AME 60614
Examination 1
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1. (30) Consider the following data

x	$f(x)$
0	1
1	3
3	6

Generate a global Lagrange interpolating polynomial to fit the data. Estimate $df/dx|_{x=0}$ with a first order method and then do the same for the highest order estimate that employs the global Lagrange interpolating polynomial.

Solution

For the data given

$$\begin{aligned} L_0(x) &= \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3), \\ L_1(x) &= \frac{x(x-1)}{(1-0)(1-3)} = \frac{1}{2}x(3-x), \\ L_2(x) &= \frac{x(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1). \end{aligned}$$

Our Lagrange interpolating polynomial is then

$$\begin{aligned} f_a(x) &= f_0 L_0(x) + f_1 L_1(x) + f_2 L_2(x), \\ &= (1)\frac{1}{3}(x-1)(x-3) + (3)\frac{1}{2}x(3-x) + (6)\frac{1}{6}x(x-1), \\ &= 1 + \frac{x}{6}(13-x). \end{aligned}$$

The simple first order estimate for the derivative is

$$\frac{df}{dx} \sim \frac{f(1)-f(0)}{1-0} = \frac{3-1}{1-0} = 2.$$

Now differentiate the Lagrange interpolating polynomial:

$$\frac{df_a}{dx} = \frac{1}{6}(13-2x).$$

At $x = 0$, we get

$$\left.\frac{df_a}{dx}\right|_{x=0} = \frac{13}{6} = 2.16667.$$

2. (40) Consider the system

$$\begin{aligned}\frac{dy_1}{dt} &= -2y_1 + y_2, \qquad y_1(0) = 1,\\ \frac{dy_2}{dt} &= y_1 - 2y_2, \qquad y_2(0) = 0.\end{aligned}$$

Determine if the exact solution is stable. With $\mathbf{y} = (y_1, y_2)^T$, and taking the step size $\Delta t = h$, cast the forward Euler approximation to the solution in the form

$$\mathbf{y}^{(n+1)} = \mathbf{B} \cdot \mathbf{y}^{(n)}.$$

Find $\mathbf{B}$. Determine a condition on h for the method to be stable.

Solution

With $\mathbf{y} = (y_1, y_2)^T$, we have

$$\frac{d\mathbf{y}}{dt} = \mathbf{A} \cdot \mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}_o.$$

Here

$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \qquad \mathbf{y}_o = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are found by insisting that

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0.$$

So

$$\begin{aligned}(-2-\lambda)^2 - 1 &= 0,\\ -2-\lambda &= \pm 1,\\ -2 \mp 1 &= \lambda.\end{aligned}$$

Thus

$$\lambda = -1, \qquad \lambda = -3.$$

The exact solution is *stable.* Calculation reveals the exact solution that satisfies the initial conditions to be

$$\begin{aligned}y_1(t) &= \frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t},\\ y_2(t) &= -\frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t}.\end{aligned}$$

The forward Euler method gives

$$\begin{aligned}\mathbf{y}^{(n+1)} &= \mathbf{y}^{(n)} + h\mathbf{A} \cdot \mathbf{y}^{(n)},\\ &= (\mathbf{I} + h\mathbf{A}) \cdot \mathbf{y}^{(n)}.\end{aligned}$$

With

$$\mathbf{B} = \mathbf{I} + h\mathbf{A} = \begin{pmatrix} 1-2h & h \\ h & 1-2h \end{pmatrix},$$

we have

$$\mathbf{y}^{(n+1)} = \mathbf{B} \cdot \mathbf{y}^{(n)}.$$

The eigenvalues of $\mathbf{B}$ are $1 + h\lambda_i$, where λ_i are the eigenvalues of $\mathbf{A}$. So they are $1 - h$ and $1 - 3h$. The method is stable if $||\mathbf{B}|| < 1$. The norm is given by the largest of the singular values

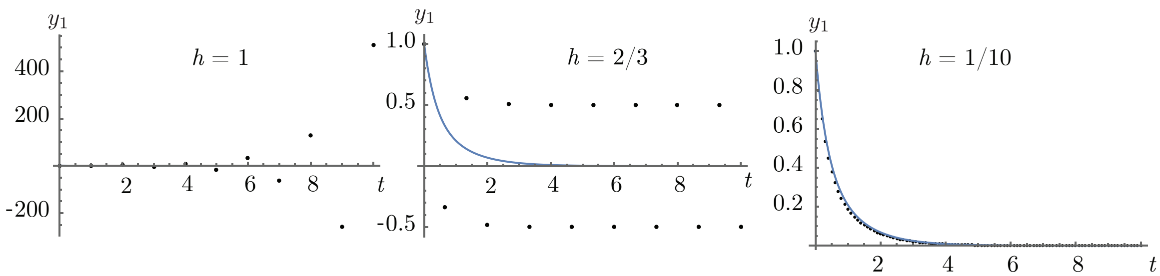

Figure 1: Plot of $y_1(t)$ for $h = 1, 2/3, 1/10$.

of $\mathbf{B}$. The singular values are the positive square roots of the eigenvalues of $\mathbf{B}^T \cdot \mathbf{B}$. Because $\mathbf{B}$ is symmetric, these simply become the magnitudes of the eigenvalues of $\mathbf{B}$:

$$\sigma_1 = |1 - h|, \qquad \sigma_2 = |1 - 3h|.$$

By inspection $\sigma_1 < 1$ if $0 < h < 2$ and $\sigma_2 < 1$ if $0 < h < 2/3$. The most restrictive requirement for stability is

$$0 < h \leq \frac{2}{3} = \frac{2}{|\lambda_{max}|},$$

where $|\lambda_{max}| = 3$. Plots of approximations to $y_1(t)$ for various h are shown in Fig. 1. One sees how the numerical approximation is either unstable, neutrally stable, or stable, depending on h.

3. (30) Consider the model problem $dy/dt = \lambda y$, $y(0) = 1$. One can apply the leapfrog method to estimate an approximate soltuion:

$$\frac{y_{n+1} - y_{n-1}}{2h} = \lambda y_n.$$

Through use of Taylor series, a) find the *modified equation* for which the leapfrog method provides a better approximation, b) show the leapfrog method is a consistent method, c) prepare an exact solution to the modified equation and find the algebraic equation whose solution is required to ascertain the stability of the leapfrog method.

Solution

Let us do a Taylor series expansion

$$\begin{aligned}
\frac{y + h\frac{dy}{dt} + \frac{h^2}{2}\frac{d^2y}{dt^2} + \frac{h^3}{6}\frac{d^3y}{dt^3} \cdots - \left(y - h\frac{dy}{dt} + \frac{h^2}{2}\frac{d^2y}{dt^2} - \frac{h^3}{6}\frac{d^3y}{dt^3} + \ldots\right)}{2h} &= \lambda y, \\
\frac{2h\frac{dy}{dt} + \frac{h^3}{3}\frac{d^3y}{dt^3} + \ldots}{2h} &= \lambda y, \\
\frac{dy}{dt} + \frac{h^2}{6}\frac{d^3y}{dt^3} + \ldots &= \lambda y.
\end{aligned}$$

As $h \to 0$, the modified equation approaches the exact equation, $dy/dt = \lambda y$, so the approximation is *consistent*. If we assume $y = ce^{rt}$, we are led to the characteristic polynomial

$$r + \frac{h^2}{6}r^3 = \lambda.$$

For small h, one root is obviously $r \sim \lambda$. This is stable for $\Re(\lambda) < 0$, But there are two other roots as well. Solving the general cubic is hard. If we solve it for $\lambda = -1$, $h = 1$, we get

$$r = -0.884622, \qquad r = 0.442311 \pm 2.5665i.$$

The first mode is stable, but the oscillatory mode is unstable as it has a positive real part. If we let h shrink to $h = 1/1000$, and keep $\lambda = -1$, we find

$$r = -1, \qquad r = 0.5 \pm 2449.49i.$$

For $h = 1/100000$, we get

$$r = -1, \qquad r = 0.5 \pm 244949i.$$

The frequency of oscillation increases, and those modes remain unstable. Detailed Taylor series analysis for $\lambda = -1$ gives the three roots in the limit as $h \to 0$ as

$$r = -1 + \frac{h^2}{6} - \frac{h^4}{12} + \ldots, \qquad r = \frac{1}{2} - \frac{h^2}{12} + \cdots \pm i\left(\frac{\sqrt{6}}{h} + \frac{1}{8}\sqrt{\frac{3}{2}}h - \frac{35}{256\sqrt{6}}h^3 + \ldots\right).$$

The first root is associated with the original ordinary differential equation. The second two are spurious and a consequence of the numerical scheme only. Worse, the presence of a positive real part shows they induce instabilities that persist as $h \to 0$.

One might also examine the amplification factor, but this was not the suggested method. As shown in lecture, the leapfrog method leads to

$$y_{n+1} = y_{n-1} + 2\lambda h y_n.$$

With $y_n = \sigma^n c$, this leads to

$$\sigma^2 - 2\lambda h\sigma - 1 = 0.$$

There are two roots to this

$$\sigma = \lambda h \pm \sqrt{\lambda^2 h^2 + 1}.$$

Taylor series of the positive root gives

$$\sigma_1 = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 - \frac{1}{8}\lambda^4 h^4 + \ldots.$$

This is the physical root, and indicates this mode has second order accuracy. For λ real and negative, we see $|\sigma| < 1$, and this is stable. However, there is a spurious root

$$\sigma = -1 + \lambda h - \frac{1}{2}\lambda^2 h^2 + \frac{1}{8}\lambda^4 h^4 - \ldots.$$

For λ real and negative, this mode has $|\sigma| > 1$, so it is an unstable mode.