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The effect of diffusion on the dynamics of unsteady detonations

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The dynamics of a one-dimensional detonation predicted by a one-step irreversible Arrhenius kinetic model are investigated in the presence of mass, momentum and energy diffusion. A study is performed in which the activation energy is varied and the length scales of diffusion and reaction are held constant. As the activation energy is increased, the system goes through a series of period-doubling events and eventually undergoes a transition to chaos. The rate at which these bifurcation points converge is calculated and shown to be in agreement with the Feigenbaum constant. Within the chaotic regime, there exist regions in which there are limit cycles consisting of a small number of oscillatory modes. When an appropriately fine grid is used to capture mass, momentum and energy diffusion, predictions are independent of the differencing scheme. Diffusion affects the behaviour of the system by delaying the onset of instability and strongly influencing the dynamics in the unstable regime. The use of the reactive Euler equations to predict detonation dynamics in the unstable and marginally stable regimes is called into question as the selected reactive and diffusive length scales are representative of real physical systems; reactive Navier–Stokes is a more appropriate model in such regimes.

Key words: bifurcation, detonations, reacting flows

1. Introduction

A detonation is a shock-induced combustion wave in which the exothermic energy release contributes to driving the shock. As mentioned by Shepherd (2009), it is a common notion in detonation theory that the effects of diffusion can be neglected in comparison with those of reaction and advection, see Fickett & Davis (1979), Fedkiw, Merriman & Osher (1997), Oran *et al.* (1998), Hu *et al.* (2004), Wang, He & Yu (2005), He & Karagozian (2006) and Walter & da Silva (2006). However, there are indications that such an assumption can be problematic. For example, using grid sizes around 10^{-6} m for their three-dimensional simulations of unsteady H_2–air detonations, Tsuboi, Eto & Hayashi (2007) report wave dynamics that show strong sensitivity to the grid. While apparent convergence of some structures was reported,

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they also note with regard to some particulars of the detonation structure: 'The present results cannot resolve such cross-hatchings in the ribbon because of a lack of grid resolution'. Deiterding (2009) also reports that the interactions between chemistry and hydrodynamic flow in inviscid detonations 'in general exhibit a strong dependency on the mesh spacing'. The presence of reaction dynamics and steep gradients at micrometre length scales suggests that in fact physical diffusion has an important role to play. Indeed, Powers (2006) reported that two-dimensional detonation patterns are strongly grid-dependent for simulations of reactive Euler equations, but relax to a grid-independent dissipative structure for a comparable reactive Navier–Stokes calculation. This suggests numerical diffusion is actually playing a significant role in the inviscid calculations and that one should consider the introduction of grid-independent physical diffusion to properly capture the dynamics.

Consideration of the reaction-advection length scales admitted by an inviscid detonation explains why such fine discretizations are necessary. Powers & Paolucci (2005) performed a spatial eigenvalue analysis on a detailed kinetic H_2–air model and showed for inviscid detonations that the length scales for a steady Chapman–Jouguet (CJ) detonation can span five orders of magnitude near the equilibrium, with the smallest length scale for an ambient mixture at atmospheric pressure being 10^{-7} m and the largest being 10^{-2} m; away from equilibrium the breadth of scales can be even larger. These fine reaction scales are a manifestation of an averaged representation of the molecular collision model in which the fundamental length scale is the mean free path, see Al-Khateeb, Powers & Paolucci (2010). In order to achieve a mathematically verified prediction, this wide range of scales must be resolved, which poses a daunting task.

The choice of a one-step kinetic model induces a single reaction scale, in contrast to the multiple reaction scales of detailed kinetic models. This allows the interplay between chemistry and transport phenomena to be studied more easily. Such a model has been studied extensively in the inviscid limit; the stability (see Sharpe 1997) and nonlinear dynamics are well understood. To briefly review, Erpenbeck (1964) originally investigated the linear stability. Lee & Stewart (1990) developed a normal-mode approach to the linear stability of the idealized detonation to one-dimensional perturbations using a shooting method to find the unstable modes. Bourlioux, Majda & Roytburd (1991) studied the nonlinear development of instability. Kasimov & Stewart (2004) also applied a normal mode approach to the linear stability problem and performed a numerical analysis using a first-order shock-fitting technique. Ng *et al.* (2005) developed a coarse bifurcation diagram showing how the oscillatory behaviour became progressively more complex as activation energy increased. Henrick, Aslam & Powers (2006) developed a more detailed bifurcation diagram using a true fifth-order shock-fitting method. In two dimensions, Watt & Sharpe (2005) concluded, for this model, that 'resolved and accurate calculations of the cellular dynamics are currently computationally prohibitive, even with a dynamically adaptive numerical scheme'. For the same model in two dimensions, Radulescu *et al.* (2007) found that the correct average chemical thickness observed in experiments cannot be realized using artificial diffusive terms alone. Seitenzahl *et al.* (2009) found that the essential length scales for detonation initiation in multidimensional stellar simulations are strikingly under-resolved and suggest one-dimensional simulations are necessary.

With regard to diffusive detonations, early work (e.g. Hirschfelder & Curtiss 1958; Wood 1963) is summarized in Fickett & Davis (1979, Chapter 5). Clarke, Kassoy & Riley (1986) and Clarke *et al.* (1990) investigated the effects of diffusion on the development of a detonation by studying one particular activation energy in the

one-step model with small energy release in one dimension. The existence of steady diffusive strong detonations in the limit of weak diffusion on the one-step model was shown by Gasser & Szmolyan (1993). Moreover, the stability of these detonations in the weak diffusion limit has been studied by Lyng & Zumbrun (2004); in addition, Texier & Zumbrun (2011) demonstrated that these detonations will undergo a transition through a Hopf bifurcation as the overdrive is lowered. Singh *et al.* (2001) studied the formation of a diffusive detonation with a detailed kinetics model in one dimension. Ziegler *et al.* (2011) investigated how diffusive effects can play a role in multiple dimensions.

The goal of the present paper is to predict the effects of diffusion on the long-time dynamics of a detonation described by the standard one-step kinetics model. The plan of the paper is as follows. In § 2, the mathematical model is presented, followed by a description of the computational method. In § 3, the difficulties in using the Euler equations are discussed, and the Navier–Stokes model is used to predict the diffusive analogue of the period-doubling phenomena predicted in the inviscid limit by Sharpe & Falle (2000), Ng *et al.* (2005) and Henrick *et al.* (2006). The convergence of the period-doubling bifurcation points is shown to be in agreement with the general theory of Feigenbaum (1979), and diffusion is seen to have a generally stabilizing effect on detonation dynamics.

2. Formulation

Here, the diffusive extension of the standard detonation model problem first studied by Erpenbeck (1964) is formulated.

2.1. *Mathematical model*

The model equations adopted are the one-dimensional reactive Navier–Stokes equations with one-step kinetics in a reference frame moving at constant velocity, D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho(u - D)) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u(u - D) + P - \tau) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}\left(\rho\left(e + \frac{u^2}{2}\right)\right) + \frac{\partial}{\partial x}\left(\rho\left(e + \frac{u^2}{2}\right)(u - D) + j_q + (P - \tau)u\right) = 0, \tag{2.3}$$

$$\frac{\partial}{\partial t}(\rho\lambda) + \frac{\partial}{\partial x}(\rho\lambda(u - D) + j_m) = \rho r, \tag{2.4}$$

where x and t are the spatial and temporal coordinates, respectively, ρ is the mass density, u is the particle velocity, P is the pressure, τ is the viscous stress, e is the specific internal energy, j_q is the diffusive heat flux, λ is the reaction progress variable, j_m is the diffusive mass flux and r is the reaction rate. Equations (2.1)–(2.3) describe the conservation of mass, linear momentum and energy; equation (2.4) describes the evolution of reaction products.

The constitutive relations chosen for mass, momentum and energy diffusion are

$$j_m = -\rho\mathscr{D}\frac{\partial \lambda}{\partial x}, \quad \tau = \frac{4}{3}\mu\frac{\partial u}{\partial x}, \quad j_q = -k\frac{\partial T}{\partial x} + \rho\mathscr{D}q\frac{\partial \lambda}{\partial x}, \tag{2.5}$$

where $\mathscr{D}$ is the mass diffusion coefficient, μ is the dynamic viscosity, k is the thermal conductivity, T is the temperature and q is the heat release of reaction. Equations (2.5) are Fick's law for binary diffusion, the Newtonian stress–strain rate relation and an

extended Fourier's law. A calorically perfect ideal gas model is adopted for an ideal mixture in which the molecular masses and specific heats of both reactant and product gases are identical:

$$P = \rho RT, \quad e = \frac{P}{(\gamma - 1)\rho} - q\lambda, \tag{2.6}$$

where R is the gas constant and γ is the ratio of specific heats.

The irreversible one-step reaction model, $A \rightarrow B$, was chosen, where A and B are reactant and product, respectively. In the undisturbed state only A is present; the mass fractions of A and B are given by $1 - \lambda$ and λ, respectively. The reaction rate, r, is given by the law of mass action with Arrhenius rate sensitivity:

$$r = a(1 - \lambda) \exp\left(-\frac{\tilde{E}}{P/\rho}\right) H(P - P_s), \tag{2.7}$$

where a is the collision frequency factor, $\tilde{E}$ is the activation energy and $H(P - P_s)$ is a Heaviside function which suppresses reaction when $P < P_s$ where P_s is a selected pressure. Also, the ambient density and pressure are taken to be ρ_o and P_o, respectively. Similar trends to those we will predict could be expected had a model with greater fidelity to realistic gas mixtures been chosen.

2.2. *Computational method*

The unsteady dynamics of the one-dimensional detonation are predicted using a temporally explicit pointwise method of lines approach. The spatial discretization of the advective terms was accomplished using a combination of a standard fifth-order weighted essentially non-oscillatory (WENO) and Lax–Friedrichs schemes in the manner of Xu, Aslam & Stewart (1997); the diffusive terms were evaluated using sixth-order central differences. In § 3.4, the predictions of this scheme are compared with those from a simple sixth-order central difference of the advective terms, and it will be seen that ordinary central differencing suffices to describe detonation dynamics. Temporal integration is accomplished using a third-order Runge–Kutta method. A standard technique of code verification, the method of manufactured solutions (see Roache 2002) was performed, and a fifth-order convergence rate was predicted as the grid was refined demonstrating that the code correctly solves the governing equations. Convergence in the presence of the Heaviside function is discussed in § 3.3.

2.3. *Initialization and problem parameters*

All calculations were performed in a single-processor environment on an AMD 2.4 GHz processor with 512 kB cache. The simulations were initialized with the inviscid Zel'dovich–von Neumann–Döring (ZND) solution in a frame travelling at the CJ speed. Each simulation is integrated in time to determine the long-time behaviour. For a calculation of 2.0 μs, the computational time required was 2 days. Some calculations took as long as 8 days for full relaxation. By selecting the diffusion coefficient, $\mathscr{D} = 10^{-4}$ m^2 s^{-1}, thermal conductivity, $k = 10^{-1}$ W m^{-1} K^{-1}, and viscosity, $\mu = 10^{-4}$ N s m^{-2}, the Lewis, *Le*, Prandtl, *Pr*, and Schmidt, *Sc*, numbers evaluated at the ambient density, $\rho_o = 1$ kg m^{-3}, are unity. These parameters are within an order of magnitude of those of gases at a slightly elevated temperature. In the inviscid detonation, the activation energy controls the stability of the system; the rate constant merely introduces a length scale, the half reaction length, $L_{1/2}$

(the distance between the inviscid shock and the location at which $\lambda = 1/2$). If $L_{1/2}$ is fixed, the effect of diffusion on the system can be explored. Using simple dimensional analysis of advection and diffusion parameters ($U = 1000$ m s^{-1} was chosen as a typical velocity scale) gives rise to an approximate length scale of mass diffusion, $\mathscr{D}/U = 10^{-7}$ m, and likewise for momentum and energy diffusion $\mu/\rho_o/U = 10^{-7}$ m, and $k/\rho_0/c_P/U = 10^{-7}$ m. Since all of the diffusion length scales are the same, let this scale be denoted as $L_\mu = 10^{-7}$ m. The chosen parameters in this study are $P_o = 0.101325$ MPa, $P_s = 0.200$ MPa, $q = 5\,066\,250$ m^2 s^{-2}, $\gamma = 6/5$, $\tilde{E} \in [2\,533\,125, 3\,232\,400]$ m^2 s^{-2} and $c_P = \gamma R/(\gamma - 1) = 1000$ J kg^{-1} K^{-1}, where c_P is the specific heat at constant pressure. With this heat release, D_{CJ} for the inviscid problem is

$$D_{CJ} = \sqrt{\gamma \frac{P_o}{\rho_o} + \frac{q(\gamma^2 - 1)}{2}} + \sqrt{\frac{q(\gamma^2 - 1)}{2}} = 2167.56 \text{ m s}^{-1}. \tag{2.8}$$

The selection of P_s is arbitrary, since there is minimal effect on the system over the range of 0.102 to 1.010 MPa. To compare directly with previous work in the inviscid limit, the activation energies will be presented in dimensionless form, $E = \tilde{E}/(P_o/\rho_o)$, thus $E \in [25, 32]$. Using these parameters allows for the interaction of diffusion and reaction effects to be studied and induces a set of scales similar to those given in reactive Navier–Stokes models with detailed chemical kinetics. Unless otherwise stated, the calculations presented are for a ratio of $L_\mu/L_{1/2} = 1/10$, such that $L_{1/2} = 10^{-6}$ m, which is similar to the finest reaction length scale of H_2–air detonations.

The coarsest scales in H_2–air detonations are much larger than the chosen $L_{1/2}$; as shown by Powers & Paolucci (2005), an ambient mixture of H_2–air at atmospheric pressure has an induction zone of approximately 2×10^{-4} m. In the more realistic detailed kinetics systems, the main heat release occurs over the coarse length scales. It must thus be recognized that the chosen length scale on which the heat is released is much finer than expected in a realistic physical system; the main reason for this choice is to reduce the stiffness of the system so as to enable a tractable computation of a fully resolved multiscale detonation. A future study employing significant computational resources will report results with no such compromise for fully resolved viscous hydrogen–air detonations.

3. Results

In this section, the reactive Euler equations are first considered, and detrimental effects of shock-capturing on predicting the convergence of unstable inviscid detonations are examined. These defects are remedied by the addition of physical diffusion. Unsteady solutions to the reactive Navier–Stokes equations are shown to converge, and it is demonstrated that ordinary central difference schemes can perform as well as more complex schemes such as WENO.

3.1. *Inviscid shock-capturing*

In addition to using the Euler equations for modelling detonations, the use of shock-capturing techniques and moving reference frames is also common. Quirk (1994) reports that when using the Euler equations with any shock-capturing technique, a shock moving slowly relative to the numerical grid will have low-frequency numerical perturbations. These low-frequency perturbations can be lessened by refining the grid.

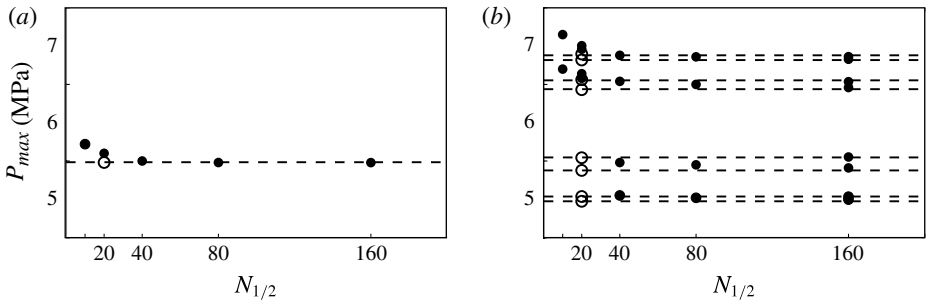

FIGURE 1. Peak inviscid detonation pressures versus $N_{1/2}$ for (*a*) $E = 26.64$ and (*b*) $E = 27.82$. Shock-capturing predictions are given by the filled circles and shock-fitting ($N_{1/2} = 20$) prediction is represented by open circles and dashed lines.

To avoid this issue completely, Henrick *et al.* (2006) used a high-accuracy shock-fitting technique to predict the behaviour of the one-step detonation. This method limits the artificial viscosity to negligible levels and, thus, enables an accurate prediction with the number of points within the so-called half reaction zone width, $N_{1/2} = 20$. For an activation energy of $E = 26.64$, a simple period-1 limit cycle detonation is predicted using shock-fitting; using shock-capturing with the same resolution, the predicted behaviour of a period-1 detonation is in agreement with that of shock-fitting with a relative difference of the peak pressure of 2.1 %. Increasing the resolution lessens this relative difference as shown in figure 1(*a*). At $E = 27.82$, shock-fitting predicts a period-8 limit cycle detonation, whereas shock-capturing, using the higher resolution of 40 points in the half reaction length, predicts a period-4 detonation. This difference can be reconciled by increasing the resolution, demonstrated in figure 1(*b*). The present study, in good agreement with Sharpe & Falle (2000), found that $N_{1/2} > 80$ was needed in this regime. The resolution requirement to accurately predict the correct dominant frequencies in high-periodicity oscillatory detonations maybe even more stringent, as seen in Ng *et al.* (2005). This suggests that numerical diffusion is playing an important role in determining the behaviour of the system at lower resolutions.

3.2. *Effect of physical diffusion*

The plausible yet erroneous predictions due to the inherent numerical diffusion in the model can be remedied by increasing the resolution of the scheme. However, for high-periodicity instabilities, the necessary resolution tends towards infinity for the inviscid model. A preferable approach is to include explicit physical diffusion and so introduce a cutoff length scale below which physical diffusion properly serves to dampen oscillations.

3.2.1. *Stability limit*

In the inviscid case, linear stability analysis by Lee & Stewart (1990) revealed that for $E < 25.26$, the steady ZND wave is linearly stable and is otherwise linearly unstable. The activation energy at this stability boundary is labelled E_0. Henrick *et al.* (2006) numerically found the stability limit, for the inviscid case, at $E_0^i = 25.265 \pm 0.005$. Here, a diffusive case well above the inviscid stability limit was examined, $E = 26.64$, which Henrick *et al.* (2006) found to relax to a period-1 limit cycle for an inviscid simulation. In the diffusive simulation, it can be seen from

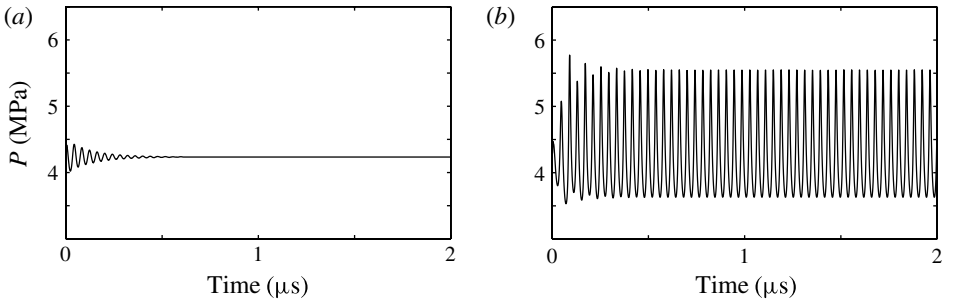

FIGURE 2. Plot of P versus t, $L_\mu/L_{1/2} = 1/10$: (*a*) $E = 26.64$, stable diffusive detonation, and (*b*) $E = 29.00$, period-1 diffusive detonation.

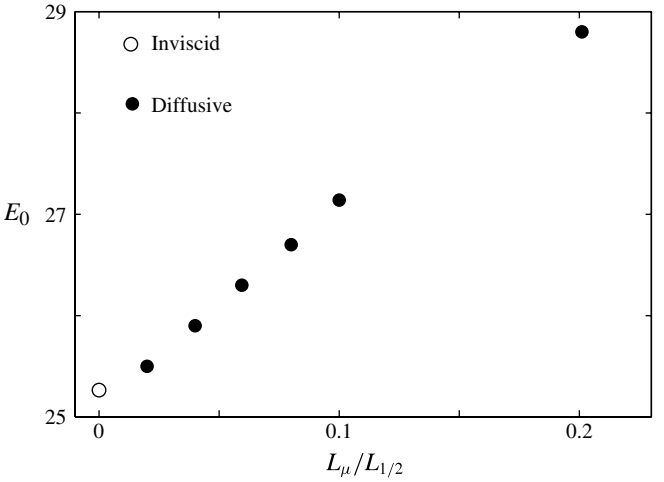

FIGURE 3. The location of the stability limit, E_0, versus the diffusion length scale, $L_\mu/L_{1/2}$.

figure 2(*a*) that there is no limit cycle behaviour, and the detonation predicted by diffusive theory is in fact a stable steadily propagating wave. The stability boundary for the diffusive case is found at $E_0^d \approx 27.14$. A period-1 limit cycle may be realized in the diffusive case by increasing the activation energy above E_0^d; an example is shown in figure 2(*b*) with an activation energy of $E = 29.00$.

One would expect that the onset of instability would be delayed more as the scale of diffusion approaches that of reaction. Figure 3 shows precisely this behaviour. Furthermore, as the two scales approach one another, the onset of instability is delayed significantly in comparison with the inviscid case.

3.2.2. *Period doubling and transition to chaos*

For higher values of E, more complicated dynamics are predicted. A period-doubling behaviour and transition to chaos for unstable detonations are found to be remarkably similar to that predicted by the simple logistic map studied by May (1976). The activation energy at which the behaviour switches from a period-2^{n-1} to a period-2^n solution is denoted as E_n, for $n \geqslant 1$. As predicted by Sharpe & Falle (2000), Ng *et al.* (2005) and shown in Henrick *et al.* (2006), transition to a period-2

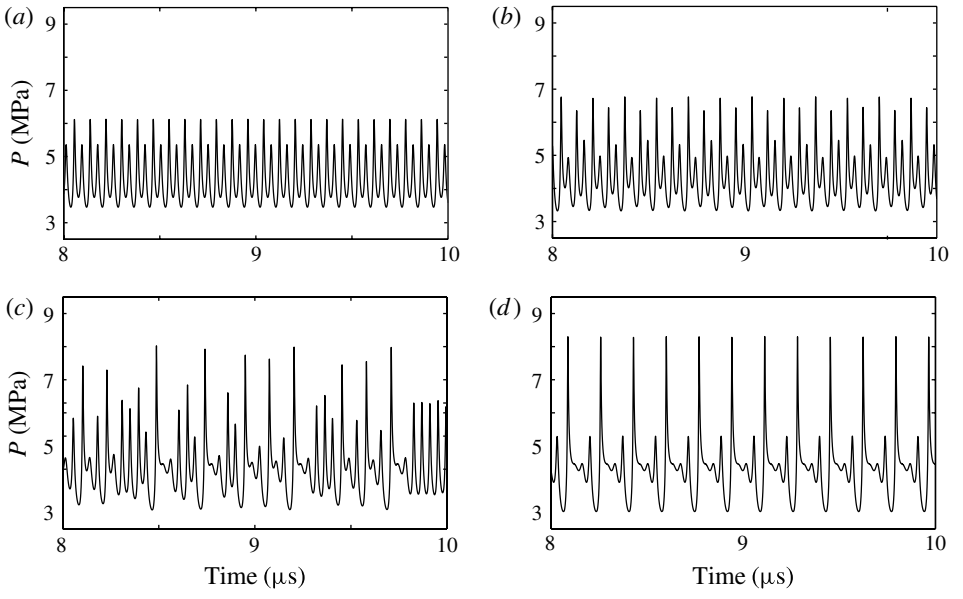

FIGURE 4. Plot of P versus t for diffusive detonation with $L_\mu/L_{1/2} = 1/10$: (a) $E = 29.50$, period-2, (b) $E = 29.98$, period-4, (c) $E = 30.74$, chaotic, and (d) $E = 30.86$, period-3.

n	Inviscid E_n^i	Inviscid δ_n^i	Diffusive E_n^d	Diffusive δ_n^d
0	25.2650	—	27.14	—
1	27.1875	3.86	29.32	3.89
2	27.6850	4.26	29.88	4.67
3	27.8017	4.66	30.00	—
4	27.82675	—	—	—

TABLE 1. Numerically determined bifurcation points for inviscid and diffusive detonation, and approximations to Feigenbaum's constant.

oscillation occurs at $E_1^i \approx 27.2$ for the inviscid case. In the diffusive case, it was found instead $E_1^d \approx 29.32$; figure 4(a) shows the time history of the detonation pressure for a higher $E = 29.50$, which shows in the long-time limit two distinct relative maxima, $P \approx 6.117$ MPa and $P \approx 5.358$ MPa. Increasing further to $E = 29.98$, another period-doubling is realized, and a period-4 oscillating detonation is achieved as seen in figure 4(b). The bifurcation points for both models are listed in table 1 along with approximations for Feigenbaum's constant, δ_∞ :

$$\delta_\infty = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{E_n - E_{n-1}}{E_{n+1} - E_n}. \tag{3.1}$$

Feigenbaum (1979) predicted $\delta_\infty \approx 4.669201$. Diffusive and inviscid models predict δ_∞ well.

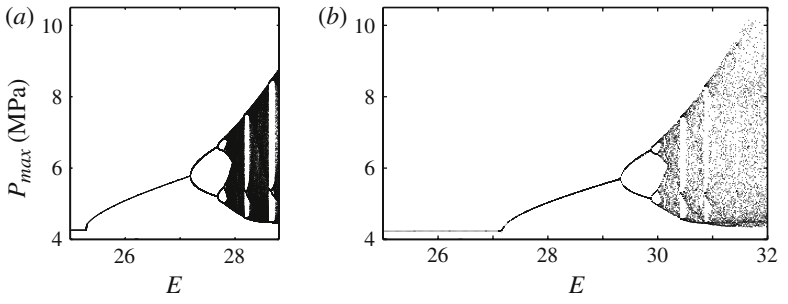

FIGURE 5. Comparison of numerically generated bifurcation diagrams: (*a*) inviscid detonation with shock-fitting and (*b*) diffusive detonation with $L_\mu/L_{1/2} = 1/10$.

Δx (m)	$P_{29.0}$ (MPa)	$r_{c_{29.0}}$	$\omega_{29.0}$ (MHz)	$r_{c_{29.0}}$	$P_{29.5}$ (MPa)	$r_{c_{29.5}}$	$\omega_{29.5}$ (MHz)	$r_{c_{29.5}}$
2.50×10^{-8}	3.864	—	24.5652	—	5.844	—	12.1833	—
1.25×10^{-8}	3.737	2.071	24.6173	3.15	4.757	2.084	12.2121	3.37
6.25×10^{-9}	3.706	—	24.6232	—	4.596	—	12.2149	—

TABLE 2. Convergence rates of pressure and frequency at two activation energies with diffusion.

3.2.3. *Chaos and order*

Figure 5(*a*) gives the bifurcation diagram for the case studied by Henrick *et al.* (2006) in the inviscid limit using a shock-fitting algorithm with negligible numerical diffusion. Figure 5(*b*) gives the diffusive analogue. It was constructed by sampling 351 points with $E \in [25, 32]$, with a spacing of $\Delta E = 0.02$. Simulations were integrated to $t = 10$ μs and relative maxima in P were recorded for $t \geqslant 7.5$ μs. In the diffusive case, the period-doubling bifurcations occur up to $E^d_\infty \approx 30.03$. Beyond this point, there exists a region that is densely populated in relative maxima which is most likely a chaotic regime. Increasing the activation energy yet further, one comes to regions with a small number of oscillatory modes with periods of 3, 5 and 6. A chaotic detonation is shown in figure 4(*c*); at a higher activation energy, a solution with period-3 is found and is shown in figure 4(*d*).

3.3. *Convergence rate*

The presence of the Heaviside function in the reaction source may preclude the expected fifth-order accuracy. The state variables do in fact converge at a lower order than that of the theoretical value. At a representative point in space/time, $x = 1.477 \times 10^{-4}$ m at $t = 1$ μs, the pressure converges at a rate of 2.071 and 2.084 for detonations with activation energies of $E = 29.0$ and $E = 29.5$, respectively. However, the state variables converge at rates larger than unity suggesting that the Heaviside function plays a small role. In addition to the state variables, the average frequency, ω, of the oscillation was examined for both cases and are listed in table 2. The average was taken starting at 1 μs, as the period-1 and period-2 detonations are within the limit cycle behaviour, and going until 3.5 μs; the convergence rates for frequency are higher, 3.15 and 3.37, respectively.

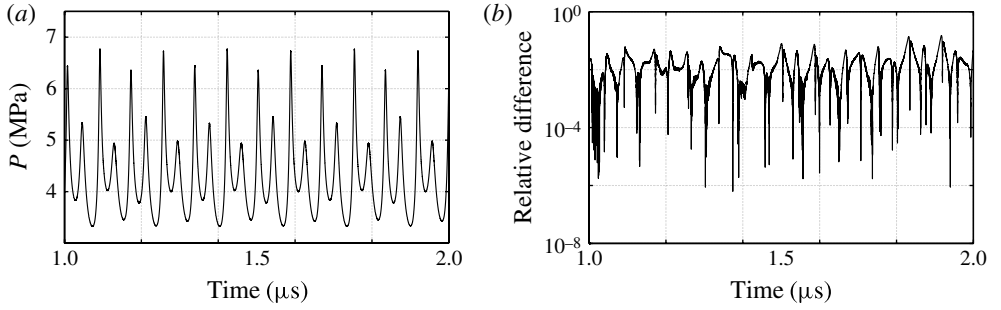

FIGURE 6. (*a*) Diffusive detonation pressure versus time for both the central differencing and WENO schemes for $E = 30.02$ and (*b*) the relative difference between the schemes.

3.4. *WENO versus central differences*

For resolved diffusive detonations, the use of the ubiquitous WENO scheme is unnecessary; a simpler central difference of the advective terms is sufficient. Even with complex behaviours, as in the period-8 behaviour predicted for $E = 30.02$ and shown in figure 6(*a*), the use of a central difference for the advective terms yields results which agree with those of the WENO scheme. The relative difference between predictions of the two methods is shown in figure 6(*b*). The values of the detonation pressure match up to a time shift which originates at the initialization.

4. Conclusions

Investigation of the one-step kinetic model of one-dimensional unsteady detonation with mass, momentum and energy diffusion shows that the dynamics are significantly influenced in the region of instability relative to its inviscid counterpart. As in the inviscid limit, bifurcation and transition to chaos are predicted and show similar behaviour to that of the logistic map. For physically motivated length scales not unlike the finest reaction scale of H_2–air detonations, where the length scale of diffusion is comparable with that of reaction, the addition of diffusion delays the onset of instability. The inviscid approximation indeed gives a good approximation when in the stable or weakly unstable regimes. However, for activation energies large enough to induce complex limit cycle behaviour, ordinary shock-capturing methods applied to inviscid models using an under-resolved grid can fail to capture the correct long-time dynamics. If the dynamics of unstable and marginally stable detonations are to be captured correctly, physical diffusion needs to dominate numerical diffusion. Lastly, it is clear that for resolved detonations with diffusion, simple central spatial differences work as well as more exotic schemes. This has been demonstrated for a WENO scheme. One might expect a similar conclusion to be drawn for any numerical filtering strategy such as other shock-capturing methods or turbulence modelling, applied to a detonation with an intrinsic physical instability.

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