Math 20580
Practice Midterm 3
April 16, 2015

Name: Solutions
Instructor: ____________________
Section: ____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. a b c d ~~e~~
2. a b ~~c~~ d e
3. a b ~~c~~ d e
4. a ~~b~~ c d e
5. a b c ~~d~~ e
6. a b c d ~~e~~
7. a b ~~c~~ d e
8. a b c ~~d~~ e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Find the closest point to $\begin{bmatrix}1\\1\\2\end{bmatrix}$ in the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\2\end{bmatrix}$.

   (a) $\begin{bmatrix}-2\\1\\1\end{bmatrix}$ (b) $\begin{bmatrix}-1\\1\\2\end{bmatrix}$ (c) $\begin{bmatrix}1\\1\\2\end{bmatrix}$ (d) $\begin{bmatrix}8/5\\1\\6/5\end{bmatrix}$ (e) $\begin{bmatrix}-3/5\\1\\6/5\end{bmatrix}$

Note that the subspace has an orthogonal basis $\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\2\end{pmatrix}$. Therefore the projection of $\begin{pmatrix}1\\1\\2\end{pmatrix}$ onto the subspace has formula

$$\frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{1}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\2\end{pmatrix}}{5}\begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{3}{5}\begin{pmatrix}-1\\0\\2\end{pmatrix}$$

2. Which of the following is a least square solution $\hat{\mathbf{x}}$ to the equation

$$\begin{bmatrix}1 & -2\\2 & 1\\1 & -2\\2 & 1\end{bmatrix}\mathbf{x} = \begin{bmatrix}1\\3\\1\\3\end{bmatrix}?$$

   (a) $\begin{bmatrix}11/9\\1/9\end{bmatrix}$ (b) $\begin{bmatrix}3/2\\1/2\end{bmatrix}$ (c) $\begin{bmatrix}7/5\\1/5\end{bmatrix}$ (d) $\begin{bmatrix}1\\-2\end{bmatrix}$ (e) $\begin{bmatrix}2\\1\end{bmatrix}$

$$A^TA = \begin{pmatrix}1 & 2 & 1 & 2\\-2 & 1 & -2 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\2 & 1\\1 & -2\\2 & 1\end{pmatrix} = \begin{pmatrix}10 & 0\\0 & 10\end{pmatrix}$$

$$A^T\begin{pmatrix}1\\3\\1\\3\end{pmatrix} = \begin{pmatrix}1 & 2 & 1 & 2\\-2 & 1 & -2 & 1\end{pmatrix}\begin{pmatrix}1\\3\\1\\3\end{pmatrix} = \begin{pmatrix}14\\2\end{pmatrix}$$

So we see that $\begin{pmatrix}7/5\\1/5\end{pmatrix}$ solves the normal eqns. and is the least square solution.

3. Which of the following functions is a solution to the initial value problem

$$\frac{dy}{dt} = (y-t)^2 + 1; \qquad y(0) = -1?$$

(a) $y = \frac{1}{t+1} - 2$ (b) $y = t$ (c) $y = \frac{-1}{t+1} + t$
(d) $y = t - 1$ (e) $y = \frac{-2}{t+1} + 1$

Eqn. is not linear or separable so we need to plug in and check.

We see that (c) solves the problem.

4. Let A be an $m \times n$ matrix. Which of the following may be *false*?

(a) The equation $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent for any $\mathbf{b}$ in $\mathbb{R}^m$.
(b) $A^T A$ is invertible.
(c) A solution to $A^T A\mathbf{x} = A^T\mathbf{b}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$.
(d) The columns of A^T lie in the column space of $A^T A$.
(e) If $A^T A\mathbf{x} = A^T\mathbf{b}$ then $A\mathbf{x} - \mathbf{b}$ is orthogonal to Col(A).

(b) is sometimes false — for example if $A = 0$.

The rest are always true.

5. Which of the following is a general solution to the differential equation

$$1 + (\frac{x}{y} - \sin y)\frac{dy}{dx} = 0?$$

(a) $xy + y\sin y - \sin y = c$ (b) $xy + y\cos y - \sin y = cy$
(c) $xy + y\sin y - \cos y = c$ (d) $xy + y\cos y - \sin y = c$
(e) $xy + y\cos y - \cos y = c$

The equation has an integrating factor $\mu = y$.
Multiplying gives an exact eqn.

$$y + (x - y\sin y)y' = 0.$$

$$\Psi_x = y \Rightarrow \Psi = xy + h(y).$$

$$h'(y) = -y\sin y \Rightarrow h = y\cos y - \sin y + C$$

$$\Rightarrow \Psi = xy + y\cos y - \sin y = C$$

6. Consider the initial value problem

$$\sin(2x) + \cos(3y)\frac{dy}{dx} = 0 \qquad y(\pi/2) = \pi/3$$

Which of the following implicitly defines the solution?

(a) $\dfrac{-\cos(2x)}{2} + \dfrac{\sin(3y)}{3} = \dfrac{-1}{2}$ (b) $-\cos(2x) + \sin(3y) = \dfrac{1}{2}$
(c) $\sin(2x) + \cos(3y) = 1$ (d) $-\cos(2x) + \sin(3y) = \dfrac{-1}{2}$
(e) $\dfrac{-\cos(2x)}{2} + \dfrac{\sin(3y)}{3} = \dfrac{1}{2}$

This is separable. For a solution

$$\int \cos 3y\,dy = -\int \sin 2x\,dx + C$$

so $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + C$

Initial condition gives $0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2}$.

7. Let $y(t)$ be the unique solution of the initial value problem

$$(t^2 - t)\frac{dy}{dt} + \cos(\pi t)y = \frac{t^2 - t}{t - 2} \qquad y(3/2) = 0$$

What is the largest interval where y is defined?

(a) $t > 0$ (b) $0 < t < 2$ (c) $1 < t < 2$ (d) $t < 1/2$ (e) $t < 2$

In standard form $p(t) = \dfrac{\cos \pi t}{t(t-1)}$ which is singular at $t = 0, 1$

and $g(t) = \dfrac{t(t-1)}{t(t-1)(t-2)} = \dfrac{1}{t-2}$ which is singular at $t = 2$.

Therefore a soln. extends to the interval $1 < t < 2$.

8. A tank initially contains $100l$ of pure water. Then, at $t = 0$, a sugar solution with concentration of $4g/l$ starts being pumped into the tank at a rate of $5l/\text{min}$. The tank is kept well mixed, and the solution is being pumped out at the rate of $4l/\text{min}$. Which of the following is the initial value problem for $y(t)$ = quantity of sugar, in grams, in the tank at time t?

(a) $\dfrac{dy}{dt} = 5y - 4(100 + t) \qquad y(0) = 0$

(b) $\dfrac{dy}{dt} = 20 - 4y \qquad y(0) = 0$

(c) $\dfrac{dy}{dy} = 4 \qquad y(0) = 100$

(d) $\dfrac{dy}{dt} = 20 - \dfrac{4y}{100 + t} \qquad y(0) = 0$

(e) $\dfrac{dy}{dt} = 20 - \dfrac{y}{(100 + t)^2} \qquad y(0) = 100$

$\dfrac{dy}{dt} = $ (rate sugar enters) $-$ (rate it leaves tank)

$= 4.5 - 4 \cdot \left(\text{conc. of sugar at time } t\right)$

$= 20 - \dfrac{4y}{100+t}$ since volume of fluid $= 100 + t$.

Part II: Partial credit questions (11 points each). Show your work.

9. Using the Gram-Schmidt Process, find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\2\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\3\\1\\3\end{bmatrix}$.

First get an orthogonal basis. Set $y_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

$$y_2 = \begin{pmatrix}1\\2\\1\\2\end{pmatrix} - \frac{\begin{pmatrix}1\\2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}{1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\1\\2\end{pmatrix}.$$

$$y_3 = \begin{pmatrix}1\\3\\1\\3\end{pmatrix} - \frac{\begin{pmatrix}1\\3\\1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}{1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \frac{\begin{pmatrix}1\\3\\1\\3\end{pmatrix}\cdot\begin{pmatrix}0\\2\\1\\2\end{pmatrix}}{9}\begin{pmatrix}0\\2\\1\\2\end{pmatrix}$$

$$= \begin{pmatrix}1\\3\\1\\3\end{pmatrix} - \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \frac{13}{9}\begin{pmatrix}0\\2\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1/9\\-4/9\\1/9\end{pmatrix}$$

Normalizing, an orthonormal basis is

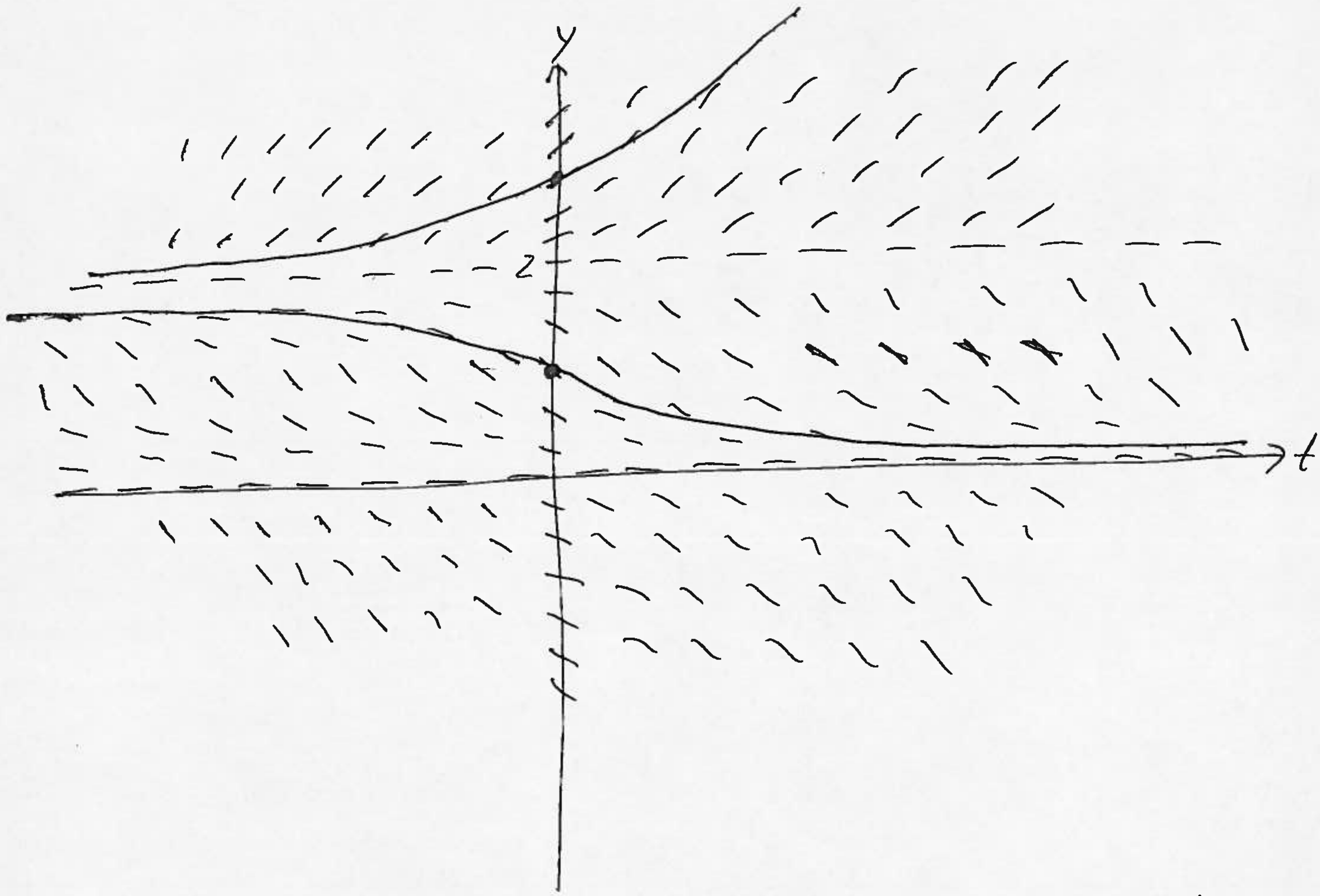
$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \frac{1}{3}\begin{pmatrix}0\\2\\1\\2\end{pmatrix}, \frac{1}{\sqrt{18}}\begin{pmatrix}0\\1\\-4\\1\end{pmatrix}.$$

10. By drawing a direction field, sketch two solutions to the ODE

$$\frac{dy}{dt} = t^2 y^2 (y-2)$$

with initial conditions $y(0) = 1$ and $y(0) = 3$.

Indicate clearly the limiting behavior $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$.



See if $y(0)=1$ then $\lim_{t\to+\infty} y(t)=0$, $\lim_{t\to-\infty} y(t)=2$

If $y(0)=3$, $\lim_{t\to-\infty} y(t)=2$.

Here $\lim_{t\to+\infty} y(t)$ is either $+\infty$ or the solution may not exist for all positive t

(ie. it may converge to $+\infty$ in finite time.)

11. Find the function $y(t)$, for $t > 0$, which solves the initial value problem

$$t\frac{dy}{dt} + 4y = \frac{e^{-t}}{t^2} \quad , \quad y(1) = 0$$

This is linear.

In standard form $y' + \left(\frac{4}{t}\right)y = \frac{e^{-t}}{t^3}$,

$p(t) = \frac{4}{t}$ and $g(t) = \frac{e^{-t}}{t^3}$.

An integrating factor is $\mu(t) = e^{\int \frac{4}{t} dt} = t^4$.

The general soln. is $y = \frac{1}{t^4}\left(\int te^{-t} dt + C\right)$

Compute:– $\int te^{-t} dt = -te^{-t} + \int e^{-t} dt$

Int. by Parts
$u = t,\ v' = e^{-t}$
$u' = 1,\ v = -e^{-t}$

$= -te^{-t} - e^{-t} + C$

So $y = \frac{-e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}$

$y(1) = 0 \Rightarrow -e^{-1} - e^{-1} + C = 0 \Rightarrow C = 2e^{-1}$

& $y = \frac{1}{t^4}\left(2e^{-1} - (t+1)e^{-t}\right)$.

12. Consider the differential equation

$$2y\frac{dy}{dx} = -e^x$$

(a) Find the general solution.
(b) Find the solution with $y(0) = 1$.
(c) What is the largest interval in which the solution in part (b) is defined?

(a) This is separable. For a soln. we have

$$\int 2y\,dy = \int -e^x\,dx + C$$

$\Rightarrow y^2 = -e^x + C$, this is an implicit general solution.

(b) If $y(0) = 1$ then $1 = -1 + C \Rightarrow C = 2$

So particular soln. is $y^2 = 2 - e^x$

or $y = \sqrt{2 - e^x}$ (positive square root because $y(0) > 0$).

(c) the solution exists only when $2 - e^x > 0$

or $x < \ln 2$.