0,\infty].$$

(a) Find the the slope of the characteristic lines, and plot three of them in the (x,t) plane.

(b) Solve this equation for the following initial distribution and boundary condition:

$$\begin{aligned} u(x,0) &= \exp[-100\pi(0.5-x)^2], \\ u(0,t) &= 0, \end{aligned}$$

i. using Crank-Nicolson scheme,

ii. using Lax-Wendroff scheme.

(c) Plot the solution at $t = 0.1, 0.5$ and 1.

5. In 1978, Leonard[1] proposed a modification to the well-known upwind scheme. With this modification the truncation error of this explicit method become $\mathcal{O}(\Delta x^2)$.

(a) By utilizing the model convection-diffusion problem, $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$, find the modified differential equation (MDE)?

(b) Using von Neumann analysis, check the stability of Leonard scheme.

(c) Use Leonard's scheme to solve the following partial differential equation,

$$\frac{\partial u}{\partial t} + (1-u)\frac{\partial u}{\partial x} = 10^{-3}\frac{\partial^2 u}{\partial x^2},$$

$$u(x,0) = 1 - \sin\left(\frac{\pi x}{5}\right), \qquad x \in [0,10],$$

which represent the viscous Burger's equation. Plot the solution at $t = 0.005, 0.05, 0.5$, and 5.

[1] B. P. Leonard, M. A. Leschziner, M. McGuirk, *Num. Meth. Laminar Turbulent Flow*, 1978, p. 807.