LECTURE NOTES ON MATHEMATICAL METHODS (exerpts for ME 334)

(full text at http://www.nd.edu/~powers/ame.561/M.ps)

Mihir Sen
Joseph M. Powers

Department of Aerospace and Mechanical Engineering
University of Notre Dame
Notre Dame, Indiana 46556-5637
USA

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Contents

Chapter 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

A first-order ordinary differential equation is of the form

$$F(x, y, y') = 0 \tag{1.1}$$

where $y' = \frac{dy}{dx}$.

1.1 Separation of variables

Equation (1.1) is separable if it can be written in the form

$$P(x)dx = Q(y)dy \tag{1.2}$$

which can then be integrated.

Example 1.1

Solve

$$yy' = \frac{8x+1}{y}, \text{ with } y(1) = -5.$$

Separating variables

$$y^2 dy = 8x dx + dx.$$

Integrating, we have

$$\frac{y^3}{3} = 4x^2 + x + C.$$

The initial condition gives $C = -\frac{140}{3}$, so that the solution is

$$y^3 = 12x^2 + 3x - 140.$$

The solution is plotted in Figure 1.1.

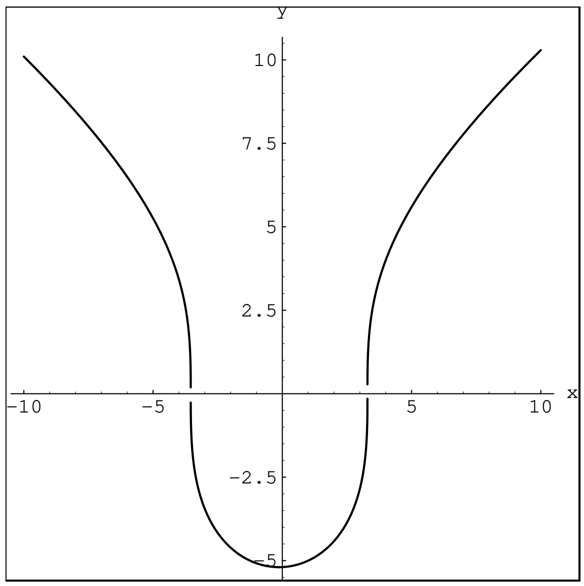

Figure 1.1: $y(x)$ which solves $yy' = (8x+1)/y$ with $y(1) = -5$.

1.2 Exact equations

A differential equation is exact if it can be written in the form

$$dF(x, y) = 0. \tag{1.3}$$

where $F(x, y) = 0$ is a solution to the differential equation Using the chain rule to expand the derivative

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$$

So for an equation of the form

$$P(x, y)dx + Q(x, y)dy = 0 \tag{1.4}$$

we have an exact differential if

$$\frac{\partial F}{\partial x} = P(x, y), \qquad \frac{\partial F}{\partial y} = Q(x, y) \tag{1.5}$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial P}{\partial y}, \qquad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial Q}{\partial x} \tag{1.6}$$

As long as $F(x, y)$ is continuous and differentiable, the mixed second partials are equal, thus,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1.7}$$

must hold if $F(x, y)$ is to exist and render the original differential equation to be exact.

Example 1.2

Solve

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{e^{x-y}}{e^{x-y}-1} \\
\left(e^{x-y}\right)dx + \left(1-e^{x-y}\right)dy &= 0 \\
\frac{\partial P}{\partial y} &= -e^{x-y} \\
\frac{\partial Q}{\partial x} &= -e^{x-y}
\end{aligned}
$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the equation is exact. Thus

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= P(x,y) \\
\frac{\partial F}{\partial x} &= e^{x-y} \\
F(x,y) &= e^{x-y} + A(y) \\
\frac{\partial F}{\partial y} = -e^{x-y} + \frac{dA}{dy} = Q(x,y) &= 1 - e^{x-y} \\
\frac{dA}{dy} &= 1 \\
A(y) &= y - C \\
F(x,y) = e^{x-y} + y - C &= 0 \\
e^{x-y} + y &= C
\end{aligned}
$$

The solution for various values of C is plotted in Figure 1.2.

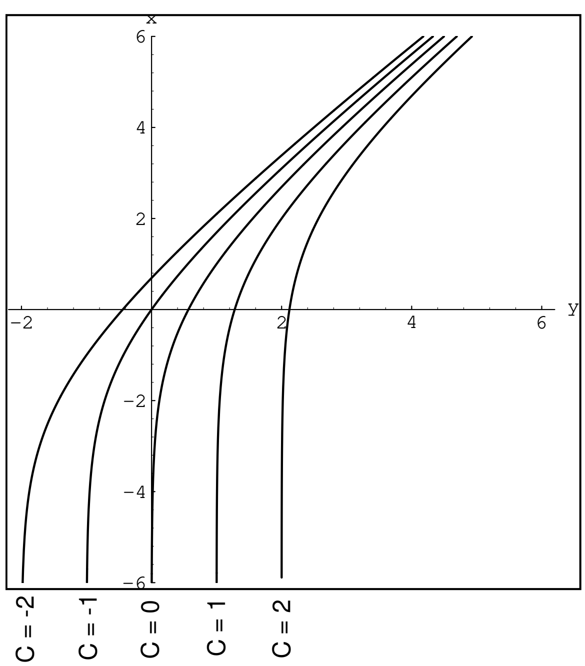

Figure 1.2: $y(x)$ which solves $y' = \exp(x-y)/(\exp(x-y)-1)$

1.3 Integrating factors

Sometimes, an equation of the form (1.4) is not exact, but can be made so by multiplication by a function $u(x,y)$, where u is called the *integrating factor*.

The general first-order linear equation

$$\frac{dy(x)}{dx} + P(x)\ y(x) = Q(x) \tag{1.8}$$

with

$$y(x_o) = y_o$$

can be solved using the integrating factor

$$e^{\int_a^x P(s)ds} = e^{(F(x)-F(a))}.$$

We choose a such that

$$F(a) = 0.$$

Multiply by the integrating factor and proceed:

$$\begin{aligned}
\left(e^{\int_a^x P(s)ds}\right)\frac{dy(x)}{dx} + \left(e^{\int_a^x P(s)ds}\right)P(x)\ y(x) &= \left(e^{\int_a^x P(s)ds}\right)Q(x) && (1.9)\\
\text{product rule:}\quad \frac{d}{dx}\left(e^{\int_a^x P(s)ds}y(x)\right) &= \left(e^{\int_a^x P(s)ds}\right)Q(x) && (1.10)\\
\text{replace } x \text{ by } t\text{:}\quad \frac{d}{dt}\left(e^{\int_a^t P(s)ds}y(t)\right) &= \left(e^{\int_a^t P(s)ds}\right)Q(t) && (1.11)\\
\text{integrate:}\int_{x_o}^x \frac{d}{dt}\left(e^{\int_a^t P(s)ds}y(t)\right)dt &= \int_{x_o}^x \left(e^{\int_a^t P(s)ds}\right)Q(t)dt && (1.12)\\
e^{\int_a^x P(s)ds}y(x) - e^{\int_a^{x_o} P(s)ds}y(x_o) &= \int_{x_o}^x \left(e^{\int_a^t P(s)ds}\right)Q(t)\ dt && (1.13)
\end{aligned}$$

which yields

$$y(x) = e^{-\int_a^x P(s)ds}\left(e^{\int_a^{x_o} P(s)ds}y_o + \int_{x_o}^x \left(e^{\int_a^t P(s)ds}\right)Q(t)dt\right). \tag{1.14}$$

Example 1.3

Solve

$$y' - y = e^{2x}; \qquad y(0) = y_o.$$

Here

$$P(x) = -1$$

or

$$P(s) = -1$$

$$\int_a^x P(s)ds = \int_a^x (-1)ds = -s|_a^x = a - x$$

So

$$F(\tau) = -\tau$$

For $F(a) = 0$, take $a = 0$. So the integrating factor is

$$e^{\int_a^x P(s)ds} = e^{a-x} = e^{0-x} = e^{-x}$$

Multiplying and rearranging, we get

$$\begin{aligned}
e^{-x}\frac{dy(x)}{dx} - e^{-x}y(x) &= e^x \\
\frac{d}{dx}\left(e^{-x}y(x)\right) &= e^x \\
\frac{d}{dt}\left(e^{-t}y(t)\right) &= e^t \\
\int_{x_o=0}^{x} \frac{d}{dt}\left(e^{-t}y(t)\right)\, dt &= \int_{x_o=0}^{x} e^t dt \\
e^{-x}y(x) - e^{-0}y(0) &= e^x - e^0 \\
e^{-x}y(x) - y_o &= e^x - 1 \\
y(x) &= e^x\left(y_o + e^x - 1\right) \\
y(x) &= e^{2x} + (y_o - 1)\, e^x
\end{aligned}$$

The solution for various values of y_o is plotted in Figure 1.3.

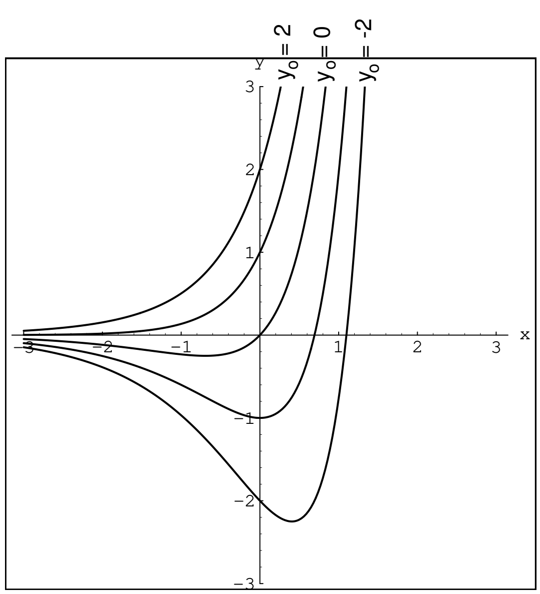

Figure 1.3: $y(x)$ which solves $y' - y = e^{2x}$ with $y(0) = y_o$

1.4 Reduction of order

There are higher order equations that can be reduced to first-order equations and then solved.

1.4.1 y absent

If

$$f(x, y', y'') = 0 \tag{1.15}$$

then let $u(x) = y'$. Thus $u'(x) = y''$, and the equation reduces to

$$f\left(x, u, \frac{du}{dx}\right) = 0 \tag{1.16}$$

which is an equation of first order.

Example 1.4

Solve

$$xy'' + 2y' = 4x^3.$$

Let $u = y'$, so that

$$x\frac{du}{dx} + 2u = 4x^3.$$

Multiplying by x

$$x^2\frac{du}{dx} + 2xu = 4x^4$$

$$\frac{d}{dx}(x^2u) = 4x^4.$$

This can be integrated to give

$$u = \frac{4}{5}x^3 + \frac{C_1}{x^2}$$

from which

$$y = \frac{1}{5}x^4 - \frac{C_1}{x} + C_2$$

for $x \neq 0$.

1.4.2 x absent

If

$$f(y, y', y'') = 0 \tag{1.17}$$

let $u(x) = y'$, so that

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\frac{dy}{dx} = \frac{du}{dy}u$$

The equation becomes

$$f\left(y, u, u\frac{du}{dy}\right) = 0 \tag{1.18}$$

which is also an equation of first order. Note however that the independent variable is now y while the dependent variable is u.

Example 1.5

Solve

$$y'' - 2yy' = 0; \qquad y(0) = y_o, \qquad y'(0) = y'_o.$$

Let $u = y'$, so that $y'' = \frac{du}{dx} = \frac{dy}{dx}\frac{du}{dy} = u\frac{du}{dy}$. The equation becomes

$$u\frac{du}{dy} - 2yu = 0$$

Now

$$u = 0$$

satisfies the equation. Thus

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ y &= C \\ \text{applying one initial condition:} \qquad y &= y_o \end{aligned}$$

This satisfies the initial conditions only under special circumstances, i.e. $y'_o = 0$. For $u \neq 0$,

$$\begin{aligned} \frac{du}{dy} &= 2y \\ u &= y^2 + C_1 \\ \text{apply I.C.'s:} \qquad y'_o &= y_o^2 + C_1 \\ C_1 &= y'_o - y_o^2 \\ \frac{dy}{dx} &= y^2 + y'_o - y_o^2 \\ \frac{dy}{y^2 + y'_o - y_o^2} &= dx \end{aligned}$$

from which for $y'_o - y_o^2 > 0$

$$\begin{aligned} \frac{1}{\sqrt{y'_o - y_o^2}} \tan^{-1}\left(\frac{y}{\sqrt{y'_o - y_o^2}}\right) &= x + C_2 \\ \frac{1}{\sqrt{y'_o - y_o^2}} \tan^{-1}\left(\frac{y_o}{\sqrt{y'_o - y_o^2}}\right) &= C_2 \\ y(x) = \sqrt{y'_o - y_o^2}\tan\left(x\sqrt{y'_o - y_o^2} + \tan^{-1}\left(\frac{y_o}{\sqrt{y'_o - y_o^2}}\right)\right) \end{aligned}$$

The solution for $y_o = 0, y'_o = 1$ is plotted in Figure 1.4.

For $y'_o - y_o^2 = 0$

$$\begin{aligned} \frac{dy}{dx} &= y^2 \\ \frac{dy}{y^2} &= dx \\ -\frac{1}{y} &= x + C_2 \end{aligned}$$

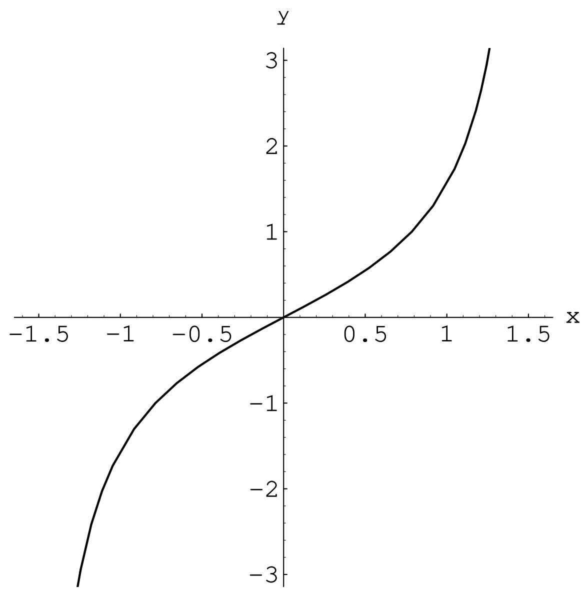

Figure 1.4: $y(x)$ which solves $y'' - 2yy' = 0$ with $y(0) = 0, y'(0) = 1$

$$\begin{aligned}
-\frac{1}{y_o} &= C_2 \\
-\frac{1}{y} &= x - \frac{1}{y_o} \\
y &= \frac{1}{\frac{1}{y_o} - x}
\end{aligned}$$

For $y_o' - y_o^2 < 0$, one would obtain solutions in terms of hyperbolic trigonometric functions.

Problems

1. Find the general solution of the differential equation $y' + x^2 y(1+y) = 1 + x^3(1+x)$. Plot solutions for $y(0) = -1, 0, 1$.

2. Solve
$$\dot{x} = 2tx + te^{-t^2}x^2$$
. Plot a solution for $x(0) = 1$.

3. Solve
$$3x^2y^2\ dx + 2x^3y\ dy = 0.$$

4. Solve
$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

5. Solve the nonlinear equation $(y' - x)y'' + 4y' = 4x$.

6. Solve $xy'' + 2y' = x$. Plot a solution for $y(0) = 2, y'(0) = 1$.

7. Solve $y'' - 2yy' = 0$. Plot a solution for $y(0) = 0, y'(0) = 2$.

8. Given that $y_1 = x^{-1}$ is one solution of $y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0$, find the other solution.

9. Solve

   (a) $y' \tan y + 2\sin x \sin(\frac{\pi}{2} + x) + \ln x = 0$

   (b) $xy' - 2y - x^4 - y^2 = 0$

   (c) $y' \cos y \cos x + \sin y \sin x = 0$

   (d) $y' + y \cot x = e^x$

   (e) $x^5 y' + y + e^{x^2}(x^6 - 1)y^3 = 0$, with $y(1) = e^{-1/2}$

   (f) $y' + y^2 - xy - 1 = 0$

   (g) $y'(x + y^2) - y = 0$

   (h) $y' = \frac{x+2y-5}{-2x-y+4}$

   (i) $y' + xy = y$

   Plot solutions for $y(0) = -1, 0, 1$ (except for part e).

10. Find all solutions of

$$(x+1)(y')^2 + (x-y)y' - y = 0$$

11. Find an a for which a *unique* solution of

$$(y')^4 + 8(y')^3 + (3a+16)(y')^2 + 12ay' + 2a^2 = 0, \text{ with } y(1) = -2$$

exists. Find the solution.

12. Solve

$$y' - \frac{1}{x^2}y^2 + \frac{1}{x}y = 1$$

Chapter 2

LINEAR DIFFERENTIAL EQUATIONS

An ordinary differential equation can be written in the form

$$\mathbf{L}(y) = f(x) \tag{2.1}$$

where $y(x)$ is an unknown function. The operator $\mathbf{L}$ is composed of a combination of derivatives $\frac{d}{dx}, \frac{d^2}{dx^2}$ etc. $\mathbf{L}$ is linear if

$$\mathbf{L}(y_1 + y_2) = \mathbf{L}(y_1) + \mathbf{L}(y_2) \tag{2.2}$$

and

$$\mathbf{L}(\alpha y) = \alpha \mathbf{L}(y) \tag{2.3}$$

where α is a scalar. The general form of $\mathbf{L}$ is

$$\mathbf{L} = P_n(x)\frac{d^n}{dx^n} + P_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + P_1(x)\frac{d}{dx} + P_0(x) \tag{2.4}$$

The ordinary differential equation (2.1) is then linear.

Definition: The functions $y_1(x), y_2(x), \ldots, y_n(x)$ are said to be *linearly independent* if $C_1 y_1(x) + C_2 y_2(x) + \ldots + C_n y_n(x) = 0$ if and only if $C_1 = C_2 = \ldots = C_n = 0$.

So, if a set of functions is linearly independent, we get

$$\begin{aligned} C_1 y_1(x) + C_2 y_2(x) + \ldots + C_n y_n(x) &= 0 \\ C_1 y_1'(x) + C_2 y_2'(x) + \ldots + C_n y_n'(x) &= 0 \\ &\vdots \\ C_1 y_1^{(n-1)}(x) + C_2 y_2^{(n-1)}(x) + \ldots + C_n y_n^{(n-1)}(x) &= 0 \end{aligned}$$

This is a linear system of the form $A_{ij} C_j = 0$. For linear *independence*, i.e. $C_j = 0$, we need $\det A_{ij} \neq 0$. This particular determinant is known as the *Wronskian* W of $y_1(x), y_2(x), \ldots, y_n(x)$ and is defined as

$$W = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} \tag{2.5}$$

$W \neq 0$ indicates linear independence of the functions $y_1(x), y_2(x), \ldots, y_n(x)$. Unfortunately, the converse is not always true; that is if $W = 0$, the equations may or may not be linearly dependent, though in most cases $W = 0$ indeed implies linear dependence.

Example 2.1

Determine the linear independence of (a) x and $2x$, (b) x and x^2.

(a) $W = \begin{vmatrix} x & 2x \\ 1 & 2 \end{vmatrix} = 0$, linearly dependent.

(b) $W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0$, linearly independent, except at $x = 0$.

A general form of linear equations of order n is

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = f(x). \tag{2.6}$$

The equation is said to be *homogeneous* if $f(x) = 0$.

A homogeneous equation of order n has n linearly independent solutions. These are called *complementary functions*. If y_i $(i = 1, \ldots, n)$ are the complementary functions of the equation, then

$$y(x) = \sum_{i=1}^{n} C_i y_i(x) \tag{2.7}$$

is the general solution of the homogeneous equation. If $y_p(x)$ is a *particular solution* of the equation (2.6), the general solution is then

$$y(x) = \sum_{i=1}^{n} C_i y_i(x) + y_p(x). \tag{2.8}$$

2.1 Complementary functions for equations with constant coefficients

This section will consider solutions to the homogeneous part of the differential equation.

2.1.1 Arbitrary order

Consider the homogeneous equation with constant coefficients

$$A_n y^{(n)} + A_{n-1} y^{(n-1)} + \ldots + A_1 y' + A_0 y = 0 \tag{2.9}$$

where A_i, $(i = 0, \ldots, n)$ are constants. To find the solution of this equation we let $y = e^{rx}$. Substituting we get

$$A_n r^n e^{rx} + A_{n-1} r^{(n-1)} e^{rx} + \ldots + A_1 r^1 e^{rx} + A_0 e^{rx} = 0. \tag{2.10}$$

Canceling the common factor e^{rx}, we get

$$A_n r^n + A_{n-1} r^{(n-1)} + \ldots + A_1 r^1 + A_0 r^0 = 0 \tag{2.11}$$

$$\sum_{j=0}^{n} A_j r^j = 0. \tag{2.12}$$

This is called the *characteristic* equation. It is an n^{th} order polynomial which has n roots (some of which could be repeated, some of which could be complex), r_i $(i = 1, \ldots n)$ from which n linearly independent complementary functions $y_i(x)$ $(i = 1, \ldots n)$ have to be obtained. The general solution is then given by equation (2.7).

If all roots are real and distinct, then the complementary functions are simply $e^{r_i x}$, $(i = 1, \ldots, n)$. If, however, k of these roots are repeated, i.e. $r_1 = r_2 = \ldots = r_k = r$, then the linearly independent complementary functions are obtained by multiplying e^{rx} by $1, x, x^2, \ldots, x^{k-1}$. For a pair of complex conjugate roots $p \pm qi$, one can use de Moivre's formula (see Appendix) to show that the complementary functions are $e^{px} \cos qx$ and $e^{px} \sin qx$.

Example 2.2

Solve

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2y = 0$$

Substituting $y = e^{rx}$, we get a characteristic equation

$$r^4 - 2r^3 + r^2 + 2r - 2 = 0$$

which can be factorized as

$$(r+1)(r-1)(r^2 - 2r + 2) = 0$$

from which

$$r_1 = -1, \qquad r_2 = 1 \qquad r_3 = 1 + i \qquad r_4 = 1 - i$$

The general solution is

$$\begin{aligned}
y(x) &= C_1 e^{-x} + C_2 e^x + C_3' e^{(1+i)x} + C_4' e^{(1-i)x} \\
&= C_1 e^{-x} + C_2 e^x + C_3' e^x e^{ix} + C_4' e^x e^{-ix} \\
&= C_1 e^{-x} + C_2 e^x + e^x \left(C_3' e^{ix} + C_4' e^{-ix} \right) \\
&= C_1 e^{-x} + C_2 e^x + e^x \left[C_3' \left(\cos(x) + i \sin(x) \right) + C_4' \left(\cos(-x) + i \sin(-x) \right) \right] \\
&= C_1 e^{-x} + C_2 e^x + e^x \left[(C_3' + C_4') \cos(x) + i (C_3' - C_4') \sin(x) \right] \\
y(x) &= C_1 e^{-x} + C_2 e^x + e^x (C_3 \cos x + C_4 \sin x)
\end{aligned}$$

where $C_3 = C_3' + C_4'$ and $C_4 = i(C_3' - C_4')$.

2.1.2 First order

The characteristic polynomial of the first order equation

$$ay' + by = 0 \tag{2.13}$$

is

$$ar + b = 0 \tag{2.14}$$

so

$$r = -\frac{b}{a} \tag{2.15}$$

thus the complementary function for this equation is simply

$$y = Ce^{-\frac{b}{a}x} \tag{2.16}$$

2.1.3 Second order

The characteristic polynomial of the second order equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \tag{2.17}$$

is

$$ar^2 + br + c = 0 \tag{2.18}$$

Depending on the coefficients of this quadratic equation, there are three cases to be considered.

- $b^2 - 4ac > 0$: two distinct real roots r_1 and r_2. The complementary functions are $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$.
- $b^2 - 4ac = 0$: one real root. The complementary functions are $y_1 = e^{rx}$ and $y_2 = xe^{rx}$.
- $b^2 - 4ac < 0$: two complex conjugate roots $p \pm qi$. The complementary functions are $y_1 = e^{px}\cos qx$ and $y_2 = e^{px}\sin qx$.

Example 2.3

Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

with solutions

$$r_1 = 1, \qquad r_2 = 2.$$

The general solution is then

$$y = C_1e^x + C_2e^{2x}$$

Example 2.4

Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

The characteristic equation is

$$r^2 - 2r + 1 = 0$$

with a repeated root

$$r = 1.$$

The general solution is then

$$y = C_1 e^x + C_2 x e^x$$

Example 2.5

Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

The characteristic equation is

$$r^2 - 2r + 10 = 0$$

with solutions

$$r_1 = 1 + 3i, \qquad r_2 = 1 - 3i.$$

The general solution is then

$$y = e^x(C_1 \cos 3x + C_2 \sin 3x)$$

2.2 Particular solutions

We will now consider particular solutions of the inhomogeneous equation (2.6).

The most common way to find particular solutions is to use the method of undetermined coefficients In this method we guess a solution with unknown coefficients and then substitute in the equation to determine these coefficients.

Example 2.6

$$y'' + 4y' + 4y = 169 \sin 3x$$

Thus

$$r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0$$

$$r = -2$$

Since the roots are repeated, the complementary functions are

$$y_1 = e^{-2x} \qquad y_2 = xe^{-2x}$$

For the particular function, guess

$$y_p = a \sin 3x + b \cos 3x$$

so

$$y_p' = 3a \cos 3x - 3b \sin 3x$$

$$y_p'' = -9a \sin 3x - 9b \cos 3x$$

Substituting in the differential equation, we get

$$(-9a \sin 3x - 9b \cos 3x) + 4\,(3a \cos 3x - 3b \sin 3x) + 4\,(a \sin 3x + b \cos 3x) = 169 \sin 3x$$

$$(-5a - 12b) \sin 3x + (12a - 5b) \cos 3x = 169 \sin 3x$$

Equating the coefficients of the sin and cos terms,

$$\begin{bmatrix} 12 & -5 \\ -5 & -12 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 169 \end{bmatrix}$$

we find that $a = -5$ and $b = -12$. The solution is then

$$y(x) = (C_1 + C_2 x)e^{-2x} - 5 \sin 3x - 12 \cos 3x.$$

Example 2.7

Solve

$$y'''' - 2y''' + y'' + 2y' - 2y = x^2 + x + 1$$

Let the particular integral be of the form $y_p = ax^2 + bx + c$. Substituting we get

$$-(2a + 1)x^2 + (4a - 2b - 1)x + (2a + 2b - 2c - 1) = 0$$

For this to hold for all values of x, the coefficients must be zero, from which $a = -\frac{1}{2}$, $b = -\frac{3}{2}$, and $c = -\frac{5}{2}$. Thus

$$y_p = -\frac{1}{2}(x^2 + 3x + 5)$$

The solution of the homogeneous equation was found in a previous example, so that the general solution is

$$y = C_1 e^{-x} + C_2 e^x + e^x (C_3 \cos x + C_4 \sin x) - \frac{1}{2}(x^2 + 3x + 5)$$

A variant must be attempted if any term of $F(x)$ is a complementary function.

Example 2.8
Solve

$$y'' + 4y = 6 \sin 2x$$

Since $\sin 2x$ is a complementary function, we will try

$$y_p = x(a \sin 2x + b \cos 2x)$$

from which

$$\begin{aligned} y_p' &= 2x(a \cos 2x - b \sin 2x) + (a \sin 2x + b \cos 2x) \\ y_p'' &= -4x(a \sin 2x + b \cos 2x) + 4(a \cos 2x - b \sin 2x) \end{aligned}$$

Substituting into the equation, we compare coefficients and get $a = 0$, $b = -\frac{3}{2}$. The general solution is then

$$y = C_1 \sin 2x + C_2 \cos 2x - \frac{3}{2} x \cos 2x.$$

Example 2.9
Solve

$$y'' + 2y' + y = xe^{-x}$$

The complementary functions are e^{-x} and xe^{-x}. To get the particular solution we have to choose a function of the kind $y_p = ax^3e^{-x}$. On substitution we find that $a = 1/6$. Thus the general solution is

$$y = C_1 e^{-x} + C_2 xe^{-x} + \frac{1}{6} x^3 e^{-x}$$

Problems

1. Find the general solution of the differential equation

$$y' + x^2 y(1 + y) = 1 + x^3(1 + x)$$

2. Show that the functions $y_1 = \sin x$, $y_2 = x \cos x$, and $y_3 = x$ are linearly independent. Find the lowest order differential equation of which they are the complementary functions.
3. Solve the following initial value problem for (a) $C = 6$, (b) $C = 4$, and (c) $C = 3$ with $y(0) = 1$ and $y'(0) = -3$.

$$\frac{d^2y}{dt^2} + C\frac{dy}{dt} + 4y = 0$$

   Plot your results.

4. Solve
   (a) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$
   (b) $\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 11\frac{d^2y}{dx^2} - 7\frac{dy}{dx} = 12$
   (c) $y'' + 2y = 6e^x + \cos 2x$
   (d) $x^2y'' - 3xy' - 5y = x^2 \log x$.
   (e) $\frac{d^2y}{dx^2} + y = 2e^x \cos x + (e^x - 2)\sin x$.
5. Find a particular solution to the following ODE using (a) variation of parameters and (b) undetermined coefficients.
$$\frac{d^2y}{dx^2} - 4y = \cosh 2x$$
6. Solve the boundary value problem
$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$$
with boundary conditions $y(0) = 0$ and $y(\pi/2) = -1$ Plot your result.
7. Solve
$$2x^2\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} - 8\frac{dy}{dx} = 1$$
with $y(1) = 4$, $y'(1) = 8$, $y(2) = 11$. Plot your result.
8. Solve
$$x^2y'' + xy' - 4y = 6x$$
9. Find the general solution of
$$y'' + 2y' + y = xe^{-x}$$
10. Find the Green's function solution of
$$y'' + y' - 2y = f(x)$$
with $y(0) = 0$, $y(1) = 0$. Determine $y(x)$ if $f(x) = 3\sin x$. Plot your result.
11. Find the Green's function solution of
$$y'' + y = f(x)$$
with $y(0) = 0$, $y'(\pi) = 0$. Verify this is the correct solution when $f(x) = x^3$. Plot your result.
12. Solve $y''' - 2y'' - y' + 2y = \sin^2 x$.
13. Solve $y''' + 6y'' + 12y' + 8y = e^x - 3\sin x - 8e^{-2x}$.
14. Solve $x^4y'''' + 7x^3y''' + 8x^2y'' = 4x^{-3}$.
15. Show that x^{-1} and x^5 are solutions of the equation
$$x^2y'' - 3xy' - 5y = 0$$
Thus find the general solution of
$$x^2y'' - 3xy' - 5y = x^2$$
16. Solve the equation
$$2y'' - 4y' + 2y = \frac{e^x}{x}$$
where $x > 0$.