AME 538
Examination 2: **Solution**
Prof. J. M. Powers
6 December 2002

1. (20) A uniform stream with velocity U in the x direction combines with a source of strength m at $(a, 0)$ and a sink of strength $-m$ at $(-a, 0)$. Plot the resulting streamlines, note any stagnation points and closed-body streamlines. If the far field pressure is p_o, find the pressure at the leading edge stagnation point.

Solution

Here we have a complex potential

$$W(z) = Uz + \frac{m}{2\pi}\ln(z-a) - \frac{m}{2\pi}\ln(z+a).$$

Using the properties of the natural logarithm, we get,

$$W(z) = Uz + \frac{m}{2\pi}\ln\left(\frac{z-a}{z+a}\right).$$

Taking the derivative, we get

$$\frac{dW}{dz} = u - iv = U + \frac{m}{\pi}\frac{a}{z^2-a^2}.$$

$$u - iv = U + \frac{m}{\pi}\frac{a}{(x+iy)^2-a^2}.$$

Realizing that because of symmetry the stagnation point will be on the x axis, where $y = 0$, we have on the x axis

$$(u-iv)|_{y=0} = U + \frac{m}{\pi}\frac{a}{x^2-a^2}.$$

Now the imaginary part of this expression is zero, so on the x axis, we have $v = 0$. So for a stagnation point, we then take

$$u = U + \frac{m}{\pi}\frac{a}{x^2-a^2} = 0.$$

Solving for x, we find stagnation points at

$$x = \pm a\sqrt{1 - \frac{m}{U\pi a}}.$$

It can be seen easily be direct visualization of the stream function, or shown algebraically that the stagnation point needs to have $|x| > a$ to allow for a closed contour streamline, which results, for $m > 0, a > 0$ in

$$U < 0.$$

A set of streamfunctions and velocity potentials is shown in Fig. 1.

Bernoulli's equation for this flow field gives

$$\left(p + \frac{1}{2}\rho|\mathbf{u}|^2\right)_{farfield} = p + \frac{1}{2}\rho(u^2+v^2).$$

In the far field $p = p_o$ and $|\mathbf{u}|^2 = U^2$. At the stagnation point $u = v = 0$, so at the stagnation point, we have

$$p_o + \frac{1}{2}\rho U^2 = p.$$

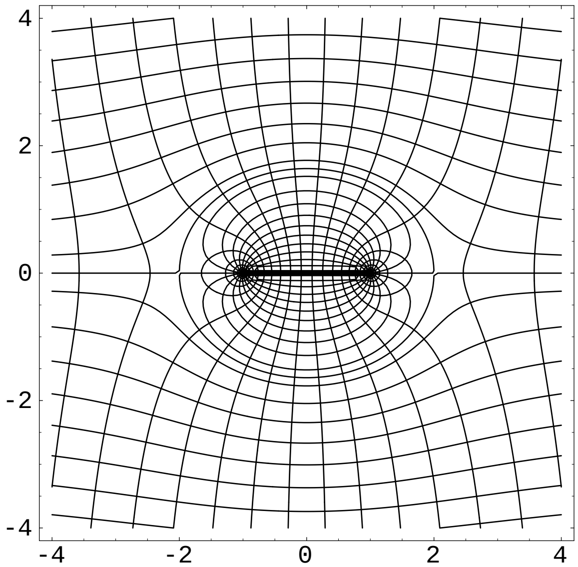

Figure 1: Streamlines and velocity potential for $m = \pi$, $U = -1/3$, $a = 1$.

2. (20) A complex potential is given by

$$W(z) = 2z + \frac{1}{z} + \frac{1}{z^2}.$$

Find the velocity vector at the point $(x, y) = (1, 1)$.

Solution

Here, we take the derivative of the complex potential to find the complex velocity field.

$$W(z) = 2z + \frac{1}{z} + \frac{1}{z^2}.$$

$$\frac{dW}{dz} = 2 - \frac{1}{z^2} - \frac{2}{z^3}.$$

$$u - iv = 2 - \frac{1}{(x+iy)^2} - \frac{2}{(x+iy)^3}.$$

At the point $(1, 1)$, we then have

$$u - iv = 2 - \frac{1}{(1+i)^2} - \frac{2}{(1+i)^3}.$$

$$u - iv = 2 - \frac{(1-i)^2}{4} - \frac{2(1-i)^3}{8}.$$

$$u - iv = 2 + \frac{i}{2} + \frac{i+1}{2}.$$

$$u - iv = 5/2 + i.$$

So

$$u = \frac{5}{2}, \qquad v = -1.$$

3. (30) A shock is driven into air by a piston moving at 300 m/s. The still air has a temperature of 300 K and a pressure of 100 kPa.

   (a) Assuming air to be a calorically perfect ideal gas with $R = 287\ J/kg/K$ and $\gamma = 7/5$, calculate the speed of the shock, and the pressure after the passage of the shock.

   (b) Assuming air to be an ideal gas with $R = 287\ J/kg/K$ and calorically imperfect with
$$e = a_0 + a_1 T + a_2 T^2,$$
   with $a_0 = 4640\ J/kg$, $a_1 = 706\ J/kg/K$, $a_2 = 0.062\ J/kg/K^2$, pose the resulting system of non-linear algebraic equations which could be used to solve for the shock speed and post-shock pressure, as well as other variables. Do not solve.

Solution

We use what is effectively the equation derived in class for the piston velocity:
$$D = \frac{\gamma+1}{4} v_p + \sqrt{\gamma R T_1 + v_p^2 \left(\frac{\gamma+1}{4}\right)^2}.$$
We take $\gamma = 7/5$, $v_p = 300\ m/s$, $p_1 = 100000\ Pa$, $T_1 = 300\ K$ to get
$$D = 571.075\ m/s.$$
We then use another equation derived in class
$$p_2 = \frac{2}{\gamma+1} \rho_1 D^2 - \frac{\gamma-1}{\gamma+1} p_1,$$
to get
$$p_2 = 298981\ Pa.$$
For the non-ideal state equation, we need to solve the following six jump conditions for the six unknowns ρ_2, u_2, D, p_2, h_2, and T_2.
$$\begin{aligned} \rho_2 u_2 &= -\rho_1 D, \\ \rho_2 u_2^2 + p_2 &= \rho_1 D^2 + p_1, \\ h_2 + \frac{u_2^2}{2} &= h_1 + \frac{D^2}{2}, \\ h_2 &= (a_o + a_1 T_2 + a_2 T_2^2) + R T_2, \\ p_2 &= \rho_2 R T_2, \\ u_2 &= v_p - D \end{aligned}$$
Here we consider h_1 and ρ_1 as knowns, which are easily computed from $\rho_1 = p_1/R/T_1$ and $h_1 = (a_o + a_1 T_1 + a_2 T_1^2) + R T_1$. Solving for the non-ideal shock state, we find that
$$D = 567.187\ m/s, \qquad p_2 = 297626\ Pa.$$
So the fact that the calorically imperfect gas has additional vibrational and rotational modes to absorb piston energy results in a slower wave at a lower pressure.

4. (30) Sir Isaac Newton, living in an era in which Boyle's Law and Charles' Law were well understood, but entropy and basic thermodynamic principles were not, was mistakenly inclined to think of gas dynamics as an isothermal process. Assuming the energy equation is replaced in favor of an isothermal condition,

(a) Write the mass and momentum equations for one-dimensional inviscid, unsteady, flow of a calorically perfect ideal gas, reducing them to be in terms of the two unknowns ρ and u.

(b) Write these equations in characteristic form.

(c) Use this flawed theory to estimate the speed of sound.

(d) Compare a Newtonian estimate of the speed of sound in calorically perfect ideal air at 300 K to that of an isentropic theory.

Solution

The isothermal equations are

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x} &= 0,\\ \rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x} &= 0.\end{aligned}$$

Since the flow that of an isothermal ideal gas, we have

$$p=\rho RT, \qquad \frac{\partial p}{\partial x}=RT\frac{\partial\rho}{\partial x},$$

so we have

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x} &= 0,\\ \rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+RT\frac{\partial \rho}{\partial x} &= 0.\end{aligned}$$

In matrix form, we then have

$$\begin{pmatrix}1 & 0\\ 0 & \rho\end{pmatrix}\begin{pmatrix}\frac{\partial\rho}{\partial t}\\ \frac{\partial u}{\partial t}\end{pmatrix}+\begin{pmatrix}u & \rho\\ RT & \rho u\end{pmatrix}\begin{pmatrix}\frac{\partial\rho}{\partial x}\\ \frac{\partial u}{\partial x}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Then, following the procedure in the course notes, we seek eigenvalues of

$$\begin{pmatrix}\lambda-u & -\rho\\ -RT & \rho(\lambda-u)\end{pmatrix}.$$

Solving for the eigenvalues, we get

$$\lambda=u\pm\sqrt{RT}.$$

We find the eigenvectors are

$$(\pm\sqrt{RT} \quad 1).$$

Carrying out the operations precisely as done in the notes for the isentropic case, we get the characteristic form

$$\pm\sqrt{RT}\frac{d\rho}{dt}+\rho\frac{du}{dt}=0,$$

on

$$\frac{dx}{dt}=u\pm\sqrt{RT}.$$

The characteristics give the speed of small disturbances, so the Newtonian estimate for the sound speed is

$$c_{Newtonian}=\sqrt{RT}.$$

Comparing results for $R=287\ J/kg/K$, $T=300\ K$, we find

$$c_{isentropic}=347\ m/s, \qquad c_{Newtonian}=293\ m/s.$$