Numerical Solutions to Scalar Conservation Laws

University of Notre Dame

Department of Aerospace and Mechanical Engineering

Prepared by: Alexander M. Davies

Prepared for: AME 48491 - Undergraduate Research, Joseph M. Powers

Notre Dame, Indiana 46556

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Abstract

This report focuses on numerical solutions to scalar conservation laws with an emphasis on convexity and the flux function. This study aims to amplify the importance and understanding of convexity and the flux function in flux conservation equations through provided numerical solutions to flux conservation equations with convex and non-convex flux functions. In particular, solutions utilizing a Lax-Friedrichs discretization are produced to the linear advection equation, the Bateman-Burgers' equation, and the Buckley-Leverett equation. For each equation, wave propagation and formation will be examined through relations to the flux function's derivatives and its convexity.

Table of Contents

1 Introduction

In this report, flux conservation equations will be described to provide context for the style of partial differential equations for which numerical solutions are given. Flux conservation equations are fundamental for modeling reality. In particular, flux conservative models are used to describe familiar events (e.g. traffic flow [5]) as well as physical quantities such as mass, momentum, or energy. Thus, conservation laws are of profound importance to understand the physical world. If a partial differential equation can be formed to express a relation between a quantity and its transport, that particular equation can be seen as flux conservative. A derivation of both integral and differential forms for a general flux conservation law will be formed in Section 2.

The concept of convexity is vital for understanding wave formation. A given convexity, convex or non-convex, is associated with a flux function for a particular conservation equation. Whether or not a flux function is convex outlines if the waveform is to be preserved. Both convexity and the flux function are to be examined in greater detail in Section 3. The Lax-Friedrichs finite differencing method is used for forming solutions to linear and non-linear flux conservation equations. The effects of using Lax-Friedrichs are outlined in Section 4.

The linear advection equation is a second-order partial differential equation used to describe the propagation of a signal at a constant wave speed. The equation is studied in detail in Section 5. The equation is linear, and thus can be solved exactly by *d'Alembert's solution* [1].

The Bateman-Burgers' equation, introduced by Bateman [2] and later studied by Burgers [3], is a nonlinear partial differential equation with importance for studying features of wave motion. The inviscid Bateman-Burgers can be transformed into a linear partial differential equation and can be solved exactly; however, the resulting solution can take on multiple values. When it does so, it is no longer of physical relevance. Thus, analysis in this report will be limited to the Bateman-Burgers' equation with intrinsic artificial viscosity harnessed from the Lax-Friedrichs method.

The Buckley-Leverett Equation, proposed by Buckley and Leverett [4], is important for understanding flow of fluid in porous media. The equation is another flux conservation equation; however, the flux function possesses non-convexity that is significant for understanding wave-splitting behavior. This equation is also nonlinear, and analysis will be limited to the Lax-Friedrichs numerical solution. Multiple sets of initial condtions will be examined to understand the nature of this equation in Section 7. For the remainder of the report, flux conservation equations, convexity, and the aforementioned equations will be examined in greater detail.

2 Flux Conservative Forms

It is useful to derive the equation for a generalized conservation principle in both integral and differential form. Let $u(x,t)$ be some conserved variable (e.g. mass, momentum or energy), and let $f(u(x,t))$ represent the flux of the conserved variable at point x and time t. Thus, the total amount of $u(x,t)$ in a given domain is given by

$$\text{Total } u \text{ in } [x_1, x_2] = \int_{x_1}^{x_2} u(x,t)\,\mathrm{d}x. \tag{1}$$

The rate of change of the conserved variable across the space defined by x_1 and x_2 is simply

$$\text{Net rate of accumulation} = f(u(x_1,t)) - f(u(x_2,t)), \tag{2}$$

or, in other words, the difference between u entering the space and u leaving the space. If we take a time derivative of (1) and note its equality to (2), the generalized integral form for a flux conservation equation can be shown below by (3) [5]:

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,\mathrm{d}x = f(u(x_1,t)) - f(u(x_2,t)). \tag{3}$$

Another practical way of expressing the integral form of the conservation law is given by providing the total amount of the conserved variable at time t_2 and relating this to the initial amount at t_1 and the rate of change of u through the space, that is:

$$\int_{x_1}^{x_2} u(x,t_2)\,\mathrm{d}x = \int_{x_1}^{x_2} u(x,t_1)\,\mathrm{d}x + \int_{t_1}^{t_2} f(u(x_1,t))\,\mathrm{d}t - \int_{t_1}^{t_2} f(u(x_2,t))\,\mathrm{d}t. \tag{4}$$

Note from the fundamental theorem of calculus and Leibniz's rule when $t_2 - t_1 \to 0$,

$$u(x,t_2) - u(x,t_1) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_1}^{t_2} u(x,t)\,\mathrm{d}t = \int_{t_1}^{t_2} \frac{\partial}{\partial t}u(x,t)\,\mathrm{d}t. \tag{5}$$

The limits of integration are a constant t_1 and t_2, so these do not play a role from Leibniz's rule. Performing a similar task for the flux function, for $x_2 - x_1 \to 0$, it is shown that

$$f(u(x_2,t)) - f(u(x_1,t)) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{x_1}^{x_2} f(u(x,t))\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial}{\partial x}f(u(x,t))\,\mathrm{d}x, \tag{6}$$

and substituting (5) and (6) into (4) provides

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} \left(\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x}\right)\mathrm{d}t\,\mathrm{d}x = 0. \tag{7}$$

For arbitrary t_1, t_2, x_1, and x_2, the only way to satisfy Eq. (7) is for the integrand to be zero. Thus the differential form of the flux conservation equation in one dimension is given as

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0, \tag{8}$$

where $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^m$ is an m-dimensional vector of conserved quantities [5]. In this report, analysis will be restricted to the scalar case where m=1. Note, the differential form of the conservation law is only valid for smooth u; however, a solution using the differential form does not require smooth initial conditions.

From Eq. (8), employing the chain rule on the spatial derivative, it can be seen that the expression for a generalized flux conservation equation can also be represented as

$$\frac{\partial u(x,t)}{\partial t} + f'(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0, \tag{9}$$

where $f'(u(x,t))$ is a speed of propagation of a disturbance in the conserved variable u. For a locally linear disturbance where $f'(u)$ is a local constant, f'_0, the solution locally takes on the form of the *d'Alembert solution* [1]:

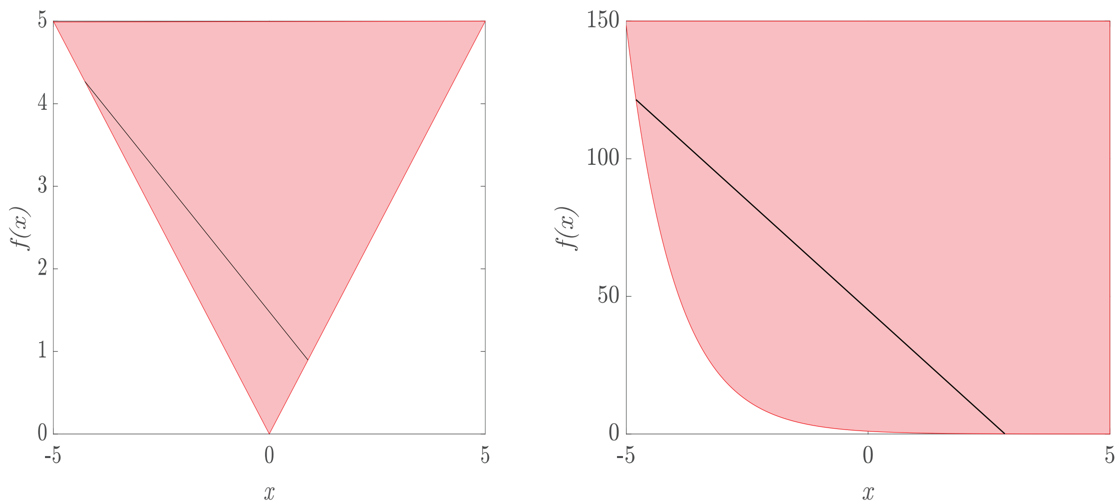
$$u(x,t) = g(x - f'_0 t).$$

Here g is an arbitrary function, and the solution represents a signal propagating at speed f'_0.

3 Convexity

Consider a function as given in Figure 1. The function's epigraph is defined as the region entirely above the curve as represented by the shaded region. Now let us define a convex function as one in which its epigraph is a convex set. A set is said to be convex if a line segment can connect any two points in the set and remain entirely within the set [6].

Note in Fig. 1(a) and Fig. 1(b) that there are no two points in the epigraph of each curve that may be connected with a line that exits the set (the black line segments are included to illustrate this). Thus, both of these functions are convex. A non-convex function displays the opposite properties. An example of a non-convex function is given in Figure 2. As shown in Fig. 2, a line may be drawn that enters the region below the curve to connect two points



(a) $f(x) = |x|$ (b) $f(x) = \mathrm{e}^{-x}$

Figure 1: Examples of convex functions.

within the epigraph.

To describe convexity through the use of an equation, it is useful to examine the curvature, κ, of a planar curve [7] given by,

$$\kappa = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}, \tag{10}$$

where the term of importance is the second derivative of the function, $f''(x)$. At any point should the second derivative of the function change sign, a non-convex function is created. To examine this property, the second derivatives of two of the aforementioned examples are given in Figure 3(a) and 3(b) for clarification. By examining the second derivative of the function, it is clear when a function is convex or otherwise. Convexity plays a large role in wave formation and motion and is primarily responsible for the splitting of wave forms into fans and discontinuities.

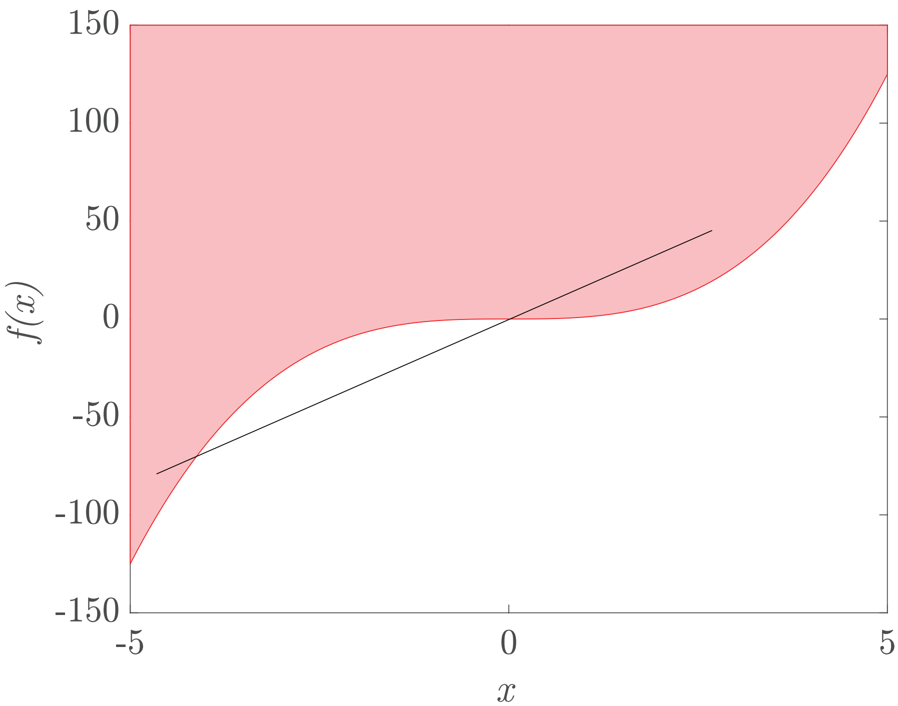


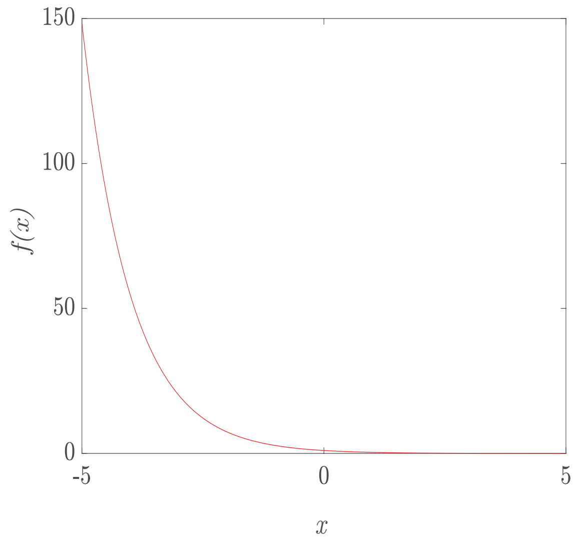
Figure 2: A non-convex, cubic function $f(x) = x^3$

4 The Lax-Friedrichs Differencing Method

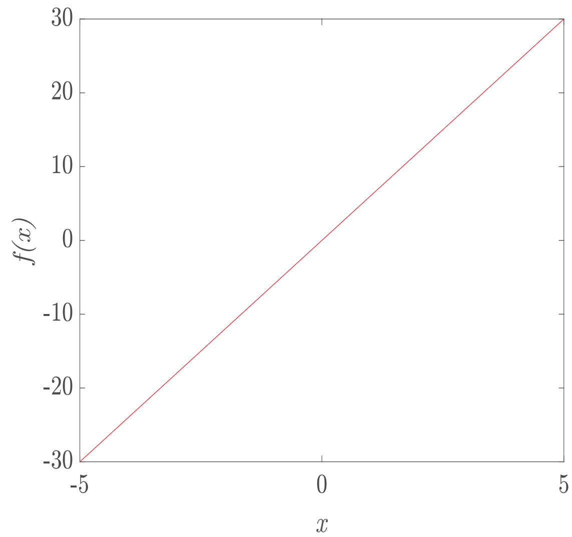
In following sections, numerical solutions will be provided for flux conservation equations. To do so, the Lax-Friedrichs numerical method is employed to provide numerical solutions. The basic form of the Lax-Friedrichs method [5] is given as

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{\Delta t}{2\Delta x}(f(u)_{j+1}^n - f(u)_{j-1}^n). \tag{11}$$

The Lax-Friedrichs method employs a central difference in space and a forward difference in time with the previous time evaluated at the spatially averaged midpoint. A sample code is provided in Appendix A. The method itself contains enough intrinsic artificial viscosity to avoid the inclusion of an extra viscous term in the numerical method. In other methods, (such as simple central differencing methods) explicit artificial viscosity needs to be added to prevent numerical instabilities found near discontinuities. Other numerical methods can



(a) $f''(e^{-x}) = e^{-x}$



(b) $f''(x^3) = 6x$

Figure 3: Second derivatives of convex and non-convex functions.

be used to recreate the solutions produced in this report, but Lax-Friedrichs was chosen to remain consistent so that comparisons may be drawn between solutions. For the entirety of the report, the discretization in space, Δx, was set to be 0.001, and the discretization in time, Δt, was 0.0005. The time discretization Δt must be far less than $\Delta x / f'(u)$ for numerically stable solutions.

5 The Linear Advection Equation

The linear advection equation can be represented as a linear, flux conservation equation with a flux function $f(u) = au$, where $a = \mathrm{d}f/\mathrm{d}u$ is a constant wave speed [5]:

$$\frac{\partial u}{\partial t} + \frac{\partial (au)}{\partial x} = 0. \tag{12}$$

Because of linearity, an exact *d'Alembert solution* [1] exists as

$$u = g(x - at).$$

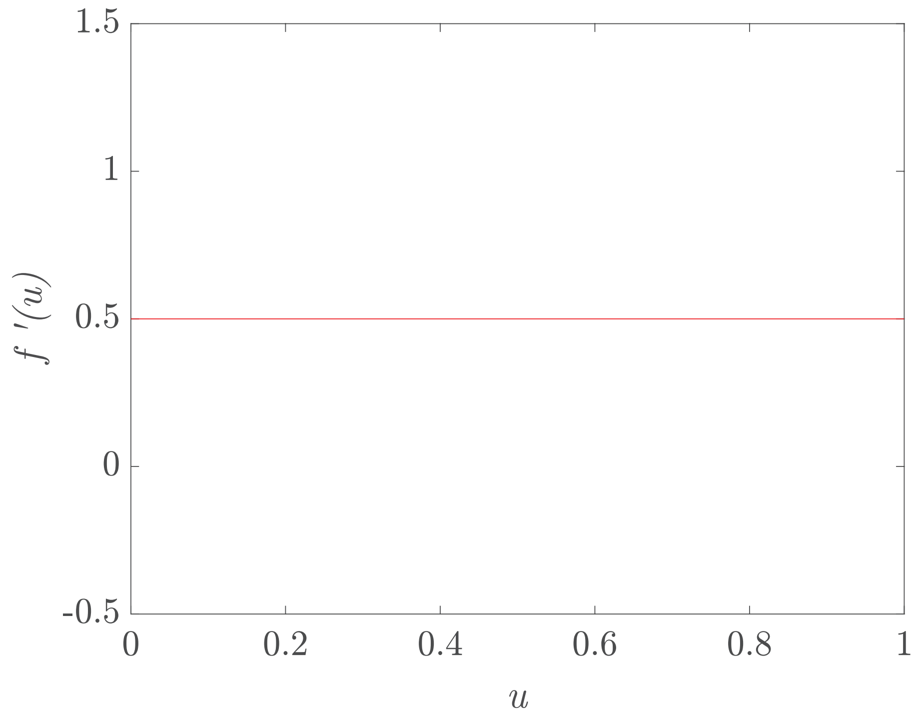


Figure 4: Derivative of the flux function for the linear advection equation with $a = 0.5$.

To predict the behavior of wave propagation, it is helpful to examine the derivative of the flux function. The derivative of the flux function for the linear advection equation is given in Figure 4. Because the derivative of the flux function with respect to u is the constant a, it should be expected that the wave propagates at that constant value. In addition, because the second derivative of the flux function is zero, we should not expect non-convexity to play a role in the shape of the wave form. Hence, it should be predicted that a traveling wave should maintain an invariant shape and speed as predicted by the characteristic, $f'(u)$. To demonstrate the above, a numerical solution with Heaviside initial conditions is provided in Figure 5. Note, the time steps for each numerical solution will remain consistent; thus, the time steps shown in Figure 5 will be left off of following plots for visual clarity.

In the numerical solution described by Fig. 5, the initial conditions are described by $u(x, 0) = 1$, for $x < 0$, $u(x, 0) = 0$, for $x > 0$. As seen in Fig. 5, there is a shallowing of the wave form as it traverses in the x-direction. This is due to artificial viscosity working

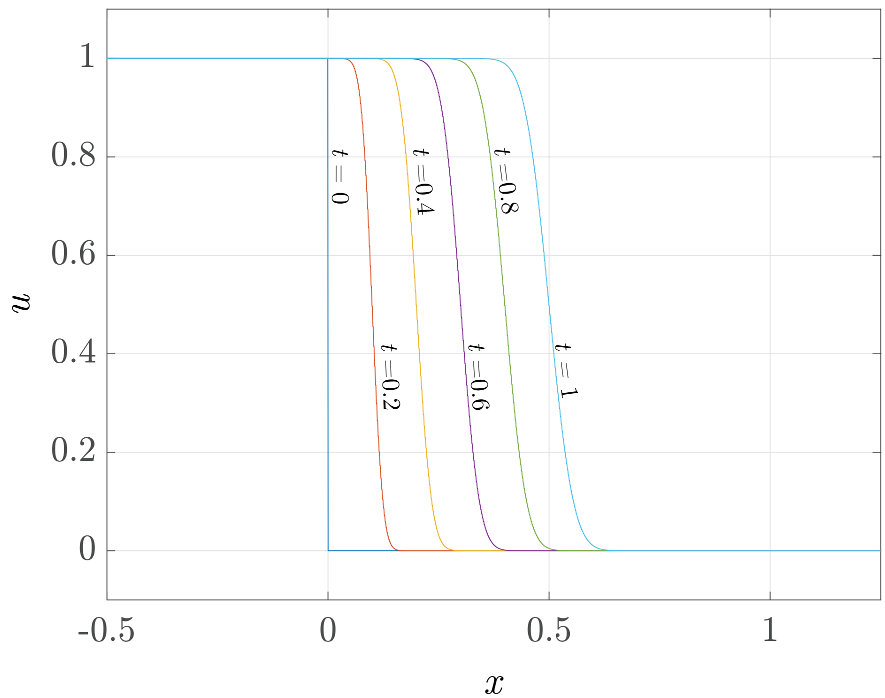


Figure 5: A numerical solution to the linear advection equation with Heaviside initial conditions for $a = 0.5$.

to eliminate the discontinuity. In the exact solution, the discontinuity holds its form as demonstrated by Figure 6.

It is helpful to examine the wave in x-t space where contours of constant u may be examined. In x-t space, it is often easier to recognize forming discontinuities and fans due to the nature of the characteristic lines stacking or "fanning out" from one another. An x-t diagram with the Heaviside initial conditions as demonstrated in Fig. 5 is given in Figure 7. In the case of the linear advection equation, the nature of the convex, constant flux function results in the constant motion of an invariant discontinuity. Thus, in the x-t diagram, it should be expected that the characteristic lines be tightly packed and nearly singular. Due to the discrete nature Lax-Friedrichs and its intrinsic viscosity, some of the jump in u is spread out over a zone of space whose width slowly increases with time.

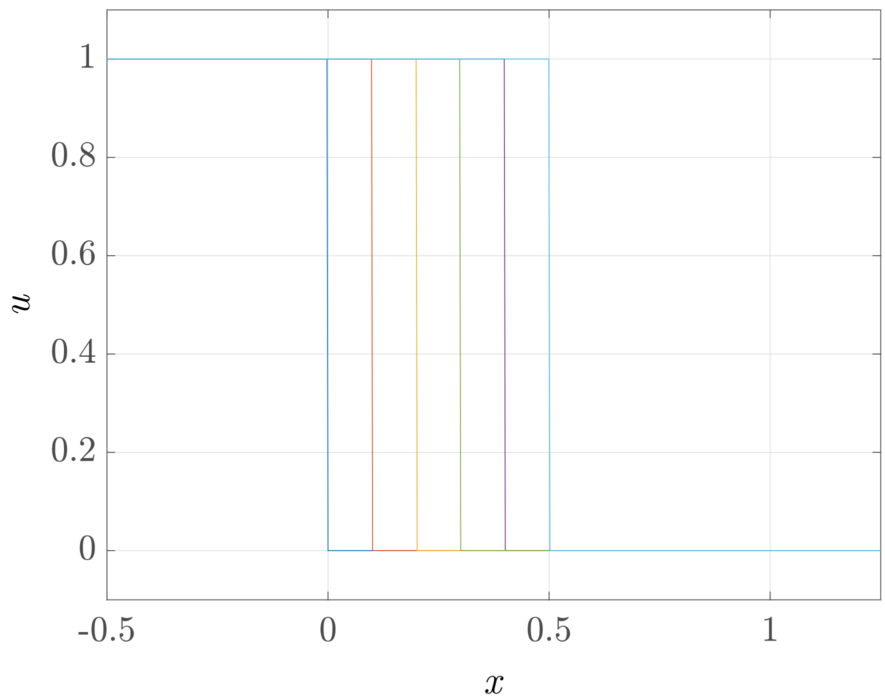


Figure 6: An exact solution to the linear advection equation with time step $\Delta t = 0.2$.

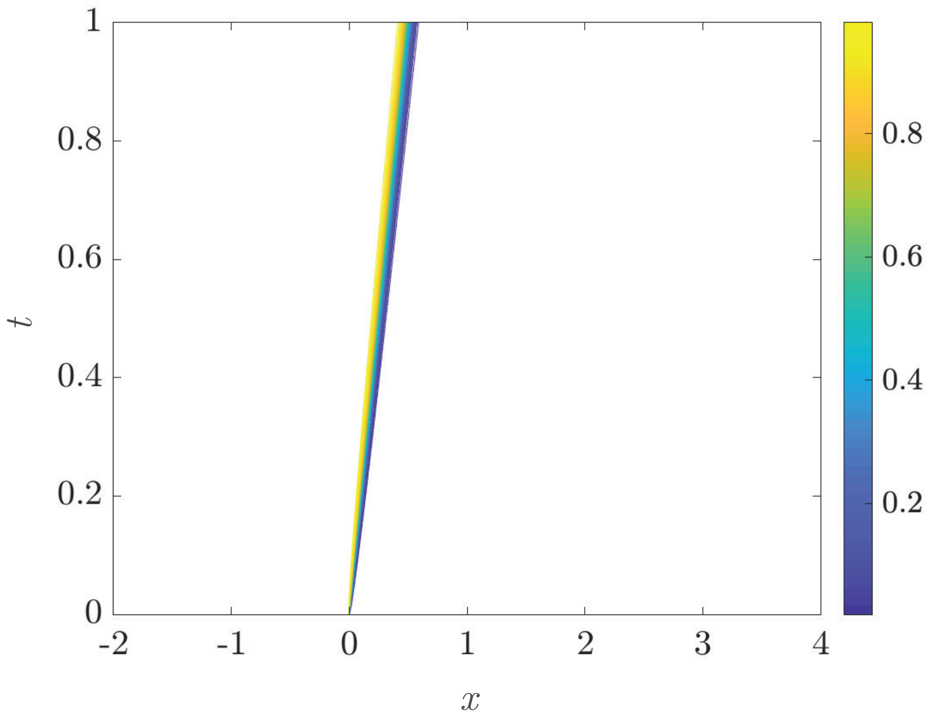


Figure 7: Contours of constant $u(x,t)$ for the linear advection equation with Heaviside initial conditions.

6 The Bateman-Burgers' Equation

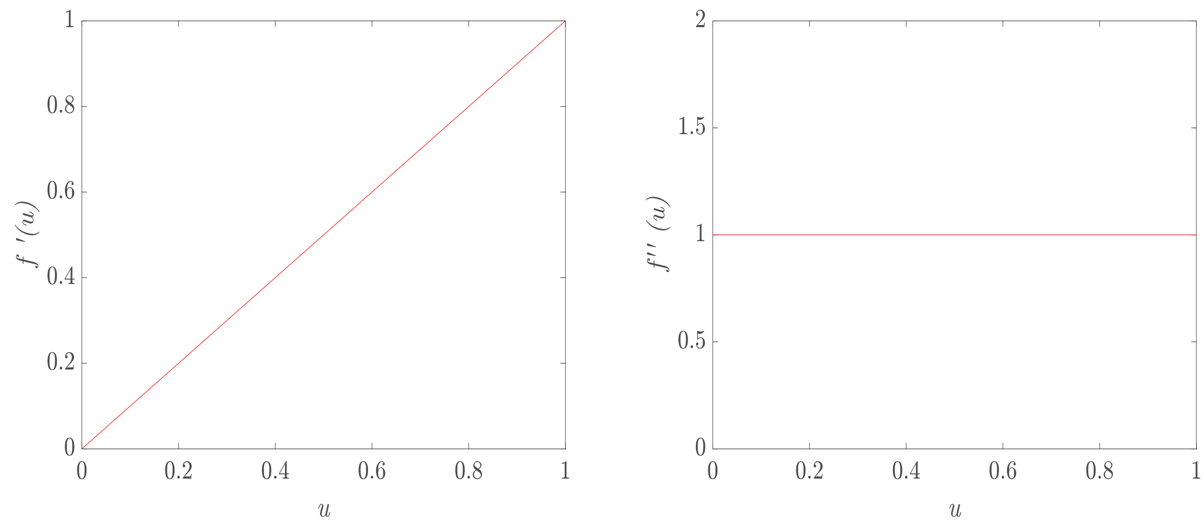
The Bateman-Burgers' equation is another example of a flux conservation equation that takes the form [5]

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \tag{13}$$

and its flux function is given by

$$f(u) = \frac{u^2}{2}. \tag{14}$$

The Bateman-Burgers' equation is useful in understanding the basics of wave motion and shock formation, although the conserved variable in the equation does not hold physical meaning. As will be seen, the solutions to Bateman-Burgers' problems often form "pseudo-shock" waves that represent a collapsing of characteristics resembling physical shock formation behavior.



(a) First derivative of the flux function with respect to u

(b) Second derivative of the flux function with respect to u

Figure 8: Derivatives of the Bateman-Burgers' flux function.

As can be seen by Eq. (13) and (14), the Bateman-Burgers' equation is nonlinear with respect to u. In many cases, the nonlinearity of the equation would prevent an exact solution; however, it is possible to create an exact solution through change of variables to the nonlinear partial differential equation [1]. The flux function's derivatives are given in Figure 8(a) and 8(b). In equation form, the derivative of the flux function is simply,

$$f'(u) = u. \tag{15}$$

For the given solutions, u will be held strictly positive; however, should different initial conditions be chosen where $u < 0$, propagation of the wave form would be leftwards rather than rightwards.

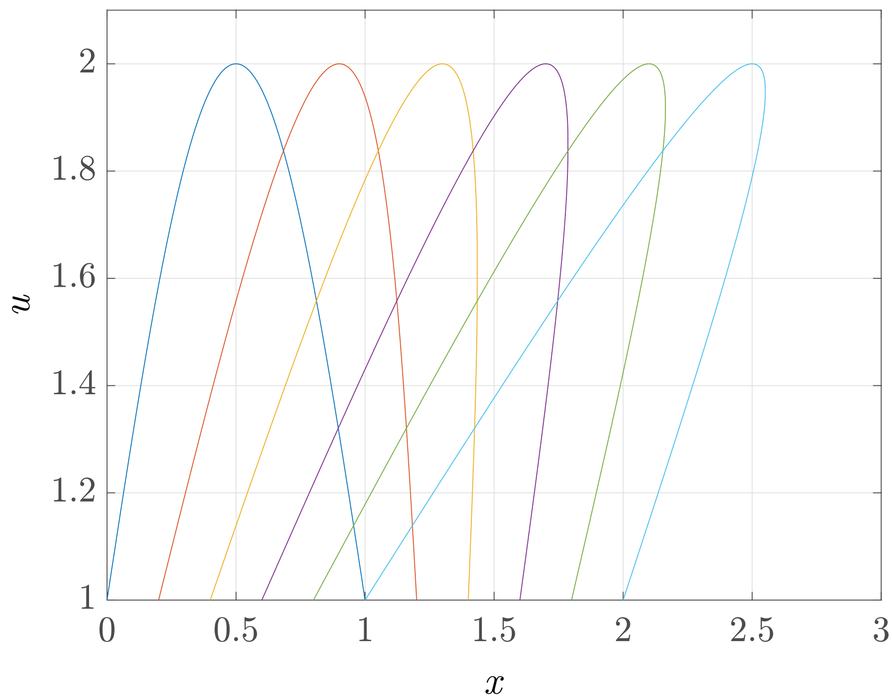
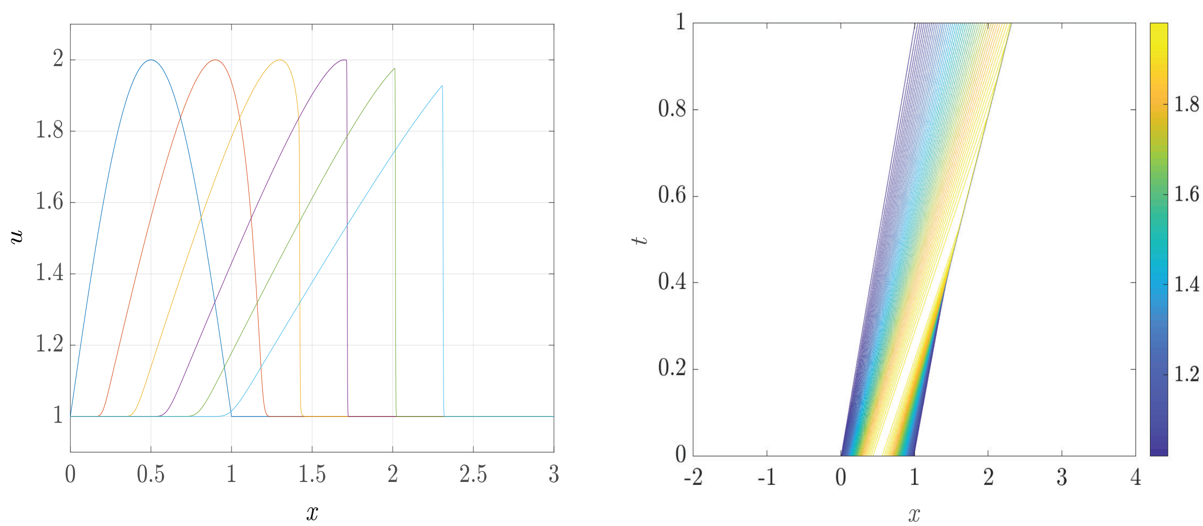


Figure 9: An exact solution with $\Delta t = 0.2$ to the inviscid Bateman-Burgers' equation with initial conditions $u(x, 0) = 1 + \sin(\pi x)$ for $0 \leq x \leq 1$.

Properties of the derivatives of the flux functions are important to understand the wave formation and propagation produced in the following solutions. Note, the curvature of the flux function is always positive (given by a positive $f''(u)$ for all u), so the flux function is purely convex. The first derivative, more interestingly, has a linear relationship with u. The linear relationship will play a large role in the propagation of the wave form as time progresses. Unlike the linear advection equation, where the flux derivative had no dependence on the conserved variable, the flux derivative's linear relation with the conserved variable will work to counteract the artificial viscosity induced by the Lax-Friedrichs numerical method. In the exact solution, there is a non-physical, "triple-valued," solution that corresponds to non-physical wave behavior [1].

Though not particularly physically useful, the triple-valued solution is helpful for understanding the effects of viscosity on the wave system. The exact solution to the inviscid Bateman-Burgers' equation is provided in Figure 9 for context. The numerical solution is given in Figure 10. The exact result is non-physical in the context of wave motion. It is obvious that as time progresses, a single point in space cannot possess two unique wave speeds. From observation, it is understood that this phenomenon does not occur due to the true nature of viscosity in physical wave motion.



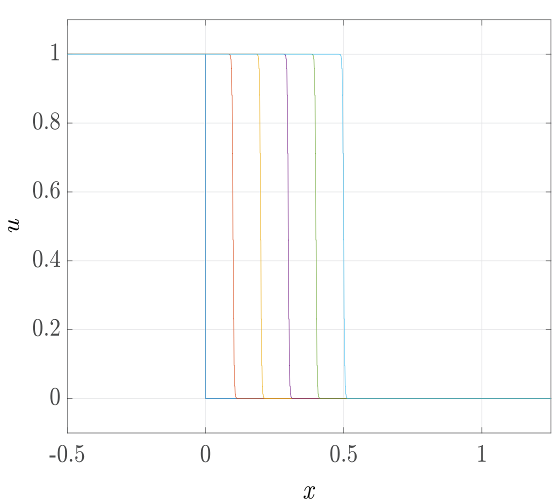
(a) u-x solution with $\Delta t = 0.2$ and conditions $u(x, 0) = 1 + \sin(\pi x)$ for $0 \leq x \leq 1$.

(b) x-t diagram for the Lax-Friedrichs solution to the Bateman-Burgers' equation.

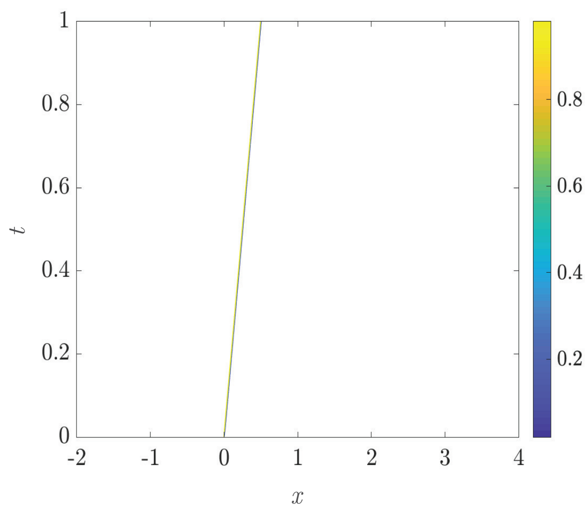
Figure 10: A numerical solution to the Bateman-Burgers' equation.

In the exact solution, however, it is easier to visualize the effects that the flux derivative has on the wave's motion. As is clearly demonstrated in Fig. 9, points at the peak of the wave possess a higher traveling wave speed. This is made clear in Fig. 8(a). The linear relationship with the flux derivative corresponds to greater particle velocities, $f'(u)$, as u increases. In the physical world, as these characteristics catch up to one another, it results in the formation of a shock wave. On the trailing side of the wave form, there is a spreading out or separation of the characteristics that results in the formation of a fan. Shock wave formation is not represented in Fig. 9 due to the absence of viscosity from the system. To correct this, the intrinsic artificial viscosity of the Lax-Friedrichs numerical method was harnessed to produce the result in Fig. 10.

In the x-t diagram provided for the Lax-Friedrichs solution, the discontinuity or "pseudo-shock" wave is represented by the single sharp line into which the characteristics converge, and the fan is represented by the characteristics to the left of the shock that appear to spread away from one another.



(a) u-x solution with $\Delta t = 0.2$ for Heaviside initial conditions.



(b) x-t diagram for the Lax-Friedrichs solution to the Bateman-Burgers' equation.

Figure 11: A numerical solution to the Bateman-Burgers' equation.

It is also beneficial to examine the solution to the Heaviside initial conditions problem solved in Section 5 for the Bateman-Burgers' Equation. Unlike the numerical solution provided in Section 5 however, the derivative of Bateman-Burgers' flux function allows the discontinuity to hold its form and avoid a shallowing that was visible in Fig. 5. In the examination of the x-t diagram, the collapsed convergence of the characteristic lines indicates the traveling "pseudo-shock" wave as predicted. The Heaviside solution is provided in Figure 11.

7 The Buckley-Leverett Equation

The Buckley-Leverett equation is a flux conservation equation of the form [8],

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{16}$$

where its flux function $f(u)$ is given by

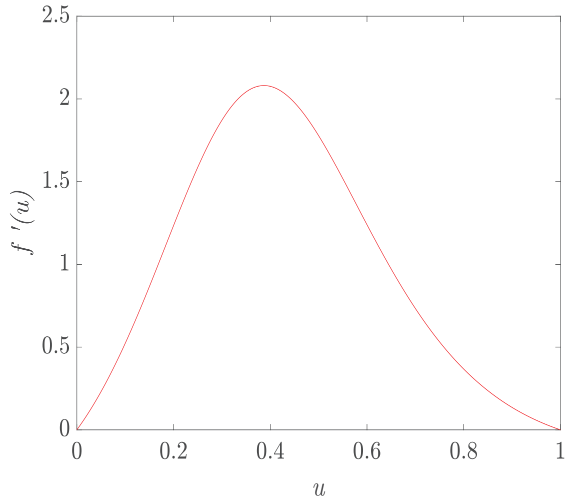
$$f(u) = \frac{u^2}{u^2 + b(1-u)^2}, \quad b < 1. \tag{17}$$

The Buckley-Leverett equation is useful for understanding fluid flow in porous media and is often used in oil reservoir simulation [8]. Note, the b in the denominator of the flux function is a constant less than one generally used to represent the viscosity of oil when compared to water. For the Heaviside initial conditions problem, the connection to the oil reservoir problem will be made.

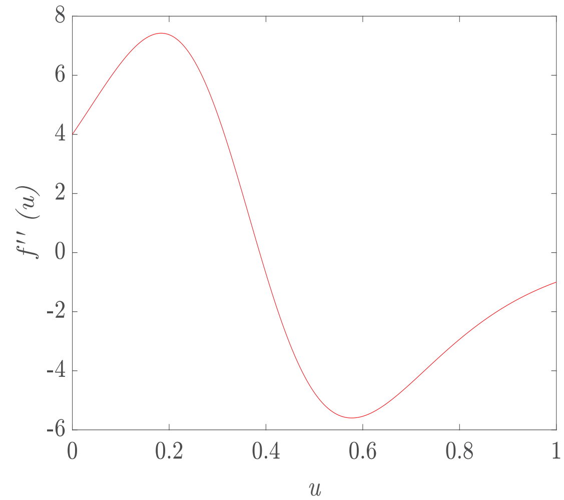
The flux function for the Buckley-Leverett equation is more complicated than both the linear advection equation and the Bateman-Burgers' equation, and the nonlinearity of the function makes an exact solution impractical. Of particular importance again, however, are the derivatives of the flux function. For these derivatives, figure representations are given by Figure 12. The derivative of the flux function in equation form is provided as,

$$f'(u) = \frac{2bu(1-u)}{(u^2 + b(1-u)^2)^2}. \tag{18}$$

Similar to Bateman-Burgers, the derivative of the flux function has a dependence on the



(a) $f'(u)$ for the Buckley-Leverett flux function.

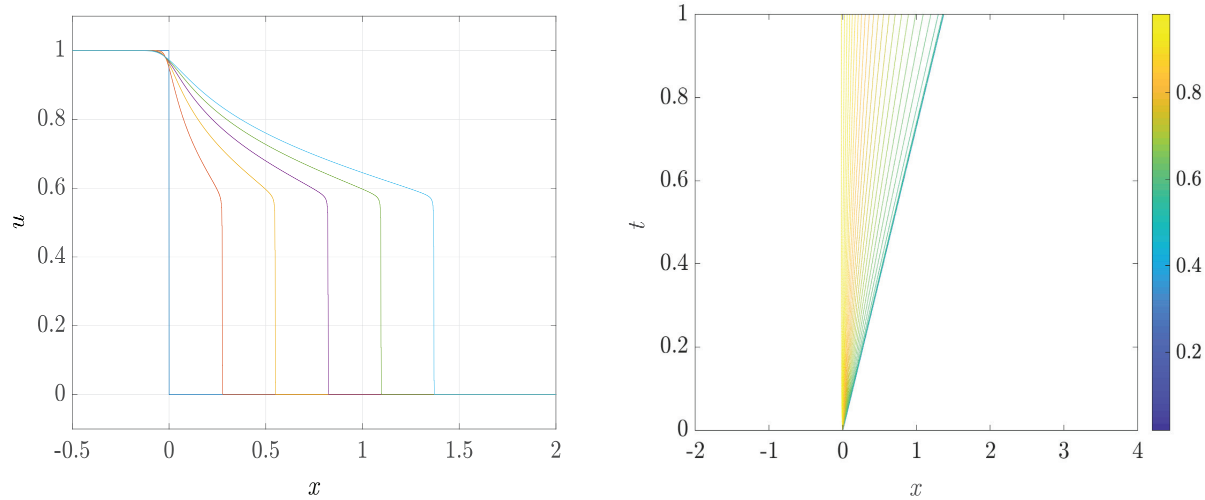


(b) $f''(u)$ for the Buckley-Leverett flux function.

Figure 12: Derivatives of the flux function for the Buckley-Leverett equation.

conserved variable. However, contrary to Bateman-Burgers, the dependence is nonlinear; thus, it should be expected that the characteristic motion of the wave form at a given u agrees with a corresponding $f'(u)$ predicted by Fig. 12(a). In Fig. 12(b), it is observed that the curvature of the flux function changes sign. This leads to a non-convexity in the flux function that creates a "wave-splitting" characteristic of non-convex flux conservation equations. In actuality, the precise number of times the curvature changes sign is equivalent to the number of "wave-splits" visible in the solution.

The Heaviside initial conditions problem is fundamental to understanding the Buckley-Leverett equation. In Fig. 13(a), it is apparent that the initial discontinuity has not held its form as time progresses. This is caused by the non-convexity of the flux function. In this case, it is apparent that a discontinuity still exists; however, a new, continuous compression is present. In the context of oil reservoir simulation, the above solution resembles firing water into a pocket of oil. If water is represented by $u = 1$, and oil by $u = 0$ (it is helpful to imagine the densities of the respective fluids), we note pure oil to the right of the discontinuity in Fig. 13(b), decreasing as t increases, followed by a combination of water and oil in the



(a) u-x solution with $\Delta t = 0.2$ for Heaviside initial conditions, $b = 0.5$.

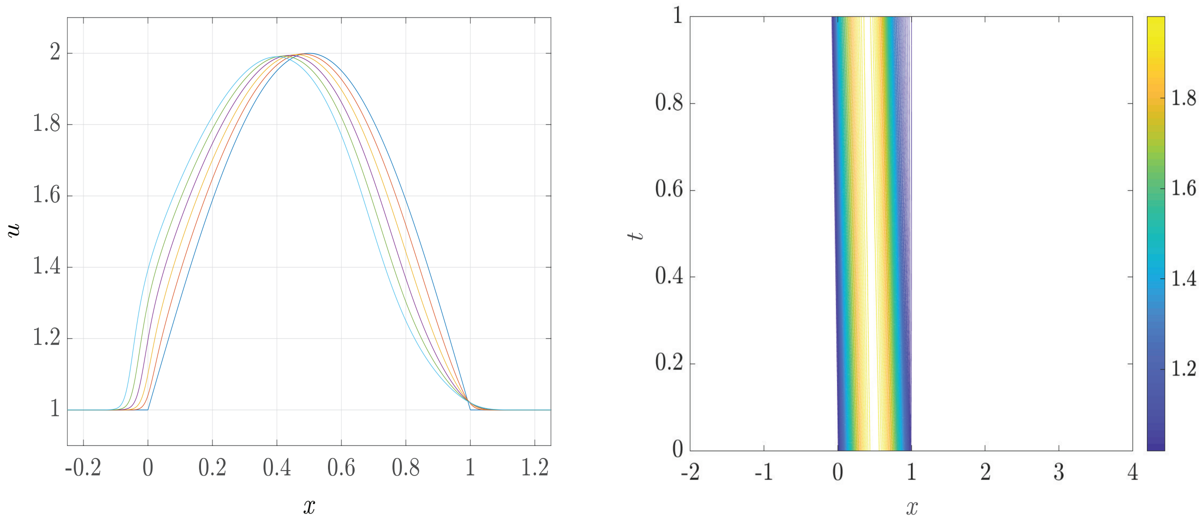
(b) x-t diagram for the Lax-Friedrichs solution to the Buckley-Leverett equation.

Figure 13: A numerical solution to the Buckley-Leverett equation.

compression fan, and pure water indicated by the yellow characteristics to the left of the fan.

Examining the case of the sine wave initial conditions introduced in Section 6, odd behavior is visualized in the solution. As the Buckley-Leverett equation is traditionally purposed for $0 \leq u \leq 1$, the set of sine wave initial conditions drive the traditional purpose out of bounds. The solution to the sine wave initial conditions is shown by Figure 14.

As is visible in Fig. 14(a), the wave propagates to the left rather than to the right as it has for all previous solutions. The solution to this issue arises due to the flux function's derivative given by Eq. (18). Through close analysis of the derivative, it is clear that for $u > 1$, the derivative of the flux function is negative. This creates the "backwards" propagation that is unexpected of traditional wave motion. Further, the nonlinearity of the flux function creates a fan on the trailing side of the wave form, as well as a steepening of the wave form directly in front of the fan. This is similar to the behavior noted in Fig. 13. The slight decrease in amplitude of the crest of the wave form is due to the Lax-Friedrichs numerical viscosity present in the solution.



(a) u-x solution with $\Delta t = 0.2$ and conditions $u(x,0) = 1 + \sin(\pi x)$ for $0 \leq x \leq 1$, $b = 0.5$.

(b) x-t diagram for the Lax-Friedrichs solution to the Buckley-Leverett equation.

Figure 14: A numerical solution to the Buckley-Leverett equation.

8 Discussions and Conclusions

In this report, properties of flux conservation equations such as the linear advection equation, Bateman-Burgers' equation, and the Buckley-Leverett equation were studied and compared to one another, and emphasis was placed on the derivatives and the convexity of their respective flux functions.

As was demonstrated through the linear advection equation and the Bateman-Burgers' equation, convex flux functions result in preserved wave shapes; whereas, in the non-convex case as witnessed in Buckley-Leverett, a "wave-splitting" effect is recognized. This wave-splitting as witnessed in Buckley-Leverett is integral for future work in non-ideal equations of state, as certain materials described by equations such as the van der Waals equation of state feature non-convex Hugoniot curves that drive non-traditional shock wave behavior.

Further, significant emphasis was placed on the derivatives of the flux function for each equation. In particular, the first derivative of the flux function is primarily responsible for the particle velocity of a point on the wave with its sign indicating the direction in which the particle will travel. In the linear advection equation, it was seen that the first derivative of the flux function is a constant wavespeed. In the Bateman-Burgers' equation, a linear relationship was observed between the flux derivative and the conserved variable. Lastly, in the Buckley-Leverett equation, a more complex equation governed the characteristic particle velocity of a given point on the wave. An understanding of the flux derivative and its relation to wave progression is vital for the creation of future flux conservation equations that may serve a physical purpose (e.g. the Buckley-Leverett equation in oil reservoir simulation) or that could be created to improve describing physical wave motion.

A general knowledge of scalar conservation laws is important for progressing into systems of conservation laws such as the Euler equations. Future work lies here, as well as in more advanced topics such as non-ideal equations of state.

References

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Appendix A

A generalized MATLAB code used for Lax-Friedrichs numerical solutions is attached below:

```
%%LAX Friedrichs
% Alex Davies
% Dr. Powers
clear all;
clc;
close all;

t=1;
% only used as a fill in for plot command,
% alternates between 1 and 2 depending on i
dx=0.001;
% 0.001 works well for 1/1000,
% 0.005 works well for most larger values
dt=0.0005;
ttot=1;
xtot=6;
pi=3.14159265;
N=ttot/dt;   %Total number of time grid points
N=N;
K=xtot/dx;   %Total number of space grid points
x=linspace(-2,4,K);  %Defining a uniform x
ti=linspace(0,ttot,N);
for j=1:K
    if x(j) < 1 && x(j) < 0
    u1(j,1)=1;  %Sampling u(x,t=0) for all x
    %u1(j,1)=1;
    elseif x(j) < 1
    u1(j,1)= 1+sin(pi*x(j));
    else
    u1(j,1)=1;
   % u1(j,1)=1;
    end
end

for i=1:N
    u1(1,i)=1;
 % Setting left endpoint based on initial condition
end

for i=1:N
```

```
    u1(K,i)=1;
 % Setting right endpoint based on initial condition
end

a=0.5;

for i=1:(N-1)  %Time loop

    for j=2:(K-1)  % Space loop
        %u1(j,i+1)=(1/2)*(u1(j-1,i)+u1(j+1,i))-(dt/(2*dx)).*a.*((u1(j+1,i))-
        %u1(j,i+1)=(1/2)*(u1(j-1,i)+u1(j+1,i))-...
          %  (dt/(2*dx))*((u1(j+1,i))^2/2-(u1(j-1,i))^2/2);
        u1(j,i+1)=1/2*(u1(j-1,i)+u1(j+1,i))-...
           ((dt)/(2*dx))*((u1(j+1,i))^2/((u1(j+1,i))^2+a*...
            (1-(u1(j+1,i)))^2)-((u1(j-1,i))^2/((u1(j-1,i))^2+...
            a*(1-(u1(j-1,i)))^2)));
    end

end


h1 = plot(x,u1(:,1));
hold on;
h2 = plot(x,u1(:,(i+t)/5));
hold on;
h3 = plot(x,u1(:,2*(i+t)/5));
hold on;
h4 = plot(x,u1(:,3*(i+t)/5));
hold on;
h5 = plot(x,u1(:,4*(i+t)/5));
hold on;
h6 = plot(x,u1(:,5*(i+t)/5));
axis([-0.25 1.25 0.9 2.1]);
grid on
xlabel('x');
ylabel('u');
string='t = ';
A1 = num2str(ti(1));
A2 = num2str(ti((i+t)/5));
A3 = num2str(ti(2*(i+t)/5));
A4 = num2str(ti(3*(i+t)/5));
A5 = num2str(ti(4*(i+t)/5));
A6 = num2str(ti(5*(i+t)/5));
%label(h1,strcat(string,A1),'location','middle');
%label(h2,strcat(string,A2),'location','middle');
```

```
%label(h3,strcat(string,A3),'location','middle');
%label(h4,strcat(string,A4),'location','middle');
%label(h5,strcat(string,A5),'location','middle');
%label(h6,strcat(string,A6),'location','middle');
%legend(strcat(string,A1), strcat(string,A2), ...
   % strcat(string,A3),strcat(string,A4),...
   % strcat(string,A5),strcat(string,A6));
set(gca,'FontSize',20,'FontName','CMU Serif');
set(gca, 'TickLabelInterpreter','latex');

figure;
u1=transpose(u1);
contour(x,ti,u1,50);
colorbar;
set(gca,'FontSize',20,'FontName','CMU Serif');
set(gca, 'TickLabelInterpreter','latex');
```