On a Complete Solution of the One-Dimensional Flow Equations of a Viscous, Heat-Conducting, Compressible Gas*

MORRIS MORDUCHOW† AND PAUL A. LIBBY†
Polytechnic Institute of Brooklyn

SUMMARY

The hydrodynamic equations describing the steady, one-dimensional flow of a viscous, heat-conducting, compressible gas are treated. The equation of state for a perfect gas is assumed valid, and the coefficients of specific heat are assumed constant. A complete integral of the energy equation is found for a Prandtl Number of $^3/_4$, which is a good approximation for the actual value of many gases over wide temperature ranges. A particular integral contained in this complete integral is $(u^2/2) + c_pT$ = a constant, which is a frequently used one-dimensional energy equation. The restrictions on its use are rigorously derived and stated. With the complete integral of the energy equation, three different types of solutions which depend on the boundary conditions are shown to be obtainable. The usual shock wave solution (satisfying the Rankine-Hugoniot relations) is only one of these three types.

Several features of the shock wave solution are shown. For example, the entropy has a maximum value at the inflection point in the velocity distribution. The validity of the continuum hypothesis is examined, and it is shown that the thickness of the wave as defined by Prandtl becomes infinite as the initial Mach Number of the flow becomes infinite provided that n, the exponent in the temperature-viscosity relation $\mu/\mu_0 = (T/T_0)^n$, is greater than $^1/_2$. Furthermore, it is concluded that the continuum theory may give reasonably correct results for flows with Mach Numbers up to 1.3.

The other two cases of flows contained in the equations are terminating flows—i.e., flows that can be valid only in a finite region. Their mathematical nature is demonstrated by two numerical examples, which indicate compression shocks followed by expansion waves. However, the physical significance, if any exists, of such solutions remains open to question.

INTRODUCTION

THE PURPOSE OF THE PRESENT PAPER is to give a concise, and yet general, theory of the one-dimensional flow of a real continuous fluid in order to derive all of the mathematical implications inherent in the equations governing such flows. Solutions with uniform and nonuniform conditions behind a shock front and the relationship between these two types of solutions are explicitly shown.

As an approach to the mathematical description of the structure of a shock wave, an analysis may first be made of the one-dimensional steady flow of a continuous medium. Such analyses have already been conducted by several investigators, but these have, in one way or another, been limited in generality. In early investigations,[1] for example, the flow of a (hypothetical) viscous fluid without heat conduction, and of a (hypothetical) nonviscous fluid with heat conduction were treated. Taylor and Maccoll[2] later gave an approximate solution for the flow of a fluid with both viscosity and heat conduction, valid only for a weak shock wave. In 1922, Becker[3] obtained an exact solution of the one-dimensional equations of a real fluid with a Prandtl Number of $^3/_4$, by assuming the expression for the temperature to be a quadratic in the velocity and then determining the coefficients by substitution into the differential equations. Thomas[4] afterwards briefly extended Becker's investigation by using variable coefficients of viscosity (μ) and of heat conduction (k), assuming μ and k to be proportional to $T^{1/2}$. The latter investigation, which introduced directly the mean free path, showed the increase of shockwave thickness due to variable values of μ and k, although, as shown in the present paper, some of the mathematical conclusions that Thomas reached must be modified if a different law for the variation of μ and k with T is assumed.

All of the aforementioned investigations had been restricted to the case of uniform-flow conditions both ahead ($x \to -\infty$) and behind ($x \to +\infty$) a shock front. Reissner and Meyerhoff[5] have recently pointed out, however, by means of a small-perturbation procedure, the existence of solutions to the shock wave equations which do not represent uniform conditions behind the shock front.

In this paper the one-dimensional flow equations are solved rigorously and completely for a constant Prandtl Number of $^3/_4$. The existence of solutions corresponding to flows with nonuniform conditions behind the wave is clearly proved and is demonstrated by two numerical examples. The thickness of a shock wave in terms of the mean free path of the gas molecules is considered.

SYMBOLS

0 = subscript denoting values at $x \to -\infty$
1 = subscript denoting values at either $x = x_1$ (or $x = 0$) or at $x \to +\infty$
$C_1, C_2, C_3, C_4, C_4', C_4''$ = constants of integration
c_p, c_v = specific heats at constant pressure and at constant volume, respectively

Received February 10, 1949.

* This paper is based on work done for the Office of Naval Research of the U.S. Navy on Contract No. N6ori-206 Task Order 1 under the supervision of Paul Lieber.

† Instructor in Aeronautical Engineering, Department of Aeronautical Engineering and Applied Mechanics.

H = the quantity $(u^2/2) + c_pT$
k = coefficient of heat conductivity
M = Mach Number
n = exponent in the viscosity-temperature relation $\mu = \mu_0(T/T_0)^n$
p = pressure
R = gas constant $R = c_p - c_v$
S = entropy
T = absolute temperature
t_1 = thickness of a shock wave, according to Prandtl definition
t_1' = value of t_1 for $n = 0$
t_2' = thickness of a shock wave, according to Taylor-Maccoll definition for $n = 0$
t'' = thickness of a shock wave (Prandtl definition) for $n \neq 0$
t_λ = thickness of a shock wave in units of local mean free paths.
u = velocity of flow (in x direction)
V = dimensionless velocity u/u_0
x = distance coordinate in direction of flow
z = $\exp\,[2\,\kappa\gamma M_0\xi/(\gamma + 1)]$
α = parameter defined by $(\gamma - 1)/(\gamma + 1) + 2/(\gamma + 1) \times M_0^2$
δ = a constant, 0.35
γ = ratio of specific heats c_p/c_v
κ = constant defined by Eq. (11b) and having the value 1.36 for air ($\gamma = 1.4$)
λ = mean free path of the gas molecules
μ = coefficient of viscosity
ρ = mass density of the fluid
τ_1 = dimensionless temperature T_1/T_0 (at $x = x_1$)
ξ = dimensionless distance coordinate x/λ_0

Equations for One-Dimensional Flow

The steady, one-dimensional flow, parallel to the x-axis, of a viscous, heat-conducting, compressible fluid is described by the following hydrodynamic equations, which, in the kinetic theory of nonuniform gases, form a second approximation to the Boltzmann equation:

Conservation of momentum:

$$\rho u \frac{du}{dx} + \frac{dp}{dx} - \frac{4}{3}\frac{d}{dx}\left(\mu \frac{du}{dx}\right) = 0 \tag{1a}$$

Conservation of mass:

$$(d/dx)(\rho u) = 0 \tag{1b}$$

Conservation of energy:

$$\rho u \frac{dE}{dx} + p\frac{du}{dx} - \frac{4}{3}\mu\left(\frac{du}{dx}\right)^2 - \frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{1c}$$

Equation of state:

$$p = p(\rho, T) \tag{1d}$$

where

ρ = mass density
u = velocity
p = hydrostatic pressure
μ = coefficient of viscosity
E = internal energy
T = absolute temperature
k = coefficient of heat conductivity

For the usual ranges of pressure, density, and temperature encountered in fluid flows, the coefficients of viscosity and heat conductivity may be assumed to be functions only of the temperature. The internal energy will be a function, in general, of p, ρ, T. The coefficients of specific heat at constant volume and pressure (c_v and c_p, respectively) are considered here as constants.* The four equations [Eqs. (1a)–(1d)] determine the flow variables u, p, ρ, and T as functions of x.

For mathematical simplicity the particular equation of state which has been used here is that for a perfect gas—namely,

$$p = \rho RT \tag{1e}$$

where R = the gas constant, $R = c_p - c_v$. With this equation of state,

$$dE = c_v dT \tag{1f}$$

The equations can be integrated as follows. The flow is assumed to be in the positive x direction, and three of the five constants of integration will be determined by conditions at minus infinity, which will be denoted by a subscript zero. Eq. (1b) can be integrated directly to give

$$\rho u = \rho_0 u_0 \tag{2}$$

Eq. (2) may be substituted in Eq. (1e) to give

$$p = \rho_0 u_0 RT/u \tag{3a}$$

or upon differentiation

$$\frac{dp}{dx} = \frac{\rho_0 u_0 R}{u}\frac{dT}{dx} - \frac{p}{u}\frac{du}{dx} \tag{3b}$$

With Eq. (3b), Eq. (1a) gives

$$\frac{p}{\rho_0 u_0}\frac{du}{dx} = u\frac{du}{dx} + R\frac{dT}{dx} - \frac{4}{3}\frac{u}{\rho_0 u_0}\frac{d}{dx}\left(\mu\frac{du}{dx}\right) \tag{4}$$

Furthermore, Eqs. (2) and (4), with Eq. (1f) and the relation $c_p = c_v + R$, can then be used to transform Eq. (1c) into

$$u\frac{du}{dx} + c_p\frac{dT}{dx} - \frac{4}{3\rho_0 u_0}\left[u\frac{d}{dx}\left(\mu\frac{du}{dx}\right) + \mu\left(\frac{du}{dx}\right)^2\right] - \frac{1}{\rho_0 u_0}\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{5a}$$

or

$$u\frac{du}{dx} + c_p\frac{dT}{dx} - \frac{4}{3\rho_0 u_0}\frac{d}{dx}\left(\mu u\frac{du}{dx}\right) - \frac{1}{\rho_0 u_0}\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{5b}$$

By a single quadrature, Eq. (5b) gives:

* Reference (6) indicates a change of 16 per cent in c_p for air for a range of temperatures from 0° to 1,600°F.

$$\frac{u^2}{2} + c_pT - \frac{k}{c_p\rho_0u_0}\left(\frac{4\mu c_p}{3k}\, u\, \frac{du}{dx} + c_p\, \frac{dT}{dx}\right) = C_1 \quad (5c)$$

where C_1 is an arbitrary constant.

If the Prandtl Number

$$\mu c_p/k = {}^3/_4 \quad (5d)$$

then Eq. (5c) becomes

$$\frac{u^2}{2} + c_pT - \frac{k}{c_p\rho_0u_0}\frac{d}{dx}\left(\frac{u^2}{2} + c_pT\right) = C_1 \quad (5e)$$

Eq. (5e) can be integrated to give

$$\frac{u^2}{2} + c_pT = C_1 + C_2 \exp\left(c_p\rho_0u_0 \int \frac{dx}{k}\right) \quad (6a)$$

Eq. (6a) may be regarded as a complete integral of the one-dimensional energy equation for a fluid with a Prandtl Number equal to $^3/_4$.* A necessary (though, as will be seen later, not sufficient) condition that the flow be uniform at plus infinity is that C_2 be equal to zero. In that case Eq. (6a) reduces to

$$(u^2/2) + c_pT = C_1 \quad (6b)$$

Thus it is seen that the well-known Eq. (6b) is an exact and complete integral of the general one-dimensional energy equation of a viscous, heat-conducting fluid under the following two conditions: (a) the Prandtl Number is $^3/_4$; (b) the flow is uniform at plus as well as at minus infinity. Condition (a) is necessary so that Eq. (6b) be exact, while condition (b) is necessary so that Eq. (6b) be complete. Although Becker[3] essentially obtained Eq. (6b) for a Prandtl Number equal to $^3/_4$ by assuming T as a general quadratic function of u and by determining the coefficients in this function so as to satisfy the governing differential equations, it appears not to have been generally recognized that this quadratic was the familiar Eq. (6b).† Moreover, by the assumption of the temperature as a quadratic function of velocity, Becker's solution was implicitly restricted throughout to the case of uniform conditions at plus infinity. Therefore, his paper does not contain Eq. (6a)—the complete integral of the general one-dimensional energy equation.

It must be noted that condition (a), requiring $P_r = {}^3/_4$, is based mathematically on the factor $^4/_3$ appearing in Eq. (5c) and, therefore, on the Navier-Stokes momentum Eq. (1a) and the energy Eq. (1c). Hence, condition (a) is based on the physical assumptions used in the derivation of the Navier-Stokes equations. These include: (i) The components of the stress tensor are related linearly to the components of the strain tensor: and (ii) the hydrostatic pressure is equal to the arithmetic average of the normal stresses.

* It may be pointed out here that Eq. (6a) is valid for variable values of μ and k provided only that the Prandtl Number remains constant and equal to $^3/_4$, k being an implicit function of x $[k = k(T) = k[T(x)] = k_1(x)]$.

† This may have been due to the somewhat obscure notation used by Becker.

The case of uniform conditions at plus and minus infinity appears to be the only one considered prior to the work of Reissner and Meyerhoff,[5] who indicated the possibility of solutions with nonuniform conditions at plus infinity. With Eq. (6a) a first order, nonlinear, differential equation for u as a function of x can be obtained; solutions of this equation with C_2 negative, zero, and positive will be discussed.

If Eq. (2) is substituted into Eq. (1a), the latter may be integrated once to give

$$\rho_0u_0u + p - ({}^4/_3)\,\mu(du/dx) = C_3 \quad (7a)$$

where C_3 is an arbitrary constant. Eqs. (3a) and (6a), combined with Eq. (7a), with $\gamma = c_p/c_v$, lead to the following first-order differential equation for u:

$$\frac{4}{3}\frac{\mu}{\rho_0u_0}u\frac{du}{dx} - \frac{\gamma+1}{2\gamma}u^2 + \frac{C_3}{\rho_0u_0}u = \frac{\gamma-1}{\gamma}C_2 \exp\left[c_p\rho_0u_0\int\frac{dx}{k}\right] + \frac{\gamma+1}{\gamma}C_1 \quad (7b)$$

C_1 and C_3 can be determined from given conditions at minus infinity, where all the flow variables are assumed to have finite values—that is, $u = u_0$, $T = T_0$, $p = p_0$, $\rho = \rho_0$, and $d/dx = 0$. Eqs. (6a) and (7a) then yield

$$C_1 = (u_0^2/2) + c_pT_0 \quad (8a)$$

and

$$C_3 = \rho_0u_0^2 + p_0 = \rho_0(u_0^2 + RT_0) \quad (8b)$$

In treating Eq. (7b) it was found convenient to nondimensionalize u with respect to u_0 and x with respect to λ_0, the mean free path of the gas molecules at minus infinity. Thus, let

$$V = u/u_0; \quad \xi = x/\lambda_0 \quad (9a)$$

Moreover, certain relations from the kinetic theory of gases may be used; thus, one can write[8]

$$\mu = \delta\rho\bar{c}\lambda \quad (9b)$$

where

δ = a constant = 0.35‡
$\bar{c}$ = mean molecular velocity = $\sqrt{8RT/\pi}$
λ = mean free path of the molecules

Furthermore, both theory and experiment indicate that $\mu = \mu(T)$; in fact, it is usually satisfactory to assume for gases that

$$\mu = \mu_0(T/T_0)^n \quad (9c)$$

where n is a positive constant depending only on the

‡ Theoretical values of δ vary from $^1/_3$ to 0.499, depending on the nature of the analysis. The value used here is that of Tait. The effect of using a different value for δ would be to change the scale of ξ (and therefore the shock-wave thickness in terms of mean free paths) in direct proportion.

gas in question. With Eqs. (8a), (8b), and (9a)–(9c), Eq. (7b) becomes

$$\left(\frac{T}{T_0}\right)^n V\frac{dV}{d\xi} - \kappa M_0 (V-1)(V-\alpha) = 2\kappa M_0\left(\frac{\gamma-1}{\gamma+1}\right)\frac{C_2}{n_0^2}\exp\left[\frac{2\kappa\gamma M_0}{\gamma+1}\int\left(\frac{T_0}{T}\right)^n d\xi\right] \quad (10)$$

where

M_0 = Mach Number at minus infinity $= u_0/\sqrt{\gamma R T_0}$

$$\alpha = \text{a parameter} = \frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{1}{M_0^2} \quad (11a)$$

$$\kappa = \text{a constant*} = \frac{3}{8\delta}(\gamma+1)\sqrt{\frac{\pi}{8\gamma}} \quad (11b)$$

Flow with Uniform Conditions at Plus Infinity

General Solutions

With $C_2 = 0$, which is a necessary condition that the flow be uniform at plus and minus infinity, Eq. (10) can be integrated directly to give

$$\xi = \frac{1}{\kappa M_0}\int\frac{(T/T_0)^n V dV}{(V-1)(V-\alpha)} + C_4 \quad (12a)$$

where C_4 is an arbitrary constant. It may be pointed out that one obvious solution satisfying Eq. (10) with $C_2 = 0$ and the boundary conditions is $V \equiv 1$. This corresponds to a uniform flow throughout, and, while it is a valid solution, it is not of particular interest here and, hence, is disregarded. From Eqs. (6b), (8a), and (9a), it follows that

$$\frac{T}{T_0} = 1 + \frac{\gamma-1}{2}M_0^2(1-V^2)$$

Hence, Eq. (12a) can be written in the form:

$$\xi = \frac{1}{\kappa M_0}\int\frac{\left[1+\frac{\gamma-1}{2}M_0^2(1-V^2)\right]^n V dV}{(V-1)(V-\alpha)} + C_4 \quad (12b)$$

For general values of n this integral must be evaluated numerically to give $\xi = \xi(V)$.

To complete the description of the flow given by this solution, it is convenient to determine the other flow variables as explicit functions of V. Thus, from Eqs. (2) and (9a),

$$\rho/\rho_0 = u_0/u = 1/V \quad (13a)$$

As before,

$$\frac{T}{T_0} = 1 + \frac{\gamma+1}{2}M_0^2(1-V^2) \quad (13b)$$

* $\kappa = 1.36$ for air with $\gamma = 1.40$.

Hence, from the gas law equation [Eq. (1e)],

$$\frac{p}{p_0} = \frac{1}{V}\left[1 + \frac{\gamma-1}{2}M_0^2(1-V^2)\right] \quad (13c)$$

Furthermore, it will be of interest to determine the entropy S and Mach Number M as functions of V. In general, for a perfect gas,

$$S - S_0 = c_p \ln(T/T_0) - R\ln(p/p_0)$$

Substituting Eqs. (13b) and (13c) in the above, one finds

$$\frac{S-S_0}{c_v} = \ln\left\{\left[1+\frac{\gamma-1}{2}M_0^2(1-V^2)\right]V^{\gamma-1}\right\} \quad (13d)$$

If the sonic velocity be denoted by a, then

$$\frac{M}{M_0} = \frac{u}{u_0}\left(\frac{a_0}{a}\right) = V\left(\frac{T}{T_0}\right)^{-1/2}$$

or

$$\frac{M}{M_0} = V\left[1+\frac{\gamma-1}{2}M_0^2(1-V^2)\right]^{-1/} \quad (13e)$$

The solution given by Eq. (12b) is essentially the one obtained in a restricted manner by Becker[3] for $n = 0$ and by Thomas[4] for $n = 1/2$. For completeness the solutions for $n = 0$ and $n = 0.768$, the experimentally determined value for air, will be given explicitly.

Constant Coefficients of Viscosity and Conductivity

If the coefficients of viscosity and conductivity are assumed constant, then $n = 0$ and Eq. (12b) becomes

$$\frac{1-V}{(V-\alpha)^\alpha} = C_4' \exp[\kappa(1-\alpha)M_0\xi] \quad (14a)$$

where C_4' is an arbitrary constant. Since by definition $V = 1$ as $\xi \to -\infty$, it is necessary from Eq. (14a) that $\alpha < 1$. This, in turn, implies that $M_0 > 1$—that is, that the flow, in order to be other than the trivial completely uniform flow, must be initially supersonic. Thus, in this formulation the necessary condition of an initial supersonic Mach Number for the existence of a shock wave is mathematically obtained without the use of the second law of thermodynamics.

For C_4' negative, Eq. (14a) yields a monotonically increasing velocity, with V becoming infinite and T becoming negatively infinite as $\xi \to +\infty$. Such a flow, the mathematical existence of which, to the authors' knowledge, has not been previously established, would be completely supersonic and would correspond to an expansion wave. However, it evidently can be valid only in a finite region wherein the temperature is positive. Moreover, it will be shown that such a flow would involve a continuous decrease in entropy and, therefore, would not be obtained physically even in a finite region. It is interesting to note that such a solution would give nonuniform conditions at plus infinity, where both V and $d/d\xi$ would become infinite. Thus, even when $C_2 = 0$ and Eq. (6b) is valid, mathematically possible solutions with nonuniform conditions at plus infinity can be obtained.

If C_4' is determined such that the origin of the coordinate system is located where the velocity profile has an inflection point—that is $\xi = 0$ where $d^2V/d\xi^2 = 0$—then C_4' is positive and Eq. (14a) becomes

$$\frac{1 - V}{(V - \alpha)^\alpha} = \frac{1 - \sqrt{\alpha}}{(\sqrt{\alpha} - \alpha)^\alpha} \exp\left[\kappa(1 - \alpha)M_0\xi\right] \quad (14b)$$

or, in another form,

$$\xi = \frac{1}{(1 - \alpha)M_0} \ln\left[\left(\frac{\sqrt{\alpha} - \alpha}{V - \alpha}\right)\frac{1 - V}{1 - \sqrt{\alpha}}\right] \quad (14c)$$

From Eq. (14c) and Eqs. (13a)–(13e) the entire characteristics of the flow can be determined. Examination of the variation of V, p, ρ, T, S, and M with ξ indicates that the flow described by this solution is the well-known compression or shock wave, which satisfies the Rankine-Hugoniot values at plus and minus infinity.[9] Figs. 1 to 3 show the distribution of V, p, and S with ξ for $M_0 = 1.1$ and for $M_0 = 2$.

The entropy distribution given by this solution may warrant a brief discussion. It will be observed from Fig. 3 that the entropy increases from zero at minus infinity to a maximum at the center of the wave ($\xi = 0$) and then decreases to a positive value at plus infinity

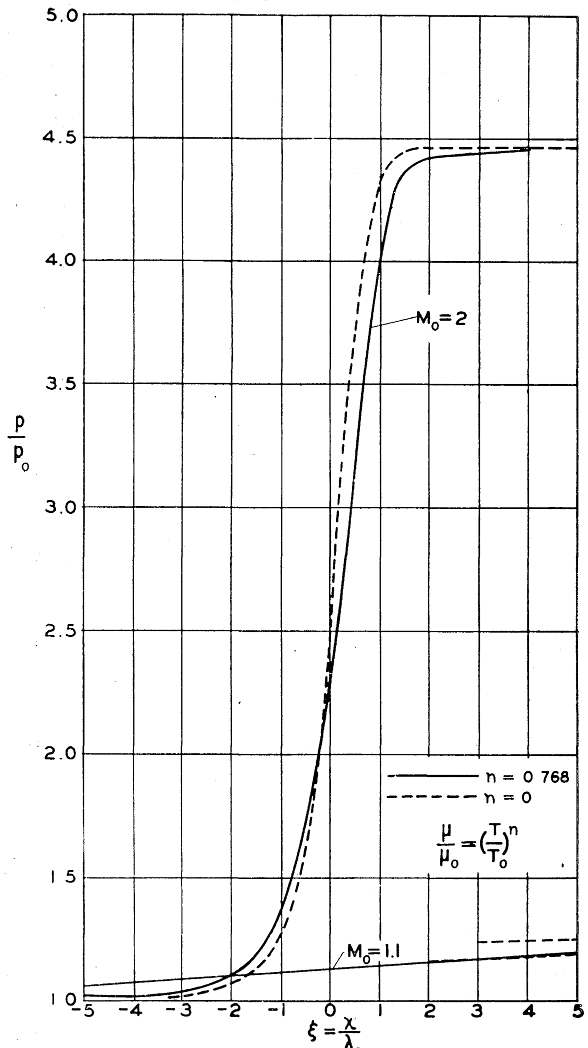


FIG 2 PRESSURE DISTRIBUTION

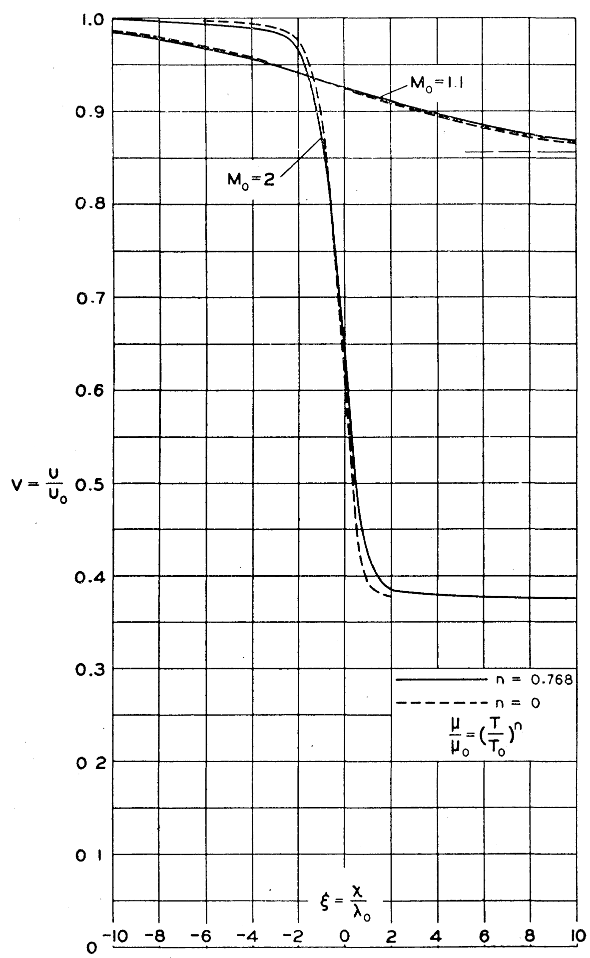


FIG. 1 VELOCITY DISTRIBUTION

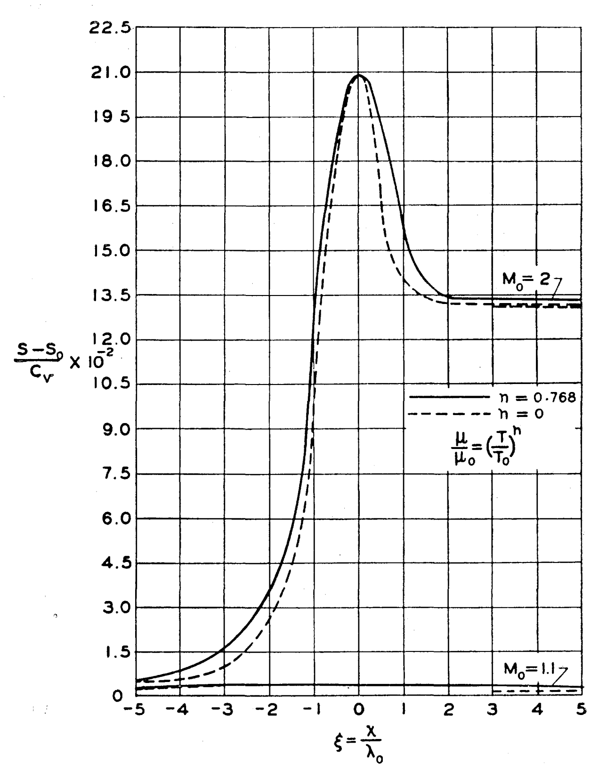


FIG. 3 ENTROPY DISTRIBUTION

This maximum in the entropy at the center of the wave can be demonstrated analytically as follows. In general,

$$\frac{dS}{dx} = R\left(\frac{1}{\gamma - 1}\frac{1}{T}\frac{dT}{dx} + \frac{1}{u}\frac{du}{dx}\right) \qquad (15a)$$

From Eqs. (9a), (10), and (13b) with $n = 0$ and $C_2 = 0$, one finds that Eq. (15a) may be written as

$$\frac{2}{\gamma + 1}\frac{1}{\kappa R M_0^3}\frac{dS}{d\xi} = -\frac{1}{V^2}\frac{(V-1)(V-\alpha)(V^2-\alpha)}{T/T_0} \qquad (15b)$$

Eq. (15b) indicates that the entropy has stationary values at three values of V—viz., $V = 1, \sqrt{\alpha}, \alpha$. It is clear that the stationary value of S at the center of the wave, where $V = \sqrt{\alpha}$ [cf. Eq. (14b)], is a maximum, since Eq. (15b) shows that $dS/d\xi$ is negative for $\alpha \leq V \leq \sqrt{\alpha}$—i.e., for V downstream of the center. It may at first sight appear that the recovery of mechanical energy on the downstream side of the center of the wave thus indicated by this solution would violate the second law of thermodynamics and thereby invalidate the solution. However, the second law applies to an entire system—that is, to the end points of the flow—and permits energy recovery in separate sections thereof. The negative entropy gradient here might also be interpreted as indicating physical effects that are not taken into account by the governing Eqs. (1a)–(1e). For example, the gas molecules in some regions may not be in thermal equilibrium.[7]

It is clear from Eq. (15b) that the expansion-wave solution discussed above must be discarded on the basis of entropy considerations. For this solution the velocity increases monotonically with ξ—that is, $V \geq 1$. It therefore follows from Eq. (15b) that $dS/d\xi < 0$ *everywhere*, which is physically impossible.

The solutions given by Eqs. (14a)–(14c) are valid solutions of the integral in Eq. (12b) provided that $\alpha < 1$ and $n = 0$. It is of interest to consider the case of $n = 0$ but $\alpha = 1$—that is, $M_0 = 1$. In that case Eq. (12b) gives

$$\xi = \frac{1}{\kappa}\left[\ln C_4''(V - 1) + \frac{1}{1 - V}\right] \qquad (16)$$

where C_4'' is an arbitrary constant. In order that $V = 1$ at minus infinity, it is necessary that $C_4'' \geq 0$, for, if $C_4'' < 0$, then it would be necessary that $V < 1$ for ξ real, and it would then follow from Eq. (16) that as $V \rightarrow 1$, $\xi \rightarrow +\infty$ instead of $\xi \rightarrow -\infty$. For a positive value of C_4'', Eq. (16) represents a flow in which the velocity increases monotonically as ξ increases. This is evidently a special case of the expansion-wave flow discussed above and must be discarded on the same grounds (entropy) as that solution. Thus, the only physically possible solution for the case $M_0 = 1$ is the solution $V \equiv 1$ everywhere.

It is thus shown that a flow that is sonic at minus infinity will remain sonic without change. More generally, it has been proved here that, if a one-dimensional flow is subsonic or sonic at minus infinity, then the flow will be uniform throughout the entire range of x. This is true for the cases of both uniform and nonuniform conditions at plus infinity, since in the latter case, the additional term (proportional to C_2) in Eq. (6a) vanishes at minus infinity, so that the mathematical argument upon which the above statement is based remains valid.

Variable Coefficients of Viscosity and Conductivity

For $n \neq 0$, the integral in Eq. (12b) can be evaluated numerically for any given n and M_0. The arbitrary constant C_4 may be determined, as in the case $n = 0$, by placing the origin ($\xi = 0$) at the inflection point ($d^2V/d\xi^2 = 0$) in the velocity-distribution curve. From this numerical solution for $\xi = \xi(V)$, it is possible to determine the variation of the other flow variables by using Eqs. (13a)–(13c).

The experimental value of n for air is 0.768. The numerical evaluation of the integral in Eq. (12b) has been carried out for this value of n and for $M_0 = 1.1$ and 2.0. The results are shown in Figs. 1–3, along with the results for $n = 0$ for ready comparison. As might perhaps have been expected, the increase in viscosity and conductivity through the shock wave decreases somewhat the absolute value of the gradients of the flow variables in these two numerical examples.

Shock Wave Thickness

Mathematically, the changes in the flow variables through a compression shock occur over the infinite distance from minus to plus infinity. Inspection of Figs. 1–3 will indicate, however, that in reality the major portion of the changes occur in a small region at the center of the wave. This small region is called the shock-wave thickness. The question which must be considered is whether the shock-wave thickness is of the order of magnitude of the mean free path of the gas molecules. If it is, then the ability of the above equations, which are based on the continuum hypothesis, to describe the structure of the shock wave is doubtful. Previous authors[2–4] have indicated that only for weak shocks ($M_0 \leq 1.3$) is the continuum theory valid. However, because of the rather arbitrary manner in which shock-wave thickness has been defined, it was thought advisable to compare the results given by the above analysis according to the various definitions.

Nondimensionalizing the space variable x by means of the mean free path (λ_0)* of the gas molecules at minus infinity is convenient for the consideration of shock-wave thickness [cf. Eq. (9a)]. The thickness is then given as a pure number—namely, the number of

* For air with $T_0 = 519°$R., $\rho_0 = 0.002378$ slugs per cu.ft.; that is, for standard sea-level conditions, $\lambda_0 = 3.46 \times 10^{-6}$ in.

mean free paths at minus infinity. If this number is large in comparison with unity, then the continuum hypothesis is valid; if this number is of the order of unity, then the continuum hypothesis is no longer a sufficiently accurate basis of determining the flow characteristics. For purposes of a more refined appraisal, the thickness may be calculated as a multiple of the local mean free path of the gas molecules at the shock wave rather than of the mean free path at minus infinity.

The point of view that may be taken here regarding the mean free path is that this quantity can be determined experimentally from the measurable quantities viscosity, density, and temperature. Thus, λ, in general, may be consistently defined by the equation [cf. Eq. (9b)]

$$\lambda = (1/\delta)(\mu\sqrt{\pi}/\rho\sqrt{8RT}) \tag{17a}$$

In particular, λ_0 will be

$$\lambda_0 = (1/\delta)(\mu_0\sqrt{\pi}/\rho_0\sqrt{8RT_0}) \tag{17b}$$

This point of view is consistent with Eqs. (9b) and (9c) regardless of the value of n.*

The various definitions of shock wave thickness will now be discussed and compared. Prandtl[10] has given a definition of shock-wave thickness t_1 shown in Fig. 4a. It may be stated mathematically as

$$\lambda_0 t_1 = -(u_0 - u_{x\to\infty})/(du/dx)_{max.} \tag{18a}$$

Since the origin of the space variables x and ξ has been placed at the inflection point in the velocity distribution and since $V = \alpha$ at plus infinity, this becomes, in terms of V and ξ,

$$t_1 = -(1 - \alpha)/(dV/d\xi)_{\xi=0} \tag{18b}$$

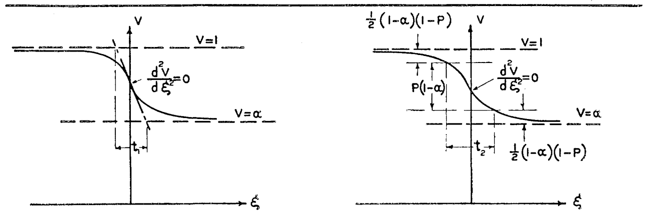
Taylor and Maccoll[2] have given another definition of shock-wave thickness t_2—namely, the distance in which a certain portion P of the total velocity change through the wave takes place (see Fig. 4b). They selected $P = 0.80$ and, according to this definition, gave a simple approximate formula for the thickness of a weak wave.

Constant Coefficients of Viscosity and Conductivity

The shock-wave thickness t_1' for $n = 0$ according to Prandtl's definition will be calculated first. Eq. (14b) gives $V = \sqrt{\alpha}$ for $\xi = 0$, and thus, from Eq. (10) with $n = 0$ and $C_2 = 0$, Eq. (18b) becomes

$$t_1' = (1 - \alpha)/\kappa M_0(1 - \sqrt{\alpha})^2 \tag{19}$$

* If the molecules of a gas are assumed to be spheres of diameter σ and mass m, then the mean free path can, by kinetic theory, be shown to be $\lambda = m/\sqrt{2}\pi\rho\sigma^2$. Hence, if σ were constant, then λ would be a function only of the density ρ at any point, and according to Eq. (17a), μ would be proportional to $T^{1/2}$ implying $n = 1/2$. Thus, by choosing $n \neq 1/2$, one is implicitly taking into account the variation of molecular diameter with temperature. In fact, with the equations used here, $\sigma \sim T^{(1-2n)/4}$. Moreover, if $n \neq 1/2$, then λ is implicitly considered to be a function of T as well as of ρ.



(a) THICKNESS ACCORDING TO PRANDTL (b) THICKNESS ACCORDING TO TAYLOR-MACCOLL

FIG. 4 DEFINITIONS OF SHOCK WAVE THICKNESS

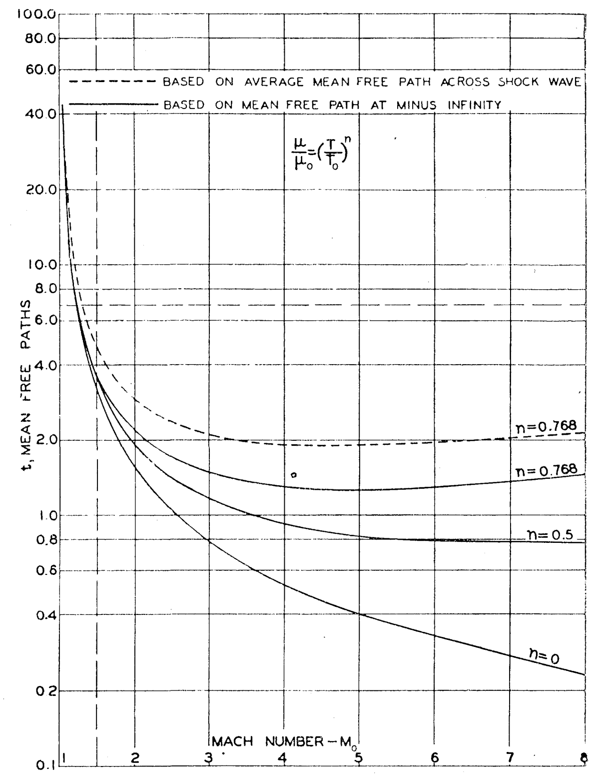


FIG. 5 SHOCK WAVE THICKNESS ACCORDING TO PRANDTL DEFINITION

Values of t_1' for a range of values of M_0 are shown in Fig. 5.

The thickness t_2' according to the Taylor-Maccoll definition for $n = 0$ is calculated for any value of P as follows: Eq. (14c) gives (cf. Fig. 4b)

$$t_2' = -\xi_1 + \xi_2 =$$

$$-\frac{1}{\kappa(1-\alpha)M_0}\ln\left[\frac{1 - V_2}{1 - V_1}\left(\frac{V_1 - \alpha}{V_2 - \alpha}\right)^{\alpha}\right] \tag{20a}$$

where

$$V_1 = \alpha + \frac{1+P}{2}(1-\alpha) = \frac{1}{2}(1+\alpha) + \frac{P}{2}(1-\alpha)$$

$$V_2 = \alpha + \frac{1-P}{2}(1-\alpha) = \frac{1}{2}(1+\alpha) - \frac{P}{2}(1-\alpha)$$

or, finally,

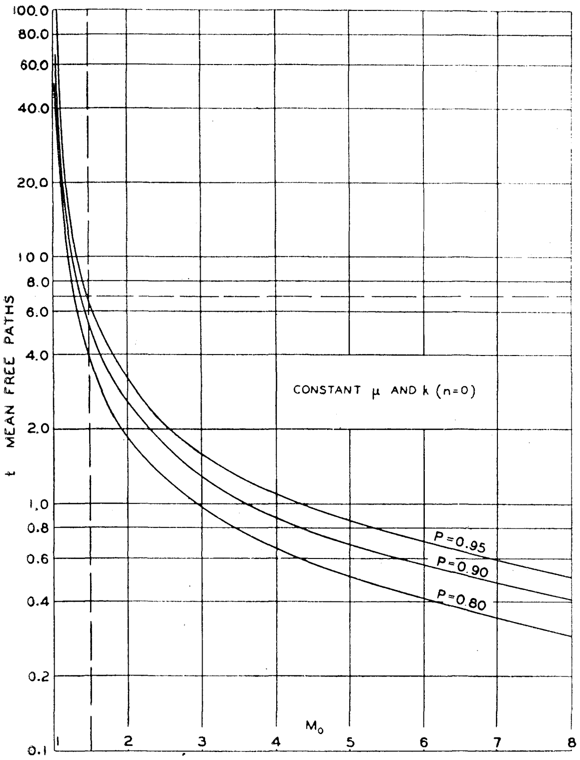


FIG 6 SHOCK WAVE THICKNESS ACCORDING TO TAYLOR-MACCOLL DEFINITION

$$t_2' = \frac{1 + \alpha}{\kappa(1 - \alpha)M_0} \ln \frac{1 + P}{1 - P} \qquad (20b)$$

Values of t_2' have been determined for $P = 0.80, 0.90, 0.95$ for a range of M_0 and are shown in Fig. 6.

Examination of the values of t_1' and t_2' shown in Figs. 5 and 6, respectively, indicate that the Taylor-Maccoll definition imposes a somewhat less severe restriction of the continuum hypothesis than does the Prandtl definition. However, if one rather arbitrarily, but reasonably, requires that $t \geq 7$ for the above governing equations to describe a normal shock wave adequately, then according to any definition it would be required that $M_0 \leq 1.3$. Thus, only for weak waves would these equations describe the actual conditions in a shock wave.

Two refinements in the calculation of these thicknesses may be made: consideration of the variability of the coefficients of viscosity and conductivity, and determination of the thickness in terms of the local mean free path at the wave.

Variable Coefficients of Viscosity and Conductivity

For determining shock wave thickness t'' when the coefficients of viscosity and conductivity of the gas are considered variable, only the Prandtl definition will be used, since it is easier to apply here than the Taylor-Maccoll definition and since, as has been seen above, the two definitions lead to similar results.

In order to apply the Prandtl definition in this case, it is necessary to calculate the value of V at the origin of the axis. By differentiating Eq. (10) ($C_2 = 0$) with respect to ξ, setting $d^2V/d\xi^2 = 0$, and eliminating $dV/d\xi$, one obtains the following equation for the value of V_c—that is, for the value of V at $\xi = 0$,

$$\left(n - \frac{1}{2}\right) V_c^4 - n\,(1 + \alpha) V_c^3 + \left(\frac{\gamma}{\gamma - 1} + n\right) \alpha\, V_c^2 - \frac{\alpha^2}{2}\frac{\gamma + 1}{\gamma - 1} = 0 \qquad (21)$$

The value of V_c may be determined from this quartic equation for any given n and α(or M_0) by Newton's[11] (approximate) or Ferrari's (exact) method.[12*] From this value and from Eq. (10) ($C_2 = 0$), $(dV/d\xi)_{\xi = 0}$ may be determined, and then t_1'' can be calculated from Eq. (18b).

The calculations outlined above were carried out for $n = 1/2$ and $n = 0.768$ for a range of M_0 values. The results are shown in Fig. 5. As might be expected, the wave thickens as n increases. Moreover, as was pointed out by Thomas,[4] for $n = 1/2$, the thickness reaches a minimum finite value as $M_0 \rightarrow \infty$. However, for $n \geq 1/2$, the thickness decreases to a minimum (about 1.5 mean free paths at $M_0 = 5$ when $n = 0.768$) and then increases without limit as $M_0 \rightarrow \infty$. This can be demonstrated analytically in the following manner: Eq. (10) ($C_2 = 0$) can be written, with the aid of Eq. (13b), as

$$\frac{dV}{d\xi} = \frac{\kappa(V - 1)(V - \alpha)}{V\left[\dfrac{1}{M_0^2} + \dfrac{\gamma - 1}{2}(1 - V^2)\right]^n M_0^{2n-1}} \qquad (22a)$$

From Eq. (12a), it is clear that as $M_0 \rightarrow \infty$, $\alpha \rightarrow (\gamma - 1)/(\gamma + 1)$. Moreover, from Eq. (21), V_c approaches a finite value, depending on n, as $M_0 \rightarrow \infty$. Thus,

$$\lim_{M_0 \rightarrow \infty} \left(\frac{dV}{d\xi}\right)_{\xi = 0} \left.\begin{array}{ll} = \dfrac{\kappa(V_c - 1)(V_c - \alpha)}{V_c\left[\dfrac{\gamma - 1}{2}(1 - V_c^2)\right]^{1/2}} & \text{if } n = 1/2 \\ = 0, & \text{if } n > 1/2; \\ \rightarrow \infty, & \text{if } n < 1/2 \end{array}\right\} \qquad (22b)$$

It follows then from Eq. (18b) that, as $M_0 \rightarrow \infty$, the shock-wave thickness given by this analysis depends strongly on the value of n; it will have a finite value if $n = 1/2$, will approach zero if $n < 1/2$, and will approach infinity if $n > 1/2$. Thus it is seen that, if $n > 1/2$ (as is actually the case for air) it may not be quite true that the stronger a normal shock wave, the thinner it will be. However, the thickness increases slowly for $n = 0.768$ as M_0 increases beyond the value ($M_0 = 5$) at which the thickness is a minimum, while phenomena such as dissociation, ionization, and con-

* One must, of course, find the (real) root that is between $V = 1$ and $V = \alpha$.

densation not considered here may then become significant.

Examination of Fig. 5 indicates, therefore, that consideration of the variability of the coefficients of viscosity and conductivity does not appreciably change the Mach Number at which the continuum hypothesis must be considered of doubtful validity. If $n = 0.768$ and t_1'' is required to be greater than 7, then M_0 must be less than 1.3.

Thickness in Terms of Local Mean Free Paths

If one is interested in the actual dimensions (say in inches) of the shock wave as given by the two definitions, the previous results are adequate. Thus, for particular conditions at minus infinity (M_0, p_0, and T_0), the thickness of the wave can be determined by calculating λ_0 from Eq. (17b) and by multiplying the appropriate value of t from Figs. 5 or 6 by this value of λ_0. However, to examine the validity of the continuum hypothesis, a more suitable appraisal of the shock-wave thickness may be obtained by determining this thickness as a multiple of the (local) mean free paths in the wave.

A good approximation to this local value may be found by taking the arithmetic mean of the mean free paths of the gas molecules at minus and plus infinity. Thus,

$$t_\lambda = 2t/[1 + (\lambda_1/\lambda_0)] \tag{23a}$$

where t_λ is the thickness of the wave in multiples of the mean free path in the wave, t is the thickness in multiples of the mean free path at minus infinity, and λ_1 is the mean free path of the gas molecules at plus infinity. From Eqs. (9c), (17a), and (17b), it can be shown that

$$\lambda_1/\lambda_0 = V_1(T_1/T_0)^{[n - (1/2)]} \tag{23b}$$

Observing that $V_1 = \alpha$ [cf. Eq. (12b)], Eqs. (13b) and (23b) may be used to write Eq. (23a) in the form

$$t_\lambda = \frac{2t}{1 + \alpha\left[1 + \frac{\gamma - 1}{2}M_0^2(1 - \alpha^2)\right]^{[n - (1/2)]}} \tag{23c}$$

$$= K(M_0, n)t$$

where $K(M_0, n)$ is a multiplicative parameter dependent only on M_0 and n. Obviously, t_λ as defined by Eq. (23a) will, in general, have a value between t and $2t$. Applying this parameter to the thickness t_1'' for $n = 0.768$ results in the thickness shown in Fig. 5.

From a consideration of Fig. 5 the conclusion may be drawn that for $t > 7$, M_0 must be less than approximately 1.3. Thus, considering the variation of mean free path through the wave does not significantly affect the conclusions based on the analysis in terms of the mean free paths at minus infinity.

Terminating Solutions

If $C_2 \neq 0$, then it can be seen from Eq. (6a) that, at $x \rightarrow \infty$, at least either u or T becomes indefinitely large. Thus, the region of the flow must in this case be finite in the direction of the positive x axis. In other words, the flow must be terminated at some point (denoted by ξ_1 in the ξ plane), at which certain conditions determining C_2 are prescribed. If, on the other hand, physically significant conditions are to be prescribed at plus infinity, then these must be the Rankine-Hugoniot conditions, and C_2 will be zero. This implies that a necessary and sufficient condition for uniform flow behind a shock wave is that the Rankine-Hugoniot relations between the thermodynamic variables before $(-\infty)$ and behind $(+\infty)$ the wave be satisfied.

For the purpose of illustrating the nature of solutions of the flow equations for which $C_2 \neq 0$, the coefficients μ and k will henceforth be assumed constant, implying $n = 0$.

Let

$$z = \exp\{2\kappa\gamma M_0\xi/(\gamma + 1)\} \tag{24a}$$

Then Eq. (10) can be written in the form:

$$Vz\frac{dV}{dz} - \frac{\gamma + 1}{2\gamma}(V - 1)(V - \alpha) - \frac{\gamma - 1}{\gamma}\frac{C_2}{u_0^2}z = 0 \tag{24b}$$

A complete numerical solution of Eq. (24b) can be obtained by prescribing the Mach Number M_0 at $\xi \rightarrow -\infty$ and by prescribing the values of two variables* (say, $T_1/T_0 = \tau_1$ and $u_1/u_0 = V_1$) at some finite point to be chosen as the origin $\xi = 0$. In the z system, the range $-\infty \leq \xi \leq 0$ corresponds to the range $0 \leq z \leq 1$. The values of the dimensionless constant C_2/u_0^2 can be obtained in terms of M_0, τ_1, and V_1 from Eqs. (6a) and (8a). One thus finds

$$\frac{C_2}{u_0^2} = \frac{1}{2}(V_1^2 - 1) + \frac{1}{\gamma - 1}\frac{1}{M_0^2}(\tau_1 - 1) \tag{24c}$$

Three types of flows can be mathematically obtained, depending on whether C_2/u_0^2 is zero, positive, or negative. If $C_2/u_0^2 = 0$, then the quantity $H \equiv (u^2/2) + c_pT$, which in adiabatic flow represents the enthalpy at the stagnation point, will be constant. Moreover, it follows from Eq. (24c) that, if the Rankine-Hugoniot relations are satisfied at point "1" for two of the thermodynamic variables (e.g., for V and for T/T_0), then $C_2/u_0^2 = 0$. If $C_2/u_0^2 > 0$, then H increases in the direction of flow, while if $C_2/u_0^2 < 0$, then H decreases. It is interesting to note, in fact, that in the one-dimensional theory an increasing H downstream is mathematically consistent with the law of the conservation of energy as stated by Eq. (1c), as well as with

* If ρ/ρ_0 and p/p_0 are given, then T/T_0 and u/u_0 can be determined by means of the continuity equation and the equation of state.

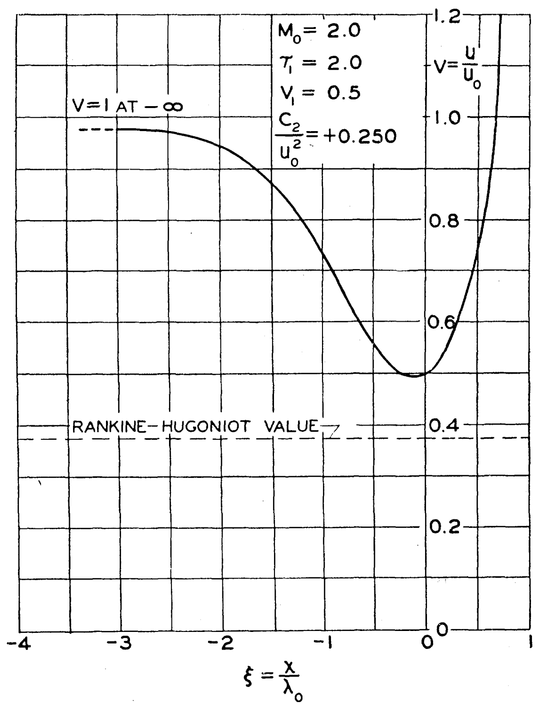


FIG. 7 VELOCITY DISTRIBUTION

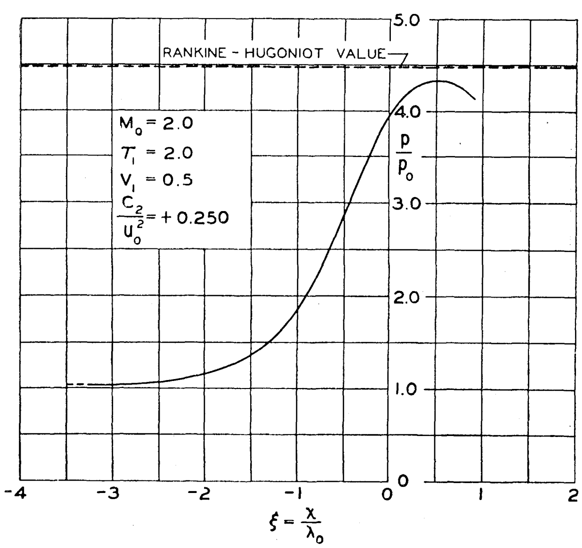


FIG. 8 PRESSURE DISTRIBUTION

the second law of thermodynamics (cf. discussion of results below).

With the numerical value of C_2/u_0^2 determined, the first-order differential Eq. (24b) can be solved numerically, without much difficulty, by standard methods,* starting from the point ($z = 1$, $V = V_1$) and proceeding backwards to $z = 0$ and forward to some finite value of $z > 1$. (As a check it should automatically be found that $V = 1$ at $z = 0$.)

The case $C_2/u_0^2 = 0$, leading to compression shocks satisfying the Rankine-Hugoniot relations before and behind a front, has already been treated here in detail.

* Cf., for example, reference 11. Also cf. method of Taylor series described below.

To indicate the nature of solutions in the case $C_2/u_0^2 \neq 0$, two numerical examples will be discussed here: one in which $C_2/u_0^2 > 0$ and one in which $C_2/u_0^2 < 0$. In both cases the values V_1 and τ_1 will be chosen so that the second law of thermodynamics will not be violated at least with respect to the points $\xi \to -\infty$ and $\xi = 0$ —i.e., so that $S_1 > S_0$.†

Numerical Examples

Example I.—Suppose $p_1/p_0 = 4$ and $T_1/T_0 \equiv \tau_1 = 2$. Also suppose $M_0 = 2$.‡ Then from Eqs. (1e) and (2), it follows that $u_1/u_0 \equiv V_1 = {}^1/_2$. Eq. (24c), with $\gamma = 1.4$, then gives

$$C_2/u_0^2 = {}^1/_4$$

From Eq. (6a), it will be found that this implies that H will increase, at $\xi = 0$, to 1.222 times its original value at $\xi \to -\infty$.

Eq. (24b) was solved numerically by starting from the given values of V and its derivatives at $z = 1$ and expanding about this point in a Taylor series. This expansion was carried out for the limited region $0.5 \leq z \leq 1$, so that only a few terms in the series were needed for sufficiently rapid convergence. For the next region, $0.2 \leq z \leq 0.5$, another Taylor series was used, this time about the point $z = 0.5$. This procedure was continued to $z = 0$. Near $z = 0$, however, it was necessary to use much smaller intervals of z in order to obtain satisfactory convergence. The expansion was similarly carried out forward for $\xi > 0$, or $z > 1$. The final results are shown in Figs. 7 and 8.

Example II.—Suppose $V_1 = 0.6$, $\tau_1 = 1.5$, and $M_0 = 2$ ($\gamma = 1.4$).

Proceeding as above, one finds

$$C_2/u_0^2 = -0.0075$$

indicating a decrease in H, at $\xi = 0$, to 0.993 times its value at $\xi \to -\infty$. Eq. (24b) was solved numerically by the Taylor series method described above, and the results are shown in Figs. 9 and 10.

The entropy $(S - S_0)/c_v$ was calculated for both examples, and it was found that for the region investigated the quantity $S - S_0$ was everywhere positive, so that the second law of thermodynamics was not violated.

It can be seen from Figs. 8 and 10 that the solutions with $C_2 \neq 0$ yield compression waves followed by large expansions.** These compressions and expansions all

† This condition is imposed so that the boundary conditions correspond to physically realizable flows.

‡ It may be objected that, as seen previously, the case $M_0 = 2$ may give unreliable results when treated by the continuum theory. The purpose of the present examples, however, is only to indicate the nature of the solutions obtainable, so that the particular value of M_0 chosen does not matter.

** Some rough calculations have indicated that the pressures in Figs. 8 and 10 must continue to drop considerably before any compression can occur. Moreover, it can be shown that $dp/d\xi \to -\infty$ as $\xi \to \infty$.

occur in a small region—namely, a region of the order of magnitude of several mean free paths—although for a smaller initial Mach Number the extent of this region increases. The physical significance, if any, of these types of solutions remains open to question.

Conclusions

Based on the one-dimensional analysis of a continuous, viscous, heat-conducting fluid given here, the following conclusions can be stated:

(1) The Eq. $(u^2/2) + c_pT$ = constant is valid, only if the Prandtl Number has a constant value of $^3/_4$. Moreover, this equation is a complete integral of the energy differential equation only under the condition of uniform flow at plus, as well as at minus, infinity. The particular value $P_r = {}^3/_4$ here is based on the Navier-Stokes equations and is therefore subject to the physical limitations of these equations.

(2) Mathematical solutions representing monotonic expansion waves can be obtained. These solutions violate, however, the second law of thermodynamics, even in the limited region in which the temperature would be positive.

(3) Under conditions of uniform flow at $\pm\infty$ and for a Prandtl Number of $^3/_4$, simple closed-form solutions of the flow equations, which represent monotonic compression shocks satisfying the Rankine-Hugoniot conditions at the end points, can be obtained. The thickness of such waves, which, in general, are of the order of magnitude of several mean free paths, will decrease as the Mach Number M_0 increases, provided that $n < {}^1/_2$. If, however, $n > {}^1/_2$ (as for air), then the thickness will diminish with M_0 until it reaches a minimum (about two local mean free paths when $M_0 = 5$ if $n = 0.768$), beyond which the thickness slowly increases to infinity as M_0 increases to infinity. The continuum hypothesis probably gives inaccurate results for those cases in which the thickness is small in comparison with the mean free path (cf. Fig. 3.)

(4) The Rankine-Hugoniot conditions are necessary and sufficient conditions for the flow to be uniform at both plus and minus infinity.

(5) A flow that is either subsonic or sonic at minus infinity will remain uniform throughout the entire field of flow.

(6) Mathematical solutions of the flow equations can be obtained for which the quantity $(u^2/2) + c_pT$ varies. Such solutions are applicable only to semi-infinite regions, since the flow variables become infinite as $\xi \rightarrow +\infty$. These solutions indicate compression waves followed by expansion waves in small regions of the order of several mean free paths. The physical significance, if any, of such solutions remains, however, open to question.

(7) Phenomena observed in actual steady normal shock waves which cannot be explained by any of the types of solutions given here must be due to either multidimensional effects or to effects not contained in the hydrodynamic continuum hypothesis of a gas with constant coefficients of specific heat and following the ideal gas law.

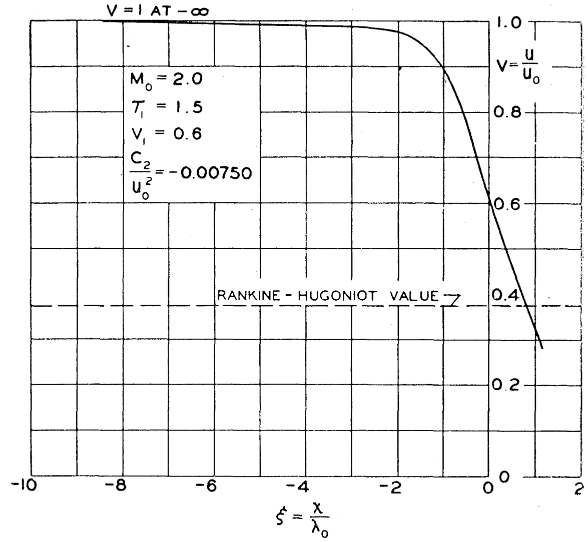


FIG. 9 VELOCITY DISTRIBUTION

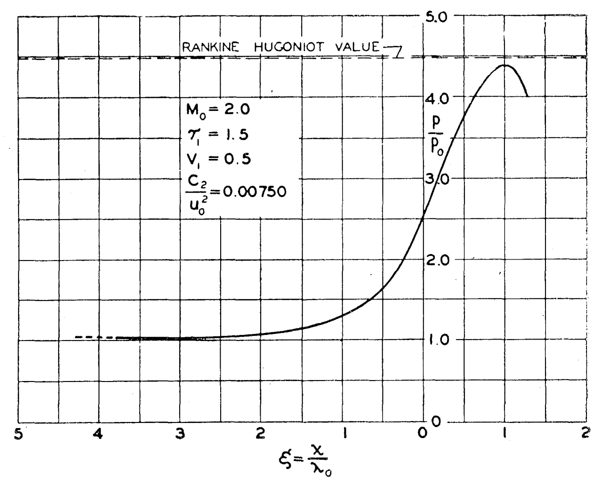


FIG. 10 PRESSURE DISTRIBUTION

Acknowledgments

The authors wish to acknowledge the helpful discussions of this paper with Prof. H. J. Reissner and Messrs. Leonard Meyerhoff and Henry G. Lew and the checking of some of the mathematical analysis by Prof. D. E. Whitford.

References

[1] Lamb, H., *Hydrodynamics*, 6th Rev. Ed., pp. 487–489 650–651; Dover Publications, New York, 1945.

[2] Taylor, G. I., and Maccoll, J. W., *The Mechanics of Compressible Fluids*, Vol. III of W. F. Durand's Aerodynamic Theory; Julius Springer, Berlin, 1934.

[3] Becker, R., *Impact Waves and Detonation*, N.A.C.A. T.M. No. 505.

(*Continued on page 704*)

$$\lim_{Z \to 0} [\partial(w/U)/\partial X] = \partial\alpha/\partial X \tag{6}$$

The order of differentiating and taking the limit has been interchanged, which is permissible provided w has a derivative. Applying Eq. (6) to Eq. (4), there is obtained, finally,

$$\frac{\partial\alpha}{\partial X} = \overline{\int\int_{\tau}} \frac{\Delta C_p(X_1, Y_1)dX_1\, dY_1}{[(X - X_1)^2 - \beta^2(Y - Y_1)^2]^{3/2}} \tag{7}$$

This equation is the basic one rather than Eq. (5) of the paper. One can now proceed with the derivation of Eq. (10) by modifying each of Eqs. (6) through (10) as follows:

Drop the operation $\lim_{Z \to 0} \frac{\partial}{\partial Z} \int_{-\infty}^{x} dx$, which appears in front of each of the double integrals, and substitute $\partial\alpha/\partial x$ for α. The derivations of Eqs. (34) and (43) should be modified in a similar manner.

Solution of the One-Dimensional Flow Equations of a Viscous, Heat-Conducting, Compressible Gas

(*Continued from page 684*)

[4] Thomas, L. H., *Note on Becker's Theory of the Shock Front*, Journal of Chemical Physics, Vol. 12, No. 11, November, 1944.

[5] Reissner, H., and Meyerhoff, L., *On the Structure of Shock Waves in a Viscous, Heat-Conducting Compressible Flow, Part I, Shock Waves in One-Dimensional Flow with Constant Coefficients of Viscosity and Conductivity*, Report submitted to Project Squid, Princeton, N.J., February, 1948.

[6] Tribus, M., and Boelter, L. M. K., *An Investigation of Aircraft Heaters, Part II, Property of Gases*, N.A.C.A. Wartime Report W-9.

[7] Bethe, H. A., and Teller, E., *Deviations From Thermal Equilibrium in Shock Waves*, Ballistic Research Laboratories Report No. X-117, Aberdeen Proving Ground, Md.

[8] Loeb, L. B., *Kinetic Theory of Gases*, p. 187; McGraw-Hill Book Company, Inc., 1927.

[9] Liepmann, H. W., and Puckett, A. E., *Introduction to Aerodynamics of a Compressible Fluid*, pp. 38–42; John Wiley and Sons, Inc., New York, 1947.

[10] Prandtl, L., *Zur Theorie des Verdichtungsstosses*, Zeitschrift für der geramte Turbinen-Wesen, Vol. 3, p. 241, 1906.

[11] Scarborough, J. B., *Numerical Mathematical Analysis;* The Johns Hopkins Press, 1930.

[12] Dickson, L. E., *New First Course in the Theory of Equations*, pp. 51–52; John Wiley and Sons, Inc., 1939.

Mixing of Any Number of Streams in a Duct of Constant Cross-Sectional Area

(*Continued from page 698*)

It is interesting to note that the same final Mach Number is obtained with any combination of initial area ratios and total pressures that satisfy the above specified condition in regions (1) and (2).

References

[1] Foa, J. V., and Rudinger, G., *On the Addition of Heat to a Gas Flowing in a Pipe at Subsonic Speed*, Journal of the Aeronautical Sciences, Vol. 16, No. 2, pp. 84–94, February, 1949.

[2] Keenan, J. H., Neumann, E. P., and Lustwerk, F., *An Investigation of Ejector Design by Analysis and Experiment*, Massachusetts Institute of Technology Meteor Report No. 18, June 1, 1948.

[3] Miller, K. D., Jr., *Flow in Ejectors Driven by Supersonic Jets*, Princeton University, Project Squid Technical Memorandum No. Pr-3, May 15, 1948.