Symplectic isotopy classes of ellipsoids and polydisks in dimension greater than four

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Abstract

In any dimension $2n \geq 6$ we show that certain spaces of ellipsoid and polydisk embeddings into a product $B^4 \times \mathbb{R}^{2(n-2)}$ of a 4-ball and Euclidean space, are not path connected. Thus a theorem of McDuff, saying that the space of symplectic embeddings of one 4-dimensional ellipsoid into another is always path connected, fails to be true in higher dimensions.

1 Introduction

We study symplectic embeddings into Euclidean space $\mathbb{R}^{2n}$, with coordinates $x_j, y_j$, $1 \leq j \leq n$, equipped with its standard symplectic form $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$. Often it is convenient to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by setting $z_j = x_j + iy_j$. The basic domains for symplectic embedding problems are ellipsoids $E$ and polydisks $P$ which we define as follows.

$$E(a_1, \ldots, a_n) = \{ \sum_j \pi |z_j|^2/a_j \leq 1 \};$$

$$P(a_1, \ldots, a_n) = \{ \pi |z_j|^2 \leq a_j \text{ for all } j \}.$$

These are subsets of $\mathbb{C}^n$ and so inherit the symplectic structure. A ball of capacity $R$ is simply an ellipsoid $B^{2n}(R) = E(R, \ldots, R)$.

Definition 1.1. Two embeddings $f_0, f_1 : E(a_1, \ldots, a_n) \to W$ are symplectically isotopic in a symplectic manifold $W$ if there exists a smooth family of symplectic embeddings $f_t : E(a_1, \ldots, a_n) \to W$ for $t \in [0, 1]$, interpolating between the original maps.

We emphasize this definition since there is a weaker equivalence relation on embeddings, asking just that there is a symplectic isotopy of $W$ mapping the image $f_0(E)$ onto $f_1(E)$. In dimension 4 the two notions are the same; this follows from Gromov’s theorem [8] that the compactly supported
symplectomorphism group of a ball is contractible. We do not know if the notions coincide in higher dimension, and our examples will be nonisotopic in the sense of Definition 1.1.

In dimension 4, that is, when \( n = 2 \), a theorem of McDuff says that the space of symplectic embeddings of one ellipsoid into another is always path connected.

**Theorem 1.2.** (McDuff [21] Corollary 1.6, see also [22]) For any \( a, b, a', b' \), the space of symplectic embeddings \( E(a, b) \to \hat{E}(a', b') \) is path connected whenever it is nonempty.

In this paper we show that McDuff’s theorem is not true in higher dimensions, that is, when \( n \geq 3 \) some spaces of ellipsoid embeddings are not path connected. For example we will show the following.

**Theorem 1.3.** There exist nonisotopic symplectic embeddings

\[
E(1.4, 5.59, 5.65, \ldots, 5.65) \to B^4(2.83) \times \mathbb{R}^{2(n-2)}.
\]

Since \( B^4(2.83) \times \mathbb{R}^{2(n-2)} \) is exhausted by ellipsoids \( E(2.83, 2.83, S, \ldots, S) \) with \( S \to \infty \), we may replace the product \( B^4(2.83) \times \mathbb{R}^{2(n-2)} \) in Theorem 1.3 by an \( E(2.83, 2.83, S, \ldots, S) \) for any \( S \) sufficiently large. Thus we obtain non isotopic ellipsoid embeddings into an ellipsoid. The theorem extends to ellipsoids with parameters satisfying certain inequalities which we state in Theorem 3.1. However, we cannot claim that all spaces of ellipsoid embeddings into an ellipsoid are disconnected in dimension greater than 4. In particular, we do not know if the space of embeddings of a ball into an ellipsoid is ever disconnected.

In dimension 4 there are results showing that spaces of embeddings of polydisks are not path connected. The first was due to Floer, Hofer and Wysocki, showing that spaces of embeddings of a polydisk into a polydisk may be disconnected.

**Theorem 1.4.** (Floer-Hofer-Wysocki [7] Theorem 4) Let \( \max(a, b) < R < a + b \). Then \( g_1 : (z_1, z_2) \mapsto (z_1, z_2) \) and \( g_2 : (z_1, z_2) \mapsto (z_2, z_1) \) give nonisotopic embeddings \( P(a, b) \to P(R, R) \).

Note that if \( R > a + b \) then the embeddings are isotopic in \( P(R, R) \) through unitary maps. The condition \( \max(a, b) < R \) ensures that the \( g_i \) have images in \( P(R, R) \).

The starting point for our approach to Theorem 1.3 is the following theorem about polydisks embedded in a ball.

**Theorem 1.5.** (Hind, [9] Theorem 1.1) There does not exist a Hamiltonian diffeomorphism \( \phi \) with support contained in \( B^4(2a+b) \) such that \( \phi(P(a, b)) \subset B^4(a+b) \).
This leads immediately to examples of nonisotopic polydisks symplectomorphic to $P(a, b)$ with $b > 2a$. Indeed, by a symplectic fold, for any $\epsilon > 0$ there exists a symplectic embedding $P(a, b) \to B^4(2a + \frac{b}{2} + \epsilon)$, see [25], Proposition 4.3.9.

It turns out that Theorem 1.5 does have a generalization to higher dimensions, not only for polydisks but also for polylke domains (products of a disk and an ellipsoid) $Q$ which we define as follows.

$$Q(b, a_2, a_3, \ldots, a_n) = \{ \pi|z_1|^2 \leq b, \sum_{j=2}^{n} \frac{\pi|z_j|^2}{a_j} \leq 1 \}.$$

Then a generalization of Theorem 1.5 is as follows. Note that by inclusion the polylke domain $Q(b, a_2, \ldots, a_n)$ sits inside $B^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$.

**Theorem 1.6.** Suppose that $a_2 < b$ and $a_j > 2a_2$ for all $j \geq 3$. There does not exist a Hamiltonian diffeomorphism $\phi$ with support contained in $B^4(2a_2 + b) \times \mathbb{R}^{2(n-2)}$ such that $\phi(Q(b, a_2, \ldots, a_n)) \subset B^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$.

Theorem 1.6 can be easily applied to give examples of nonisotopic polydisks inside products $B^4(R) \times \mathbb{R}^{2(n-2)}$. The folding mentioned above applied to the first two complex coordinates gives a symplectic embedding $Q(b, a_2, \ldots, a_n) \to B^4(2a_2 + \frac{b}{2} + \epsilon) \times \mathbb{R}^{2(n-2)}$ for any $\epsilon > 0$, and if $2a_2 < b$ then $\epsilon$ can be chosen such that $2a_2 + \frac{b}{2} + \epsilon < a_2 + b$. Hence Theorem 1.6 implies that this embedding cannot be symplectically isotopic to the inclusion. Similarly, if $a_3 < b$ then by switching the $z_1$ and $z_3$ coordinates we get another embedding $Q(b, a_2, \ldots, a_n) \to B^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$. Therefore we have the following corollary about embeddings of polylke domains.

**Corollary 1.7.** Let $a_3, \ldots, a_n > 2a_2$ and choose $R$ with $a_2 + b < R < 2a_2 + b$. Suppose either $2a_2 < b$ or $a_3 < b$. Then there exists a symplectic embedding $Q(b, a_2, \ldots, a_n) \to B^4(R) \times \mathbb{R}^{2(n-2)}$ which is not symplectically isotopic to the inclusion inside $B^4(R) \times \mathbb{R}^{2(n-2)}$. Furthermore, even the images are not symplectically isotopic.

We note that our bound on $R$ is sharp in the sense that if $R > 2a_2 + b$ then the folding operation in the $(z_1, z_2)$ plane can be carried out in $B^4(R)$, see [9], section 3. Similarly if $R > a_3 + b$ then a rotation between the $z_1$ and $z_3$ coordinates can be carried out in $B^4(R) \times \mathbb{R}^{2(n-2)}$. We can also produce examples of nonisotopic polydisks from Theorem 1.6 by observing that $Q(b, a_2, \ldots, a_n) \subset P(b, a_2, \ldots, a_n)$, and so embeddings $P(b, a_2, \ldots, a_n) \to B^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$ are also not isotopic to the inclusion. However it is also possible to work directly with higher dimensional polydisks and obtain a similar result.

**Theorem 1.8.** If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $a_1 + a_3 < R < 2a_1 + a_3$ then the space of embeddings $P(a_1, \ldots, a_n) \to B^4(R) \times \mathbb{R}^{2(n-2)}$ is not path connected.
More precisely, the embedding $f : (z_1, z_2, z_3, \ldots, z_n) \mapsto (z_1, z_3, z_2, z_4, \ldots, z_n)$ is not isotopic to any map with image contained in $B^4(a_1 + a_3) \times \mathbb{R}^{2(n-2)}$. In particular, the inclusion is not isotopic to $f$.

We remark that each of Theorems 1.6 and 1.8 generalize both of Theorems 1.4 and 1.5. For example, in the case of Theorem 1.8, the map $f$ and the inclusion would be isotopic in $B^4(2a_1 + a_3) \times \mathbb{R}^{2(n-2)}$ if we could find an isotopy of the $(z_2, z_3)$ plane rotating the polydisk $P(a_2, a_3)$ inside $B^4(2a_1 + a_3) \times \mathbb{R}^{2(n-2)}$ switching the $z_2$ and $z_3$ coordinates. Setting $a_1 = a_2 = a$ and $a_3 = b$ we therefore have a corollary to Theorem 1.8 generalizing Theorem 1.4.

**Corollary 1.9.** Let $a < b$ and $b < R < a + b$. Then the two embeddings $g_1 : (z_1, z_2) \mapsto (z_1, z_2)$ and $g_2 : (z_1, z_2) \mapsto (z_2, z_1)$ from $P(a, b)$ into $B^2(R) \times \mathbb{R}^2$ are not symplectically isotopic.

The proof of Theorem 1.8 is very similar to that of Theorem 1.6, although there are more closed orbits to analyze on the boundary of a polydisk itself. In this paper we focus on the proof of Theorem 1.6 because polylike domains are in some sense closer to ellipsoids, and indeed we will use some analysis from the proof of Theorem 1.6 to deduce the existence of nonisotopic ellipsoids.

**Outline of the paper.**

The proof of Theorem 1.6 is contained in section 2. The techniques borrow heavily from the proof of Theorem 1.5, but with some additional technicalities due to working in higher dimension. The rough outline is as follows.

In section 2.1 we describe the basic arrangement. The product $B^4(R) \times \mathbb{R}^{2(n-2)}$ for some $R < 2a_2 + b$ is partially compactified to $\mathbb{C}P^2 \times \mathbb{R}^{2(n-2)}$ and the polylike domain $Q$ is approximated by a smooth domain $W$. We argue by contradiction and assume that there exists a symplectic isotopy $W_t$, $0 \leq t \leq 1$, with $W_0 = W$ and $W_1 \subset \tilde{B}^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$.

Next, in section 2.2 the symplectic manifolds $X_t = (\mathbb{C}P^2 \times \mathbb{R}^{2(n-2)}) \setminus W_t$ are given almost-complex structures with cylindrical ends and we compute index and area formulas for finite energy holomorphic curves. We refer to the series of papers of Hofer, Wysocki and Zehnder, [13], [14], [15], [16], for the definitions and theory of finite energy curves.

In section 2.3 we study moduli spaces $\mathcal{M}_t$ of holomorphic curves corresponding to the $W_t$. The constituent curves have area bounded above by $R - (a_2 + b)$ and a monotonicity theorem as in [9] implies that $\mathcal{M}_1$ is empty. On the other hand we show that $\mathcal{M}_0$ has a single element. To complete the proof of Theorem 1.6 we prove a compactness theorem, following
[3], showing that \( M_0 \) and \( M_1 \) must be cobordant. This gives the required contradiction.

Finally then we address the case of ellipsoids. The strategy is to show that two nonisotopic embeddings of polylike domains extend to embeddings of ellipsoids. We give the ellipsoid embeddings in section 3.1, the second one is an extension of an embedding of a polylike domain \( Q(1.5, 1, c, \ldots, c) \) into \( B^4(S) \times \mathbb{R}^{2(n-2)} \) with \( S < 1+1.5 \) and \( c > 2 \). However we do not know whether the first embedding restricts to an embedding of the polylike domain which is isotopic the inclusion and therefore Theorem 1.6 does not apply directly to give Theorem 1.3. More precisely the construction of a nonempty moduli space of holomorphic curves in section 2.3 does not apply. The remainder of the proof consists in showing that a relevant moduli space is nevertheless nonempty and this is covered in section 3.2, where the nontriviality of the moduli space is reduced to a Proposition 3.7. This proposition is proven in section 3.3. Throughout the proofs we rely on the specific parameters of the ellipsoids involved to exclude the existence of curves not in our moduli spaces, and hence to establish compactness of these spaces.

2 Finite energy curves.

This section gives a proof by contradiction of Theorem 1.6. Some preliminary analysis is carried out in subsections 2.1 and 2.2, then we complete the proof in subsection 2.3.

2.1 Approximation of \( Q \).

Here we describe our smooth approximation \( W \) of \( Q = Q(b, a_2, \ldots, a_n) \), together with the closed characteristics on the boundary \( \partial W \). The analysis is similar to that in [9], section 2.1.

We start by fixing \( \delta \) and \( \epsilon \) with \( 0 < \delta << \epsilon \). Recall that our argument will be by contradiction and so we are assuming that there exists a symplectic isotopy \( Q_t \subset B^4(R) \times \mathbb{R}^{2(n-2)} \) with \( Q_0 = Q \) and \( Q_1 \subset B^4(S) \times \mathbb{R}^{2(n-2)} \), where \( R < 2a_2 + b \) and \( S < a_2 + b \). We will need to assume that \( \epsilon \) is small relative to both \( a_2 + b - S \) and \( 2a_2 + b - R \). Also, by perturbing the \( a_j \) if necessary, we may assume that \( \epsilon, 1/\epsilon \) and the \( 1/a_j \) are linearly independent over the rationals.

Now we choose a function \( f : [0, b] \to [0, 1] \) with \( f(0) = 0, f(b) = 1, f'(x), f''(x) \geq 0 \) and with the property that there exists an \( x_0 \) such that \( f'(x) = \epsilon \) for \( x < x_0 - \delta \) and \( f'(x) = \frac{1}{\epsilon} \) for \( x > x_0 + \delta \).

Given this, we define \( W \) as follows. It will be convenient to use symplectic polar coordinates on \( \mathbb{R}^{2n} = \mathbb{C}^n \), so we set \( R_j = \pi |z_j|^2 \) and \( \theta_j = \arg z_j \in S^1 \).
\[ W = \{ f(R_1) + \sum_{j=2}^{n} \frac{R_j}{a_j} \leq 1 \}. \]

The boundary \( \partial W \) is foliated by the Lagrangian tori \( L_c = \{ R_j = c_j \} \) which degenerate precisely when some of the \( R_j = 0 \). However, using the coordinates \( \theta_j \) we can identify the nondegenerate \( L_c \) with a fixed torus \( T^n \) and the integer homology with \( H_1(T^n, \mathbb{Z}) = \mathbb{Z}^n \).

The characteristic foliation \( \ker \omega|_{\partial W} \) is generated by the (Reeb) vector-field
\[
R_W = f'(R_1) \frac{\partial}{\partial \theta_1} + \sum_{j=2}^{n} \frac{1}{a_j} \frac{\partial}{\partial \theta_j}.
\]
In particular the Reeb vectorfield is tangent to the Lagrangian toric fibers \( L_c \).

The Reeb vectorfield has two kinds of periodic orbits. The first are the elliptic orbits \( \gamma^k = \{ z_j = 0, j \neq k \} \cap \partial W, \ k = 1, \ldots, n \). We use the notation \( r\gamma^k \) to denote the \( r \)-fold cover of \( \gamma^k \).

Since the \( 1/a_j \) are linearly independent all other periodic orbits lie in one of the complex 2-planes \( P_k = \{ z_j = 0, j \neq 1, k \} \) for \( 2 \leq k \leq n \). As \( \epsilon, \frac{1}{\epsilon} \) and \( \frac{1}{a_k} \) are linearly independent orbits in these planes are either elliptic or are called hyperbolic and lie in the region where \( x_0 - \delta < R_1 < x_0 + \delta \).

Suppose there exists such an \( R_1 \) and a rational number written in lowest terms as \( \frac{m}{n} \) such that \( f'(R_1) = \frac{m}{na_k} \). Then the corresponding torus fiber over \( c = (R_1, 0, \ldots, 0, a_k(1 - f(R_1)), 0, \ldots, 0) \) (the nonzero entries are in positions 1 and \( k \)) is foliated by a 1-parameter family of periodic Reeb orbits in the homology class \( (m, 0, \ldots, 0, n, 0, \ldots, 0) \). We denote these orbits by \( \gamma^k_{m,n} \). The \( r \)-fold cover of \( \gamma^k_{m,n} \) is written as \( \gamma^k_{r m,r n} \).

Now, if we fix a symplectic trivialization of \( T \mathbb{R}^{2n} \), the tangent bundle of \( \mathbb{R}^{2n} \) restricted to a closed orbit \( \gamma \) of \( R \) of period \( T \), then the derivative of the Reeb flow (extended to act trivially normal to \( \partial W \)) gives a map \( \psi : [0, T] \to \text{Symp}(2n, \mathbb{R}) \), where \( \text{Symp}(2n, \mathbb{R}) \) is the group of \( 2n \times 2n \) symplectic matrices. Associated to such a path is a Conley-Zehnder index \( \mu(\gamma) \) defined in this case by Robbin and Salamon in [24]. The analogue of Lemma 2.2 in [9] is the following.

**Lemma 2.1.** With respect to the standard basis of \( \mathbb{R}^{2n} \) the Conley-Zehnder indices are as follows.

\[
\begin{align*}
\mu(r\gamma^k) &= 2r + n - 1 + 2[cr_k] + 2 \sum_{j \neq k} \lfloor \frac{ra_k}{a_j} \rfloor, \text{if } k \neq 1 \\
\mu(r\gamma^1) &= 2r + n - 1 + 2 \sum_{j} \lfloor \frac{c r}{a_j} \rfloor \\
\mu(\gamma^k_{m,n}) &= 2(m + n) + \frac{1}{2} + (n - 2) + 2 \sum_{j \neq k} \lfloor \frac{na_k}{a_j} \rfloor.
\end{align*}
\]
2.2 Index and area formulas.

We compactify the open ball \( \hat{B}^4(R) \) by identifying it with the affine part of \( \mathbb{C}P^2(R) \), the complex projective plane with its Fubini-Study form scaled so that lines have area \( R \). We are considering a symplectic isotopy 
\[
Q_t \subset \hat{B}^4(R) \times \mathbb{R}^{2(n-2)} \subset \mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)}
\]
which restricts to an isotopy \( W_t \subset \mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)} \) of \( W \).

Let \( X_t = \mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)} \setminus W_t \) equipped with the restricted symplectic form. We can choose tame almost-complex structures with cylindrical ends \( J_t \) on \( X_t \) as in [6] and then study finite energy curves asymptotic to closed Reeb orbits as in [13], [14], [15], [16]. It is convenient to define the degree \( d \) of a finite energy curve to be its intersection number with \( \mathbb{C}P^1(\infty) \times \mathbb{R}^{2(n-2)} \), where \( \mathbb{C}P^1(\infty) \) is the line at infinity in \( \mathbb{C}P^2(R) \). The basic arrangement has been described in [9], section 2.2.1, but here we work in \( \mathbb{C}P^2(R) \times \mathbb{R}^{2(n-2)} \) rather than \( \mathbb{C}P^2 \).

In this subsection we give an approximate formula for the area and the virtual index formula for finite energy curves, the analogues of Lemmas 2.3 and 2.7 in [9].

Let \( C \) be a genus 0 finite energy plane with \( e^k \) punctures asymptotic to multiples of \( \gamma^k \), \( 1 \leq k \leq n \), the \( i \)th one asymptotic to \( r_i^k \gamma^k \), \( 1 \leq i \leq e^k \). (Here \( r_i^k \) is a natural number depending upon \( i \) and \( k \), hopefully this is not too confusing.) Also, let \( C \) have \( h^k \) punctures asymptotic to hyperbolic orbits in \( P_k \), \( 2 \leq k \leq n \), with the \( i \)th one asymptotic to \( \gamma_{m_i^k,n_i^k} \), \( 1 \leq i \leq h^k \).

Proposition 2.2. Up to an error of order \( \epsilon \), the symplectic area of \( C \) is given by

\[
\text{area}(C) = \int_C \omega = dR - \sum_{i=1}^{e^1} r_i^1 b - \sum_{k=2}^{n} \sum_{i=1}^{e^k} r_i^k a_k - \sum_{k=2}^{n} \sum_{i=1}^{h^k} (m_i^k b + n_i^k a_k).
\]

Note that the formula immediately implies that any nonconstant curves (which have positive area) must have degree \( d \geq 1 \).

Proof. To see this we can glue a disk in \( \partial W_t \) to each asymptotic end to produce a closed cycle of degree \( d \) in \( \mathbb{C}P^2 \), which has area \( dR \). The areas of these disks are roughly the negative terms in our formula (the error term comes because our hyperbolic orbits lie on \( \partial W_t \) rather than the singular part of \( \partial Q_t \)). \( \square \)
Proposition 2.3. The virtual index of $C$ in the space of curves with asymptotic limits allowed to vary is given by

$$\text{index}(C) = (n - 3)(2 - \sum_{k=1}^{n} e^k - \sum_{k=2}^{n} h^k) + 6d$$

$$- \sum_{i=1}^{e^1} (2r_i^1 + n - 1 + 2 \sum_{j} \lfloor \frac{e_i^1 a_j}{a_j} \rfloor)$$

$$- \sum_{k=2}^{n} \sum_{i=1}^{e^k} (2r_i^k + n - 1 + 2 \lfloor \frac{e_i^k a_k}{a_j} \rfloor + 2 \sum_{j \neq k} \lfloor \frac{e_i^k a_k}{a_j} \rfloor)$$

$$- \sum_{k=2}^{n} \sum_{i=1}^{h^k} (2(m_i^k + n_i^k) + (n - 2) + 2 \sum_{j \neq k} \lfloor \frac{n_i^k a_k}{a_j} \rfloor).$$

Note here that each elliptic limit not a cover of $\gamma^1$ contributes a negative term on the third line of the index formula, the limits asymptotic to $\gamma^1$ contribute negative terms on the second line, and the hyperbolic limits each contribute a negative term on the last line.

Proof. The general index formula for genus 0 curves with $s$ negative ends is

$$\text{index}(C) = (n - 3)(2 - s) + 2c_1(C) - \sum_{i=1}^{s} (\mu(\gamma_i) - \frac{1}{2} \dim V_i).$$

For this formula, see [2]. Here $c_1(C)$ is the Chern class which we have normalized to be $3d$, where $d$ is the degree, $\mu(\gamma_i)$ is the Conley-Zehnder index of the limiting Reeb orbit $\gamma_i$ corresponding to the $i$th end, and $\dim V_i$ is the dimension of the family of orbits containing $\gamma_i$. In our case this dimension is 0 for an elliptic orbit and 1 for a hyperbolic orbit. Substituting the Conley-Zehnder indices from Lemma 2.1 we get the formula as required.

In the remainder of this subsection we record a few algebraic consequences of the area and index formulas.

Lemma 2.4. Suppose that a finite energy curve $C$ has degree 1 and $\text{area}(C) \leq a_2$ (up to an error of order $\epsilon$). Then $C$ either has a single hyperbolic asymptotic limit $\gamma^2_{1,1}$, or all asymptotic limits are elliptic and satisfy

$$b < \sum_{i=1}^{e^1} r_i^1 b + \sum_{k=2}^{n} \sum_{i=1}^{e^k} r_i^k a_k < 2a_2 + b.$$
As \( a_2 + b < R < 2a_2 + b \) (and \( \epsilon \) is small relative to the differences) this gives

\[
b < \sum_{i=1}^{e_1} r_i b + \sum_{k} \sum_{i=1}^{e_k} r_i^k a_k + \sum_{k=2}^{n} \sum_{i=1}^{b_k} (m_i^k a_k + n_i^k b) < 2a_2 + b.
\]

Since \( a_k > 2a_2 \) for all \( k \geq 3 \) we see that if there exists a hyperbolic orbit it must be of type \( \gamma_{1,1}^2 \) and be the only asymptotic limit. On the other hand, if all limits are elliptic then they satisfy the inequality of the lemma.

**Lemma 2.5.** Suppose that a finite energy curve \( C \) has degree 1, virtual index at least \(-1\), and only elliptic asymptotic limits. Then it has only a single asymptotic limit, that is, \( C \) is a finite energy plane.

**Proof.** Let \( E \) be the total number of elliptic asymptotic limits. Since all terms in the sums on the second and third lines of the index formula of Proposition 2.3 are at least \( n + 1 \), we have the formula

\[
-1 \leq \text{index}(C) \leq (n - 3)(2 - E) + 6 - (n + 1)E = 2(n - (n - 1)E).
\]

Hence \( (n - 1)E \leq n \) and so as \( n \geq 3 \) we have \( E \leq 1 \) as required.

Putting the previous two lemmas together we have the following, which describes the curves we will be interested in.

**Lemma 2.6.** Suppose that a finite energy curve \( C \) has degree 1 and \( \text{area}(C) \leq a_2^2 \) and \( \text{index}(C) \geq -1 \). Then \( C \) is a finite energy plane asymptotic to either \( \gamma_{1,1}^2 \), \( 2\gamma^1 \) or \( 2\gamma^2 \).

**Proof.** By Lemmas 2.4 and 2.5, if the curve \( C \) is not asymptotic to \( \gamma_{1,1}^2 \) then it is a finite energy plane asymptotic to a cover of one of the \( \gamma^k \), say asymptotic to \( r\gamma^k \).

Suppose first that \( k = 1 \). Then by Lemma 2.4 we have \( b < rb < 2a_2 + b \) and Proposition 2.3 again implies that \( r \leq 2 \). Putting the two together we have \( r = 2 \).

Next suppose that \( k = 2 \). Again by Lemma 2.4 we have \( b < ra_2 < 2a_2 + b \) and by Proposition 2.3 we have

\[
\text{index}(C) \leq (n - 3) + 6 - (2r + (n - 1)).
\]

As \( \text{index}(C) \geq -1 \) this implies that \( r \leq 2 \). By our original hypothesis in Theorem 1.6 we have \( a_2 < b \), and combining the two inequalities gives \( r = 2 \).

Finally suppose that \( k \geq 3 \). By hypothesis \( a_k > 2a_2 \) and so the term \( 2\lfloor \frac{ra_k}{a_2} \rfloor \) in the index formula contributes at least 4. Hence

\[
\text{index}(C) \leq (n - 3) + 6 - (2r + (n - 1) + 4) \leq -2r
\]
a contradiction as required. \( \square \)
2.3 Moduli spaces of finite energy planes.

Let us fix an orbit $\eta_t$ of type $\gamma_{1,1}^2$ in each $\partial W_t$. Consider the corresponding moduli space

$$\mathcal{M}_t = \{ u : \mathbb{C} \to X | \text{degree}(u) = 1, \partial J_t u = 0, u \sim \eta_t \}/G$$

where $u \sim \eta$ means that $u$ is asymptotic at infinity to $\eta$, and $G$ is the reparameterization group of $\mathbb{C}$. The area formula of Proposition 2.2 says that curves in $\mathcal{M}_t$ have area roughly $R - (a_2 + b)$.

We will need to choose the almost-complex structure $J_0$ such that the line at infinity $\mathbb{C}P^1(\infty) \times \mathbb{R}^{2(n-2)}$ is complex and such that it is invariant with respect to the $T^{n-2}$ action rotating the $(z_3, \ldots, z_n)$ planes. This is possible since $W = W_0$ is invariant under the same action. A genericity assumption will also be made as explained in Lemma 2.8. The almost-complex structure $J_1$ can be assumed to be the standard product integrable structure on $(\mathbb{C}P^2(R) \setminus B^4(S)) \times \mathbb{R}^{2(n-2)}$ for some $S < a_2 + b$, as $W_1 \subset B^4(a_2 + b) \times \mathbb{R}^{2(n-2)}$.

**Lemma 2.7.** The virtual dimension of $\mathcal{M}_t$ is 0.

**Proof.** Proposition 2.3 gives virtual dimension 1 for finite energy planes of degree 1 asymptotic to an orbit of type $\gamma_{1,1}^2$. However a curve lies in $\mathcal{M}_t$ only if it is asymptotic to the specific orbit $\eta_t$, and this imposes a 1-dimensional constraint.

The moduli spaces when $t = 0, 1$ are easily described.

**Lemma 2.8.** There exists an almost-complex structure $J_0$ such that the moduli space $\mathcal{M}_0$ consists of a single, regular curve.

As this is a direct generalization of Lemma 2.8 in [9], utilizing the analysis in [11] to extend the results to higher dimension, we restrict here to an outline.

**Outline of proof.** As $J_0$ is invariant under rotations of the $(z_3, \ldots, z_n)$ planes, the $(z_1, z_2)$-plane $P_1 = \{ z_3 = \cdots = z_n = 0 \}$ is $J_0$-invariant. Hence $J_0$ can be restricted to $Y = X_0 \cap P_1$ to give an almost-complex manifold with a cylindrical end over $\partial Y := \partial W_0 \cap P_1$. The almost-complex manifold $Y$ is exactly the one studied in [9], and elements of $\mathcal{M}_0$ lying in $Y$ form a moduli space $\tilde{\mathcal{M}}_0$ in their own right. In particular Lemma 2.8 from [9] implies that $\tilde{\mathcal{M}}_0$ is nonempty, that is, there exists an element of $\mathcal{M}_0$ lying in $Y$. To complete the proof we will show first that there can be no more than one element of $\tilde{\mathcal{M}}_0$ and second that, for a generic choice of invariant $J_0$, all elements of $\mathcal{M}_0$ must lie in $Y$. Lemma 3.17 in [11] shows that, for invariant almost-complex structures, curves in $\tilde{\mathcal{M}}_0$ are regular in $\mathcal{M}_0$.

For the first part, we argue by contradiction and suppose that two distinct curves $u_0$ and $u_1$ represent equivalence classes in $\tilde{\mathcal{M}}_0$. Automatic regularity in dimension 4 (see [27], Theorem 1, or the discussion after Theorem
2.9 in [9]) implies that \( u_0 \), say, can be included in a local 1-parameter family of curves \( u_t, -\epsilon < t < \epsilon \), with a single curve in the family asymptotic to each \( \gamma_{1,1}^2 \) orbit close to \( \eta_0 \). Meanwhile, as \( u_0 \) and \( u_1 \) are both asymptotic to \( \eta_0 \), on a suitable subset of the cylindrical end \( (-\infty, S_0] \times \partial Y \) we can represent \( u_1 \) as a section \( \xi \) of the normal bundle to the image of \( u_0 \). Furthermore, if \( S_0 \) is sufficiently negative, the section \( \xi \) has no zeros and so defines a winding of \( u_1 \) about \( u_0 \). For this see [15]. This winding is the same as the winding of an eigenvector of an asymptotic operator associated to the orbit \( \eta_0 \), and as we are dealing with a negative puncture the associated eigenvalue must be positive.

Now, the asymptotic operator acts on sections of the normal bundle to \( \eta_0 \) in \( \partial Y \), which has an induced complex structure still called \( J_0 \). With respect to a basis of the normal bundle extending a tangent vector to the space of \( \gamma_{1,1}^2 \) orbits, the asymptotic operator takes the form

\[
-J_0 \frac{d}{dt} - T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

where \( T \) is the period of \( \eta_0 \). We see that the only eigenvectors with winding number 0 have eigenvalues 0 or \(-T\) and so can conclude that in this basis \( u_1 \) must wind around \( u_0 \). Hence \( u_1 \) must intersect the \( u_t \), which have winding 0 because they are asymptotic to different orbits. However, by gluing planes inside \( W_0 \) the images of \( u_1 \) and the \( u_t \) can be included in cycles of degree 1 in \( \mathbb{C}P^2 \), which therefore have intersection number 1. The added planes can be assumed to have a unique (positive) intersection point at the origin and so the intersections of \( u_1 \) and \( u_t \) contribute 0. This contradicts positivity of intersection.

For the second part of the proof we must exclude curves in \( M_0 \) not lying in \( Y \). As \( J_0 \) is \( T^{n-2} \)-invariant, any such curves appear in \((n-2)\)-dimensional families and so are certainly not regular. Hence if we are able to assume that \( J_0 \) is regular for \( M_0 \) and at the same time \( T^{n-2} \)-invariant then no such curves exist. The proof of the existence of regular invariant almost-complex structures follows the usual regularity argument working with invariant rather than general almost-complex structures. For this to work, instead of assuming that our holomorphic curves are somewhere injective we need the stronger assumption that corresponding to each curve in \( M_0 \) there exists an orbit of the \( T^{n-2} \) action which intersects the curve in a single point, see the proof of Proposition 3.16 in [11]. This is automatic in our case since by positivity of intersection a degree 1 curve must intersect \( \mathbb{C}P^1(\infty) \times \mathbb{R}^{2(n-2)} \) exactly once transversally, and hence intersect exactly one \( T^{n-2} \) orbit in \( \mathbb{C}P^1(\infty) \times \mathbb{R}^{2(n-2)} \), in a single point.

\[ \square \]

Lemma 2.9. For \( J_1 \) chosen as above, the moduli space \( M_1 \) is empty.
Proof. This is identical to the proof of Lemma 2.11 in [9]. Indeed, the image of any curve in $M_1$ can be restricted to a curve in $(\mathbb{C}P^2(R) \setminus B(S)) \times \mathbb{R}^{2(n-2)}$. As the complex structure is assumed to be a product the curve projects to a holomorphic curve in $\mathbb{C}P^2(R) \setminus B(S)$, and then by a monotonicity theorem, see [9], Lemma 2.12, we see that it has area at least $R - S$. This is a contradiction as curves in any $M_t$ have area $R - (a_2 + b)$.

Next we consider the universal moduli space

$$M = \{(u,t)|u: \mathbb{C} \to X_t, \text{degree}(u) = 1, \overline{\partial}_J u = 0, u \sim \eta, t \in [0,1]\}/G.$$ 

This has virtual dimension 1, but to show that it is a compact 1-dimensional manifold (the source of our contradiction) we will need some assumptions on the family of almost-complex structures $J_t$. First of all, since the curves in $M$ have degree 1 they are not multiply covered and so we may choose a family $J_t$ so that $M$ is a 1-dimensional manifold giving a cobordism between $M_0$ and $M_1$. The $J_t$ can be chosen to coincide with those we already have when $t = 0, 1$. Indeed, $J_0$ is regular by Lemma 2.8, and since no curves in $M$ lie entirely in $(\mathbb{C}P^2(R) \setminus B(S)) \times \mathbb{R}^{2(n-2)}$ we are free to take $J_1$ standard here and perturb elsewhere to obtain regularity if necessary.

Second, a collection of families $\{J_t\}$ of the second category is regular in the sense that any somewhere injective $J_{t_0}$-holomorphic finite energy curve, for $t_0 \in [0,1]$, has deformation index at least $-1$ (amongst $J_{t_0}$ curves). We will also assume then that our $J_t$ are regular in this sense.

Finally, the cylindrical ends of the $X_t$ are all symplectomorphic, and after identifying them by a symplectomorphism we may assume that all $J_t$ are identical outside of a compact set. This implies that they induce identical translation invariant almost-complex structures on the symplectization $S(\partial W) = \mathbb{R} \times \partial W$. Holomorphic curves in $S(\partial W)$ are either translation invariant, which means they are covers of cylinders over Reeb orbits, or come in families of dimension at least 1. Therefore, if an almost-complex structure is regular, somewhere injective finite energy curves are either trivial cylinders or have deformation index at least 1. As above such almost-complex structures form a subset of the second category and we will assume our $J_t$ induce a structure in this class.

The final lemma is the following, which contradicts Lemmas 2.8 and 2.9.

**Lemma 2.10.** The universal moduli space $M$ is sequentially compact.

Proof. The general compactness theorem for finite energy curves can be found in [3]. In our situation, it implies that a sequence of finite energy curves $u_i$ representing classes in $M_{t_i}$ with $t_i \to t_\infty$, after taking a subsequence, converge in the sense of [3] to a holomorphic building in $X_{t_\infty}$. For components in $X_{t_\infty}$ to be nonconstant they must have positive degree (see
the comment after Proposition 2.2), and so since degree is preserved in the limit and the \( u_i \) have degree 1 our limit must consist of a single curve \( u \) in \( X_{t_\infty} \) of degree 1. Therefore the curve is also somewhere injective. By regularity of the family of \( J_t \) we have \( \text{index}(u) \geq -1 \), and as the \( u_i \) have area roughly \( R - (a_2 + b) \) the area of \( u \) is bounded above by \( R - (a_2 + b) \). As \( a_2 < b \) by assumption, this excludes planes asymptotic to \( 2\gamma^2 \) and hence by Lemma 2.6, the curve \( u \) is a finite energy plane asymptotic to either an orbit \( \gamma^2_{1,1} \) or to \( 2\gamma^1 \). In the first case, as the limit preserves area, it must be asymptotic to \( \eta_{t_\infty} \) itself (as otherwise we would see symplectization components of positive area). Hence \( (u, t_\infty) \) represents a class in \( \mathcal{M} \) and we have compactness as required.

It remains to exclude limiting planes asymptotic to \( 2\gamma^1 \), which have index 0 by Proposition 2.3. If a curve with such a limit exists then we have \( R - 2b > 0 \) and so \( 2b < R < 2a_2 + b \) and \( b < 2a_2 < a_j \) for \( j \geq 3 \).

We look at components of the limit mapping to the symplectization layers \( S(\partial W) \). There is a single curve in the highest level with positive end asymptotic to \( 2\gamma^1 \). If this curve is a cylinder then the negative end is an asymptotic orbit with action between \( a_2 + b \) and \( 2b \). Given the inequalities above, the only possibilities are negative ends on orbits of type \( \gamma^2_{1,1} \) or \( 3\gamma^2 \) or \( \gamma^j \) for some \( j \geq 3 \). In all three cases the greatest common divisor of the covering degrees of the positive and negative ends is 1 and so the cylinder is somewhere injective. A variation of Proposition 2.3 (or simply using the fact that the total index is preserved in a limit) shows that cylinders asymptotic to \( 3\gamma^2 \) or \( \gamma^j \) for \( j \geq 3 \) have deformation index at most \(-2\) and so we do not expect such cylinders to exist for regular almost-complex structures. Cylinders asymptotic to \( \gamma^2_{1,1} \) have deformation index 1, but by translation invariance we do not expect such a cylinder to have negative end on \( \eta_{t_\infty} \). By area reasons, such a cylinder cannot be connected to any lower level curves, and so this possibility can also be excluded.

Finally, suppose the highest level curve in \( S(\partial W) \) has several ends. As we take a limit of curves of genus 0 exactly one of these ends is connected in our limiting building to \( \eta_{t_\infty} \) and it has action at least \( a_2 + b \). The remaining ends have action less than \( 2b - (a_2 + b) = b - a_2 < a_2 \) by the inequality above. Since no such periodic orbits exist we have a contradiction.

\[ \square \]

3 Isotopies of ellipsoids.

The main result of this section is the following theorem, from which Theorem 1.3 follows directly by checking the inequalities.

**Theorem 3.1.** Let \( A, B, C_3, \ldots, C_n, R \) be parameters satisfying the following inequalities.

\[ \text{Theorem 3.1. Let } A, B, C_3, \ldots, C_n, R \text{ be parameters satisfying the following inequalities.} \]
i. $\frac{11}{8} < A < \frac{13}{8}$;

ii. $\frac{3A}{2A-2} < B < 4A < C_3 < 2R < \frac{17}{3}$;

iii. $C_3 < C_i$ for all $i > 3$.

Then there exist nonisotopic embeddings $E(A,B,C_3, \ldots C_n) \rightarrow B^4(R) \times \mathbb{C}^{n-2}$.

We note an immediate consequence of the inequalities (i) and (ii) is that $2.75 < R < 3$.

In subsection 3.1 we describe our ellipsoid embeddings and state a result on isotopies of polylike domains which implies that the ellipsoids are nonisotopic in the sense of Definition 1.1. In section 3.2 we prove the result on polylike isotopies modulo an existence result for certain holomorphic curves which is reserved for section 3.3.

### 3.1 Construction of ellipsoid embeddings.

Suppose we are given an ellipsoid $E(B, A, C_3, \ldots C_n)$ with parameters satisfying the inequalities of Theorem 3.1. Note that for notational convenience we have reversed the order of the first two factors. Before describing our embeddings we need to choose a $c$ with $4 < c < 5$ such that the polydisk $Q(1.5, 1, c, \ldots, c)$ is a subset of our ellipsoid $E(B, A, C_3, \ldots C_n)$.

**Choice of $c$ and the polylike subset.**

The first inequality of condition (ii) can be written as $1.5B + \frac{1}{A} < 1$. Therefore we can find a $c$ with $c > \frac{C_3}{A}$ and $1.5B + \frac{c}{C_3} < 1$. By condition (ii) again we have $\frac{C_3}{A} > 4$ but as $5A > \frac{55}{8} > 6$, by condition (i), and $C_3 < 6$, by condition (ii), we have $\frac{C_3}{A} < 5$ and so we may take $4 < c < 5$. We claim that $Q(1.5, 1, c, \ldots, c) \subset E(B, A, C_3, \ldots C_n)$. Indeed, suppose $(z_1, \ldots, z_n) \in Q(1.5, 1, c, \ldots, c)$, that is, $\pi|z_1|^2 \leq 1.5$ and $(z_2, \ldots, z_n) \in E(1, c, \ldots, c)$. Then we have

\[
\frac{\pi|z_1|^2}{B} + \frac{\pi|z_2|^2}{A} + \frac{\pi|z_3|^2}{C_3} + \ldots + \frac{\pi|z_n|^2}{C_n} \leq 1.5 \frac{C_3}{B} + c \left( \frac{\pi|z_2|^2}{Ac} + \frac{\pi|z_3|^2}{c} + \ldots + \frac{\pi|z_n|^2}{c} \right) \leq 1
\]

by our bounds on $c$. For the inequality between lines one and two we are using $\pi|z_1|^2 \leq 1.5$ and condition (iii), and for the inequality between lines two and three we are using $\frac{C_3}{A} < 1$.

We will produce two embeddings of the ellipsoid which will be shown to restrict to nonisotopic polylike domains.
The embedding $f_0$.

First note that we have a symplectic embedding $g_0$ given by the composition

\[ E(A, B) := AE(1, \frac{B}{A}) \subset AE(1, 4) \rightarrow AB^4(2) = B^4(2A) \subset B^4(R). \]

The first inclusion here follows from the upper bound on $B < 4A$ in condition (ii). The next map $E(1, 4) \rightarrow B^4(2)$ can be read off from the classification of ellipsoid embeddings into balls contained in [20], although this particular embedding was also known at least to Opshtein, [23] Lemma 2.1. Finally the inclusion $B^4(2A) \rightarrow B^4(R)$ also holds by condition (ii). Taking our map to be the identity in coordinates $z_3, \ldots, z_n$ we get an embedding

\[ f_0 : E(B, A, C_3, \ldots, C_n) \rightarrow B^4(R) \times \mathbb{C}^{n-2}, \]

\[ (z_1, \ldots, z_n) \mapsto (g_0(z_2, z_1), z_3, \ldots, z_n). \]

The embedding $f_1$.

The construction of the second embedding is more subtle. In particular, to get the restriction to $Q$ we require, we will need to invoke Theorem 1.2.

First we observe again from [20], Theorem 1.1.2, or by inspection of Figure 1.1, that since $C_3 > 4A$ by condition (ii) we have an embedding

\[ \tilde{g}_1 : E(A, C_3) := AE(1, \frac{C_3}{A}) \rightarrow AB^4(\frac{C_3}{2}) := B^4(\frac{C_3}{2}) \subset B^4(R) \]

where the final inclusion is also a consequence of condition (ii). As $c < C_3$ this embedding restricts to an embedding of $E(1, c)$, which is precisely the intersection of our polylkike domain with the $(z_2, z_3)$ plane.

Now as $4 < c < 5$ there also exists an embedding

\[ E(1, c) \rightarrow B^4(\frac{c}{2}) \subset \hat{B}^4(2.5) \subset B^4(R) \]

as $R > 2A > \frac{11}{4}$ by condition (ii). Hence by Theorem 1.2 we may replace our embedding $\tilde{g}_1 : E(A, C_3) \rightarrow B^4(R)$ by an embedding $g_1$ which restricts to one sending $E(1, c) \rightarrow B^4(2.5)$. Extending this map to be the identity in coordinates $z_3, \ldots, z_n$ and composing with a linear map interchanging the first and third coordinates we get another embedding

\[ f_1 : E(B, A, C_3, \ldots, C_n) \rightarrow B^4(R) \times \mathbb{C}^{n-2}, \]

\[ (z_1, \ldots, z_n) \mapsto (g_1(z_2, z_3), z_1, z_4, \ldots, z_n). \]

This maps the polylkike domain to $\hat{B}^4(2.5) \times \mathbb{C}^{n-2}$. 

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Remark 3.2. There is in fact no symplectic embedding of the whole ellipsoid $E(B, A, C_3, \ldots C_n) \to B^4(2.5) \times \mathbb{C}^{n-2}$. This is a consequence of the Ekeland-Hofer capacities, see [4]. Indeed, the upper bound on $A$ in condition (i) together with the first inequality in condition (ii) implies that $B > 2A$. Therefore the second Ekeland-Hofer capacity of the ellipsoid $E(B, A, C_3, \ldots C_n)$ is $2A > 11/4$ while the corresponding capacity of $B^4(2.5) \times \mathbb{C}^{n-2}$ is 2.5.

Of course, if $f_0$ and $f_1$ were isotopic in the sense of Definition 1.1 then the images of the polylike domains would also be isotopic. Hence Theorem 3.1 is a consequence of the following.

Theorem 3.3. Let $R, A, B, C_i$ satisfy the inequalities of Theorem 3.1 and $4 < c < 5$ be chosen as above. Let

$$f_0 : Q(1.5, 1, c, \ldots, c) \to B^4(R) \times \mathbb{C}^{n-2}$$

be a symplectic embedding which is a restriction of an embedding $E(B, A, C_3, \ldots C_n) \to B^4(R) \times \mathbb{C}^{n-2}$ of the form

$$f_0(z_1, \ldots, z_n) = (g_0(z_1, z_2), z_3, \ldots, z_n)$$

and let

$$f_1 : Q(1.5, 1, c, \ldots, c) \to B^4(S) \times \mathbb{C}^{n-2}$$

be a symplectic embedding for some $S < 2.5$ which is a restriction of an embedding $E(B, A, C_3, \ldots C_n) \to B^4(R) \times \mathbb{C}^{n-2}$ of the form

$$f_1(z_1, \ldots, z_n) = (g_1(z_2, z_3), z_1, z_4, \ldots, z_n).$$

Then $f_0$ and $f_1$ are not isotopic through symplectic embeddings into $B^4(R) \times \mathbb{C}^{n-2}$ which extend to $E(B, A, C_3, \ldots C_n)$.

The proof of Theorem 3.3 is the subject of subsections 3.2 and 3.3. We observe here though that as $R < 3$, Theorem 3.3 would be a special case of Theorem 1.6 if we knew that the map $g_0|_{P(1.5,1)}$ were isotopic to the identity inside $B^4(R)$.

3.2 Another isotopy obstruction.

3.2.1 Preliminaries.

Here we begin the proof of Theorem 3.3. The proof will follow the same scheme as that of Theorem 1.6. As coordinates $z_4, \ldots z_n$ play no role in the proof, for notational convenience we will restrict to the case of $n = 3$, that is, dimension 6. Then we study embeddings of a polylike domain $Q(1.5, 1, c)$ which lies in an ellipsoid $E(B, A, C)$. As in section 2 our method is to replace the image of $Q(1.5, 1, c)$ under $f_0$ by a smooth domain $W = W_0$ and argue
by contradiction assuming there exists a symplectic isotopy \( W_t \) for \( 0 \leq t \leq 1 \) with \( W_1 \subset B^4(S) \times \mathbb{C} \). The Reeb orbits in \( \partial W \) were described in section 2.1 and we will follow that notation.

We make a choice of a specific orbit \( \eta_t \subset \partial W_t \) of type \( \gamma^2_{1,1} \) and \( \sigma_t \subset \partial W_t \) of type \( \gamma^2_{1,4} \) for each \( t \). We also choose compatible almost-complex structures \( J_t \) on \( X_t = \mathbb{C}P^2(R) \times \mathbb{C} \setminus W_t \) exactly as in section 2.2. Again we assume that our \( J_t \) all coincide on the cylindrical ends after identifying them with subsets of \( \partial W \times (-\infty, 0] \). Furthermore we assume that this almost-complex structure, when extended to be translation invariant on the symplectization \( S(\partial W) \), is regular (see the discussion in section 2.3 preceding Lemma 2.10).

Now there are two corresponding moduli spaces of curves we will need to examine, a moduli space \( M_t \) defined as in section 2.3 and a second moduli space \( N_t \). Define

\[
M_t = \left\{ u : \mathbb{C} \to X| \deg(u) = 1, \overline{\partial}_J u = 0, u \sim \eta_t \right\} / G
\]

where \( u \sim \eta \) means that \( u \) is asymptotic at infinity to \( \eta \), and \( G \) is the reparameterization group of \( \mathbb{C} \), and analogously

\[
N_t = \left\{ u : \mathbb{C} \to X| \deg(u) = 2, \overline{\partial}_J u = 0, u \sim \sigma_t \right\} / G.
\]

By Proposition 2.3 both moduli spaces have dimension 0. By Proposition 2.2 curves in \( M_t \) have area roughly \( R - 2.5 \) and curves in \( N_t \) have area approximately \( 2R - 5.5 \), which is less than \( R - 2.5 \) since \( R < 3 \). Hence the monotonicity argument in Lemma 2.9 implies that both moduli spaces are empty for a suitable choice of \( J_1 \) which is the standard product on \( X_1 \setminus (B^4(2.5) \times \mathbb{C}) \).

We will choose \( J_0 \) as in section 2.3 so that it is invariant under rotations in the \( z_3 \) plane. Then in subsection 3.2.3 we prove the following.

**Proposition 3.4.** For any choice of invariant \( J_0 \) either the moduli space \( M_0 \) or the moduli space \( N_0 \) is nonempty.

The uniqueness part of Lemma 2.8 applies again here to show that if these moduli spaces are nonempty then they consist of a single curve which appears transversally, in particular they represent a nontrivial cobordism class. This follows exactly as in Lemma 2.8 for the degree 1 curves, that is, a positivity of intersection argument shows that there exists at most one element of the moduli space in \( \{ z_3 = 0 \} \), and regularity implies that all elements lie in \( \{ z_3 = 0 \} \). The same argument applies to the degree 2 curves once we show that there exists a regular, invariant \( J_0 \). The proof that such almost-complex structures exist follows from a stretching argument as in [11], Proposition 3.16.

At this point the proof of Theorem 3.3 breaks into two parts according to whether \( M_0 \) or \( N_0 \) is nonempty. If \( M_0 \) is nonempty for any \( J_0 \) then the
proof proceeds exactly as Theorem 1.6. Indeed, Lemma 2.9 implies that \( \mathcal{M}_1 \) is empty for a suitable choice of \( J_1 \) and Lemma 2.10 implies that \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) are cobordant. This gives our contradiction.

We do not know if the analogue of Lemma 2.10 applies in full generality for the moduli spaces \( \mathcal{N}_t \). However we will see in section 3.2.2 that it does hold for a particular family of \( J_t \). As we already have a proof if \( \mathcal{M}_0 \) is nonempty for any invariant \( J_0 \), this will complete the proof.

### 3.2.2 The moduli space of degree 2 curves.

In this section we analyze the moduli spaces of degree 2 curves

\[
\mathcal{N}(J_t) = \{u : \mathbb{C} \to X_t | \text{degree}(u) = 2, \overline{\partial}_J u = 0, u \sim \sigma_t \}/G
\]

where \( J_t \) is an almost-complex structure on \( X_t \).

Recall that our isotopy, say \( \phi_t \), of \( W \) extends to one of \( E(B,A,C) \). Set \( E_t = \phi_t(E(B,A,C)) \). We will choose \( J_t = J_t^N \) to be an almost-complex structure on \( X_t \) stretched to length \( N \) along \( \partial E_t \), and when restricted to \( E_t \setminus W_t \) to be independent of \( t \) (up to a biholomorphism). We may assume that \( A, B, C \) are rationally independent. Then there are three distinct closed Reeb orbits on \( \partial E(B,A,C) \), namely the intersection of \( \partial E(B,A,C) \) with the coordinate planes. We label the closed Reeb orbits in the respective coordinate planes by \( \delta_k \) for \( 1 \leq k \leq 3 \).

As \( E_0 \) and \( E_1 \) are invariant under rotations in the \( z_3 \) plane we may take \( J_0^N \) and \( J_1^N \) to be invariant and also, following [11], to be regular.

**Lemma 3.5.** For some \( N \) sufficiently large, the universal moduli space

\[
\mathcal{N} = \{(u,t) | u : \mathbb{C} \to X_t, \text{degree}(u) = 2, \overline{\partial}_J u = 0, u \sim \sigma_t, t \in [0,1]\}/G
\]

is sequentially compact.

**Proof.** We suppose that there exists a sequence of planes \( (u_i, t_i) \in \mathcal{N} \) which degenerates to a \( J_t \) holomorphic building. Since \( 2R - 5.5 < 1 \) there are no planar components in the symplectization of \( \partial W_t \) and so a degree 2 plane can degenerate in \( X_t \) into either two planes of degree 1 or a single degree 2 plane of smaller area.

Degree 1 planes are necessarily somewhere injective and as in Lemma 2.6 (see also Lemma 2.10) have index at least \(-1\) and area bounded by \( 2R - 5.5 < R - 2.5 \) only if they are asymptotic to \( \gamma^2_{1,1} \). Thus they have area \( R - 2.5 \) and so the sum of the areas is \( 2R - 5 \), a contradiction as the \( u_i \) have area \( 2R - 5.5 \).

Suppose then that the limiting building has a degree 2 curve \( C \) in \( X_t \). The plane has area at most \( 2R - 5.5 \), and by Proposition 2.2 this immediately excludes planes asymptotic to elliptic orbits.
Next suppose the curve is asymptotic to a hyperbolic orbit $\gamma_{m,n}^2$, then, for a nontrivial degeneration into a curve of smaller area, the area formula Proposition 2.2 implies that $(m, n) = (3, 1)$. By Proposition 2.3 the plane has index 3. Similarly the only nontrivial symplectization component can be a cylinder with positive end on an orbit $\gamma_{3,1}^2$ and negative end at an orbit $\sigma_t$ of type $\gamma_{1,4}^2$. Such cylinders are necessarily somewhere injective and have deformation index $-1$, a contradiction if our almost-complex structure is assumed to induce regular structures on the symplectizations $\partial W_t$.

The remaining possibility is that $C$ is a degree 2 curve asymptotic to an orbit $\gamma_{m,n}^3$, and as $c > 4$ our area inequality implies that $(m, n) = (1, 1)$. For generic families of almost-complex structure we do not know whether or not such planes can exist. By Proposition 2.3 they have index $-1$. However we will show that at least they do not exist if $N$ is chosen sufficiently large.

Arguing by contradiction, suppose such planes exist for all $N$. Then we can take a limit as $N \to \infty$ (stretching the neck along $\partial E_t$) and the limit will be a holomorphic building with components in $X_t \setminus E_t$ and $E_t \setminus W_t$, and perhaps the symplectizations $S(\partial E_t)$ and $S(\partial W_t)$. We will show that the sum of the indices of these components is nonnegative, a contradiction since we are taking a limit of planes of index $-1$.

An analysis similar to that above implies that the components in $X_t \setminus E_t$ consist either of two planes of degree 1 or a single plane of degree 2. The index formula for planes $u_k$ of degree $d$ in $X_t \setminus E_t$ asymptotic to $r\delta^k$, for $1 \leq k \leq 3$ are given respectively by

\[
\text{index}(u_1) = 6d - (2r + 2 + 2\left\lfloor \frac{rB}{A} \right\rfloor + 2\left\lfloor \frac{rB}{C} \right\rfloor)
\]

\[
\text{index}(u_2) = 6d - (2r + 2 + 2\left\lfloor \frac{rA}{B} \right\rfloor + 2\left\lfloor \frac{rA}{C} \right\rfloor)
\]

\[
\text{index}(u_3) = 6d - (2r + 2 + 2\left\lfloor \frac{rC}{A} \right\rfloor + 2\left\lfloor \frac{rC}{B} \right\rfloor)
\]

We note that these indices are always even.

Let us identify any matching ends of components in $X_t \setminus E_t$ and any components in $S(\partial E_t)$. The new components still consist of either a single plane of degree 2 or two planes of degree 1. Suppose such a disk has its negative end on a cover of $\delta^1$. By the first inequality of condition (ii) and then condition (i) we have $B > \frac{5A}{2} + \frac{3}{2} > \frac{3}{2} + \frac{1}{2} = \frac{3}{2}$ and so to have a plane of positive area the plane must be of degree 2 and the asymptotic orbit simply covered. Also from condition (ii) we have $2A < B < 4A < C_3$ and so by the formula above the index of our plane is at least 2.

Next suppose we have a plane asymptotic to a cover of $\delta^2$. By condition (i) we see that $3A > 3 > R$ but $5A > \frac{25}{8} > 2R$ and so for the plane to have positive area it is either of degree 1 and covers $\delta^2$ at most twice or is of degree 2 and covers $\delta^2$ at most 4 times. In either case our plane has nonnegative index.
Suppose we have a plane asymptotic to a cover of $\delta^3$. Then as in the case of covers of $\delta^1$ the limit is simply covered and the plane has degree 2. Such planes have index $-2$ and so for a generic 1-parameter family of almost-complex structures we do not expect to see such planes in $X_t \setminus E_t$. Hence such a plane can only arise by matching ends with symplectization components. But as the negative end is simply covered we can check that any symplectization components of our plane must be somewhere injective and so have nonnegative index. Also, if there are nontrivial symplectization components then the components in $X_t \setminus E_t$ must be asymptotic to covers of $\delta^1$ and $\delta^2$ and have nonnegative index as above.

There are no symplectization components in $S(\partial W_t)$. Indeed, the area considerations above show that $\gamma^3_{1,1}$ has maximal action amongst orbits of action bounded by $2R$.

Finally then we consider the component in $E_t \setminus W_t$. As the almost-complex structure here is chosen to be independent of $t$, somewhere injective curves can be assumed to have nonnegative index. This includes our curve since its negative end is a simply covered orbit of type $\gamma^3_{1,1}$. Hence the sum of the indices of all components is nonnegative and we have a contradiction as desired.

Lemma 3.5 implies, for large $N$, that $N_0(J_0)$ is cobordant to $N_1(J_1)$. If we assume $N_0(J_0)$ is nontrivial this implies $N_1(J_1)$ is also nontrivial for a regular, $S^1$-invariant almost-complex structure $J_1 = J_1^N$. Such almost-complex structures can be shown to exist by following [11], Proposition 3.16. This is not an immediate contradiction to monotonicity however since $J_1$ is chosen to be stretched along $\partial E_1$ rather than the standard product on $X_1 \setminus (B^4(2.5) \times \mathbb{C})$. To derive a contradiction we need to show that such degree 2 curves persist as we deform $J_1$ to be a product outside of $B^4(2.5) \times \mathbb{C}$.

The proof of this follows the first part of the proof of Lemma 3.5. However in this isotopy the complex structure will no longer be stretched along $\partial E_1$ and so Lemma 3.5 may not exclude bubbling of degree 2 planes asymptotic to orbits $\gamma^3_{1,1}$. We rule out this possibility in our current situation using $S^1$-invariance. Recall that the $S^1$ action is by rotation in the $\mathbb{C}$ component of $\mathbb{C}P^2(R) \times \mathbb{C}$. Given our embedding $f_1$, this action acts transitively on hyperbolic orbits of a given type. In particular, for an $S^1$ invariant almost-complex structure, if such a plane asymptotic to a $\gamma^3_{1,1}$ orbit exists, then there exists at least an $S^1$ family of such planes, with a plane asymptotic to each such orbit. Now, as the $S^1$ action is transverse to the hyperbolic Reeb orbits, the asymptotic behavior of finite energy planes implies that planes asymptotic to $\gamma^3_{1,1}$ intersect some orbits of the $S^1$ action exactly once. Then, arguing as in [11], section 3.3, standard transversality arguments imply that we can find families $J_t$, $1 \leq t \leq 2$, of $S^1$ invariant almost-complex structures connecting $J_1$ to an almost-complex structure $J_2$.
which is the standard product on \( X_1 \setminus (B^4(2.5) \times \mathbb{C}) \), such that the family \( J_t \) is regular for degree 2 planes asymptotic to \( \gamma_{1,1}^3 \) orbits. But such planes have index \(-1\) (for a fixed almost-complex structure) and by \( S^1 \) invariance arise in \( S^1 \)-parameter families. Thus there will be no such \( J_t \)-holomorphic planes for any \( 1 \leq t \leq 2 \).

Given such a family \( J_t \) we can approximate it by a family \( J'_t \) which is regular for \( N_t \). If \( J'_t \) is chosen sufficiently close to \( J_t \) then by compactness there can be no bubbling of \( J'_t \)-holomorphic planes asymptotic to \( \gamma_{1,1}^3 \) orbits either. This completes the proof.

### 3.2.3 The proof of Proposition 3.4

Here we prove Proposition 3.4, or rather reduce it to another existence result, namely Proposition 3.7. We recall that the proof in Lemma 2.8 that \( \mathcal{M}_0 \) was nonempty, coming from [9], Lemma 2.8, consisted of essentially writing down an explicit curve. Now, since the map \( g_0 \) defining \( f_0 \) is nonstandard, a new, indirect, method is required. On the other hand, as \( J_0 \) is invariant under rotations in the \( z_3 \) plane it will suffice as in Lemma 2.8 to prove the following in dimension 4.

**Proposition 3.6.** Let \( W \subset B(R) \subset \mathbb{CP}^2(R) \) be a smoothing of the image of a symplectically embedded polydisk \( P(1.5, 1) \) which extends to an ellipsoid \( E(B, A) \) with our parameters satisfying the bounds of Theorem 3.1. Then \( Y = \mathbb{CP}^2(R) \setminus W \) admits either a finite energy plane of degree 1 asymptotic to an orbit \( \gamma_{1,1} \) or a finite energy plane of degree 2 asymptotic to an orbit \( \gamma_{1,4} \).

We note that as we are now working entirely in the \((z_1, z_2)\) plane there is no need for superscripts on our hyperbolic orbits.

**Proof.** Rescaling slightly we may assume that the image of the polydisk \( P(1.5, 1) \) lies in the interior of \( W \). Using coordinates induced from the polydisk, let \( L = \{\pi|z_1|^2 = 1.5, \pi|z_2|^2 = 1\} \) be the Lagrangian torus in its distinguished boundary. Now, for any compatible almost-complex structure on \( \mathbb{CP}^2(R) \setminus L \), with a cylindrical end symplectomorphic to the complement of the zero-section in the unit cotangent bundle of \( L \), we can study finite energy curves asymptotic to geodesics on \( L \). The key proposition which implies Proposition 3.4 is the following.

**Proposition 3.7.** Any almost-complex structure on \( Y = \mathbb{CP}^2(R) \setminus W \) extends to an almost complex structure on \( \mathbb{CP}^2(R) \setminus L \) such that \( \mathbb{CP}^2(R) \setminus L \) admits either a degree 1 finite energy plane of area roughly \( R - 2.5 \) or a degree 2 finite energy plane of area roughly \( 2R - 5.5 \). In either case, the planes have deformation index 1.
In the statement of Proposition 3.7, \( Y \) is thought of as a subset of \( \mathbb{C}P^2(R) \) rather than as an almost-complex manifold with a cylindrical end as above. The proof of Proposition 3.7 will involve the use of finite energy foliations following [12] and we postpone this until section 3.3.

Given Proposition 3.7, to complete the proof of Proposition 3.4 we consider a sequence of finite energy planes from Proposition 3.7 with respect to a sequence of almost-complex structures \( J^N \) stretched to length \( N \) along \( \partial W \).

We divide our proof into cases. First suppose that for each \( N \) in a sequence \( N \to \infty \) Proposition 3.7 produces finite energy planes of degree 1. Then in the limit we have a finite energy curve \( u \) of degree 1 in \( Y \). Proposition 3.4 is completed in this case by the following lemma.

**Lemma 3.8.** The curve \( u \) has a single end asymptotic to a hyperbolic orbit of type \( \gamma_{1,1} \).

**Proof.** Suppose that \( u \) has \( e^k \) punctures asymptotic to multiples of \( \gamma^k \), now with \( k = 1, 2 \), the \( i \)th one asymptotic to \( r_i^k \gamma^k \), \( 1 \leq i \leq e^k \). Also, suppose \( u \) has \( h \) punctures asymptotic to hyperbolic orbits with the \( i \)th one asymptotic to \( \gamma_{m_i,n_i} \), \( 1 \leq i \leq h \). Then as in the area formula of Proposition 2.2, we have

\[
\text{area}(u) = R - \sum_{i=1}^{e^1} \frac{1.5r_1^1}{i} - \sum_{i=1}^{e^2} r_1^2 - \sum_{i=1}^{h} (1.5m_i + n_i)
\]

and since we are taking limits of curves of area \( R - 2.5 \) this implies

\[
2.5 \leq \sum_{i=1}^{e^1} 1.5r_1^1 + \sum_{i=1}^{e^2} r_1^2 + \sum_{i=1}^{h} (1.5m_i + n_i) \leq R.
\]

As \( R < 3 \) we see that if there is a hyperbolic limit then it is the only limit and is of type \( \gamma_{1,1} \).

For elliptic orbits, we first claim that the only possibility is two ends asymptotic to the elliptic orbits \( \gamma^1 \) and \( \gamma^2 \) respectively.

**Proof of claim.** For this, we note that as \( R < 3 \) the limits cannot cover \( \gamma^2 \) more than twice in total, and cannot cover \( \gamma^1 \) more than once. Furthermore, if we have ends covering \( \gamma^2 \) twice in total the lower bound above implies that there must still be another end, which then contradicts our upper bound.

To conclude, if one end is asymptotic to a cover of \( \gamma^1 \), it is the only end asymptotic to \( \gamma^1 \) and the other ends cover \( \gamma^2 \). Then to satisfy the area inequalities we see that there can only be one more end, which is asymptotic to \( \gamma^2 \). Similarly, if one end is asymptotic to a cover of \( \gamma^2 \) it covers \( \gamma^2 \) exactly once and to satisfy the inequalities we must have another end asymptotic to \( \gamma^1 \). This justifies our claim.

Finally, to exclude these elliptic orbits, recall that as we are taking limits of finite energy planes the limiting building has genus 0, and so only one of
the ends can be connected in \( W \) to a finite energy curve with an asymptotic limit on \( L \). The other end is then connected to components of area at least 1, and this is a contradiction as the original planes have area \( R - 2.5 < 1 \). 

In the second case of Proposition 3.4 we now suppose that for each \( N \) in a sequence \( N \to \infty \) Proposition 3.7 produces finite energy planes of degree 2. We will deduce from this the existence of a degree 2 plane in \( Y \) asymptotic to an orbit \( \gamma_{1,4} \).

By the same area argument as in Lemma 3.8 the components of the limit in \( Y \) can each have only a single negative end, and so the limit contains either a single plane of degree 2 or two planes of degree 1 in \( Y \). We deal with these possibilities separately, starting with the limit containing two planes.

Degree 1 planes are necessarily somewhere injective and have nonnegative index only if they are asymptotic to either an elliptic orbit \( r\gamma^1 \) or \( r\gamma^2 \) with \( r \leq 2 \) or to a hyperbolic orbit of type \( \gamma_{1,1} \). As \( R < 3 \), planes asymptotic to \( 2\gamma^1 \) are excluded, and so by Proposition 2.2 all possible planes have area at least \( R - 2.5 \). But then the sum of the areas is at least \( 2R - 5 \), giving a contradiction as the degree 2 planes from Proposition 3.7 have area \( 2R - 5.5 \).

With two planes now excluded, our limit has a single plane \( u \) of degree 2 in \( Y \). First suppose the plane is asymptotic to \( r\gamma^2 \). As its area lies between 0 and \( 2R - 5.5 \) we have

\[
5.5 \leq r \leq 2R
\]

which is a contradiction as \( 2R < 6 \).

Next suppose the plane is asymptotic to \( r\gamma^1 \). Then the area inequality is

\[
\frac{5.5}{1.5} \leq r \leq \frac{2R}{1.5}
\]

But as the lower bound is greater than 3, and as \( R < 3 \) the upper bound is less than 4, this is another contradiction.

Thus we can conclude that the plane \( u \) is asymptotic to a hyperbolic orbit, say \( \gamma_{m,n} \). A multiply covered curve can be ruled out using area considerations as above, more precisely, the underlying curve must have degree 1 and thus area at least \( R - 2.5 \) and so the plane itself would have area at least \( 2R - 5 \), which exceeds \( 2R - 5.5 \). Hence the plane is somewhere injective and so has nonnegative index. By the index formula in dimension 4 this means

\[
m + n \leq 5.
\]

We have \( \text{area}(u) = 2R - (1.5m + n) \), and as this is bounded by 0 and \( 2R - 5.5 \) we also have the inequality

\[
5.5 \leq 1.5m + n \leq 2R.
\]

Proposition 3.4 holds if we can show that \( m = 1 \), for then the lower bound in the area inequality implies that \( n \geq 4 \) and we get equality as the index inequality says \( m + n \leq 5 \). Arguing by contradiction then, suppose \( m \geq 2 \).
If \( m = 2 \) then the area inequality gives \( 2.5 \leq n \leq 2R - 3 < 3 \), a contradiction.

Similarly, if \( m \geq 4 \) then \( 1.5m + n \geq 6 > 2R \), another contradiction.

It remains to exclude the case \( m = 3 \). Here the area bounds give \( 1 \leq n \leq 2R - 4.5 < 2 \) and we deduce that \( n = 1 \) and the plane is asymptotic to an orbit \( \gamma_{3,1} \).

Recall that \( u \) is a component of a holomorphic building arising as the limit of a sequence of degree 2 planes in \( \mathbb{CP}^2(R) \setminus L \) of index 1. The sum of the virtual indices of the components of our limit, minus any matching conditions, must also be 1, and we will derive our contradiction from this. The limit has a single component \( u \) in \( Y \) but may also have components in symplectization \( S(\partial W) \) and in \( W \setminus L \).

As planar components in \( W \) have area at least 1, and as the total area of the building is \( 2R - 5.5 < 1 \), all components of our (genus 0) limiting building in the symplectization \( S(\partial W) \) and in \( W \setminus L \) are cylinders. Moreover there is a unique component in \( W \setminus L \) with its negative end on \( L \). Further, all asymptotic limits of all components are necessarily hyperbolic. Indeed, if a component has elliptic ends we can abstractly glue it to higher level components to produce a degree 2 plane with an elliptic limit. This contradicts our area inequalities as above.

Now, the curve \( u \) has virtual index 3. We will see in the next section (in particular a refined version Proposition 3.7 of Proposition 3.9) that the curve in \( W \setminus L \) has negative end on an indivisible Reeb orbit (that is, the Reeb orbit is not homologous to a multiple cover of a shorter orbit). As all components in \( W \setminus L \) are cylinders, this implies that that the curve is not multiply covered and so has nonnegative index \( I \geq 0 \).

A cylindrical component in \( S(\partial W) \) with positive end asymptotic to an orbit \( \gamma_{m,n} \) and negative end asymptotic to an orbit \( \gamma_{m',n'} \) has virtual index

\[
\text{index} = 2(m + n - m' - n') + 1.
\]

For nontrivial somewhere injective curves this is at least 1 by translation invariance of the symplectization, and we see that the index is exactly 1 for trivial cylinders. Hence \( m + n - m' - n' \geq 0 \) for somewhere injective cylinders. If the cylinder is an \( r \) times multiple cover of a somewhere injective cylinder as above then the virtual index is \( 2r(m + n - m' - n') + 1 \) which therefore again is at least 1.

Let \( x \) be the number of nontrivial symplectization components in the limiting building, so we have \( x + 1 \) matching asymptotic orbits. Then invariance of the index gives

\[
1 \geq 3 + x + I - (x + 1) \geq 2
\]

and this is our contradiction, completing the proof of Proposition 3.4. \( \square \)
3.3 Proof of Proposition 3.7.

We recall that we are studying a Lagrangian $L$ which is the distinguished boundary of a polydisk $P(1.5,1) \subset B^4(R) \subset \mathbb{C}P^2(R)$. Our assumption that the embedding of $P(1.5,1)$ extends to $E(B,A)$ implies that there is a ball $B^4(A) \subset E(B,A) \subset B^4(R)$ which we may assume is disjoint from $L$. (Indeed, reducing $B$ if necessary, we can take $L \subset \partial E(B,A)$.) However only the ball of capacity 1 lies in the interior of $W$. We will study symplectic forms $\omega_w$ and corresponding almost-complex structures on $Z = \mathbb{C}P^2(R) \# \mathbb{C}P^1(w) \setminus L$ given by blowing-up a ball of some capacity $w$ in the interval $[1, A]$, and will prove a refined version of Proposition 3.7. We will always assume that our almost-complex structures leave the exceptional divisor $E$ and the line at infinity $\mathbb{C}P^1(\infty)$ complex. In fact this is precisely the arrangement considered in [12] and we follow those methods closely. Using coordinates $(z_1, z_2)$ on the polydisk we can describe the homology class of an oriented geodesic on $L$ by a pair $(k, l) \in \mathbb{Z}^2$.

**Proposition 3.9.** Let $Z$ be as above with $w \in [1, A]$. Then either there exists an embedded degree 1 finite energy plane asymptotic to a geodesic in the class $(-1, -1)$ and disjoint from $E$, or there exists an embedded degree 2 finite energy plane asymptotic to a geodesic in the class $(-1, -4)$ and again disjoint from $E$.

By taking $w = 1$ and extending an almost-complex structure on $\mathbb{C}P^2(R) \setminus W$ to all of $Z$, this immediately implies Proposition 3.7 (after blowing the ball back down).

In the course of the proof of Proposition 3.9 we will occasionally need the index formula for curves in $Z$, which is as follows.

**Proposition 3.10.** (Hind-Lisi, [12], Proposition 3.1) Let $C$ be a curve in $X \setminus L$ of degree $d$, with $e$ intersections with $E$, and with $s$ negative ends asymptotic to geodesics in the classes $(k_i, l_i)$ respectively for $1 \leq i \leq s$.

The index of $C$ (as an unparametrized curve, allowing the asymptotic ends to move in the corresponding $S^1$ family of Reeb orbits) is given by

$$\text{index}(C) = s - 2 + 6d - 2e + 2 \sum_{i=1}^{s} (k_i + l_i).$$

**Proof.** (of Proposition 3.9.) We first record the following.

**Lemma 3.11.** Let $J_t$ be a 1-parameter family of almost-complex structures on $Z$ tamed by a family of symplectic forms $\omega_w(t)$.

The universal moduli spaces of degree 1 planes asymptotic to $(-1, -1)$ geodesics and disjoint from $E$, and the universal moduli space of degree 2 planes asymptotic to $(-1, -4)$ geodesics and disjoint from $E$, are both compact for generic 1-parameter families $J_t$. 

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Proof. We will deal only with the case of degree 2 planes. As $2R - 5.5 < 0.5$ a limiting building can contain no degree 0 curves, and similarly there is not enough area for bubbling of a sphere mapping into the exceptional divisor. Hence the only possible nontrivial degeneration of a degree 2 plane is into two planes of degree 1, which of course are necessarily somewhere injective. Suppose these are asymptotic to geodesics in the classes $(k_1, l_1)$ and $(k_2, l_2)$ respectively. By Proposition 3.10, the deformation index of such a plane is

$$\text{index} = 5 + 2(k_i + l_i) \geq -1$$

and so $k_i + l_i \geq -3$ for each $i$. However as the total homology class of geodesics is preserved in any limit we also have $(k_1 + l_1) + (k_2 + l_2) = -1 - 4 = -5$ and so we may assume that $k_1 + l_1 = -3$ and $k_2 + l_2 = -2$. Focusing on the first plane, it has area

$$\text{area} = R + 1.5k_1 + l_1 = R - 3 + 0.5k_1.$$

As $R < 3$ this is negative unless $k_1 \geq 1$, and in this case the area is at least $R - 2.5$. This implies that the area of the second plane is at most

$$(2R - 5.5) - (R - 2.5) = R - 3 < 0,$$

a contradiction. \[\square\]

Now, embedded finite energy planes with nonnegative index are automatically regular, see [27], Theorem 1 (or there is a summary of the facts we need at the start of section 2.3.2 in [9]). This implies that if we find planes in our moduli spaces for one $J_t$ then such planes exist for all $t$. In other words it suffices to construct our planes for a specific almost-complex structure, and we will work with one tamed by the symplectic form $\omega_A$.

An important tool for studying holomorphic curves in $\mathbb{C}P^2(R)\sharp\mathbb{C}P^1(A)$ is the following result on foliations of holomorphic curves.

**Proposition 3.12.** (see [10], Proposition 4.1) For a generic $J$, the manifold $\mathbb{C}P^2(R)\sharp\mathbb{C}P^1(A)$ is foliated by $J$ holomorphic spheres in the class $[\mathbb{C}P^1(\infty)] - [E]$.

We are interested in the behavior of this foliation as we stretch the neck along $L$ to produce a finite energy foliation of $Z$. This process was first described in [17], see also section 2 of [12] for the analysis in this particular case. Leaves of our finite energy foliation will contain closed curves and also broken curves, which we define to be two finite energy planes asymptotic to the same geodesic in $L$ with opposite orientations.

Before studying the finite energy foliation of $Z$, we note that there is at least one finite energy foliation, of $\mathbb{C}^2$, which can be described explicitly.
Proposition 3.13. Identify $\mathbb{C}^2$ with $\mathbb{C}P^1 \times \mathbb{C}P^1 \setminus (\mathbb{C}P^1 \times \infty \cup \infty \times \mathbb{C}P^1)$ and let $L \subset \mathbb{C}^2$ be a product Lagrangian $L_{a,b} = \{ \pi |z_1|^2 = a, \pi |z_2|^2 = b \}$. Let $J$ be an almost-complex structure on $M = \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus L_{a,b}$ with a cylindrical end along $L_{a,b}$ and coinciding with the standard product complex structure away from $L_{a,b}$. We may assume that $J$ is chosen such that the planes $\{ z_1 = \sqrt{\pi} e^{i\theta} \} \cap M$ remain complex.

Then $M$ has a finite energy foliation consisting of spheres in the class $[p \times \mathbb{C}P^1]$ and broken curves asymptotic to $(0,1)$ geodesics whose images coincide with $\{ z_1 = \sqrt{\pi} e^{i\theta} \} \cap M$ for some $\theta$. The leaves can be parameterized by their intersection with the line at infinity $\mathbb{C}P^1 \times \infty$.

This foliation is obtained by taking a limit of the standard foliation by spheres in the class $[p \times \mathbb{C}P^1]$ as we stretch the neck along $L_{a,b}$. The broken curves are as described since they can be arranged to remain complex throughout the stretching.

The main technical result in our study of finite energy curves will be the following.

Theorem 3.14. Either $Z$ admits an embedded degree $1$ finite energy plane disjoint from $E$ and asymptotic to a geodesic in the class $(-1,-1)$, or there exists a finite energy foliation of $Z$ whose leaves consist of holomorphic spheres in the class $[\mathbb{C}P^1(\infty)] - [E]$ together with an $S^1$-family of broken curves.

Each broken curve consists of a plane of degree $0$ and a plane of degree $1$. The planes of degree $0$ are disjoint from $E$ and asymptotic to geodesics in the class $(1,0)$ or $(0,1)$.

Proof. We need to analyze the possible holomorphic buildings in the limit of the foliation from Proposition 3.12 as we stretch the neck along $L$. Therefore the following lemma will be useful. We are assuming our almost-complex structure on $Z$ is regular.

Lemma 3.15. All embedded degree $0$ finite energy planes in $Z$ which intersect $E$ at most once have area strictly greater than $0.5$.

Proof. We argue by contradiction and suppose that such a plane $C$ exists with area at most $0.5$. As our almost-complex structure is regular $C$ has nonnegative index. Since the exceptional divisor $E$ has area $A > 1$, and $0.5$ is the minimal possible area of degree $0$ symplectic curves in $Z$ asymptotic to geodesics on $L$ and disjoint from $E$ (recall that $L$ is the distinguished boundary of a polydisk $P(1.5,1)$), we see that the moduli space of planes containing $C$ is compact under deformations of the almost-complex structure. Indeed, any degenerate limiting holomorphic building must contain either degree $0$ curves disjoint from $E$ or the exceptional sphere itself, and both are impossible as they have too much area. Combining with Wendl’s automatic regularity for embedded finite energy planes, [27], if we deform
the almost-complex structure to be a standard product, we can find a similar plane in $\mathbb{C}^2\#\mathbb{CP}^2 \setminus L_{1,5,1}$. Note that holomorphically we can identify the blow-up of a ball in $\mathbb{C}^2$ with the usual one-point blow-up $\mathbb{C}^2\#\mathbb{CP}^2$.

Now, in $\mathbb{C}^2\#\mathbb{CP}^2 \setminus L_{1,5,1}$ we have the standard finite energy foliation of Proposition 3.13 (with one curve now replaced by a cusp-curve containing the exceptional divisor). If our plane $C$ is asymptotic to a geodesic of type $(k,l)$ with $k < 0$ then the curve must intersect leaves of the standard foliation parameterized by points $z_1 \in \mathbb{CP}^1 \times \infty$ with $|z_1|^2 > 1.5$. By positivity of intersection our curve must then intersect all such leaves, and this is a contradiction since $C$ has degree 0.

The same argument in the second coordinate shows that $C$ is asymptotic to a geodesic of type $(k,l)$ with $k,l \geq 0$. As $1 < A < 1.5$, we see that the only possible degree 0 plane of area at most 0.5 would be asymptotic to a $(1,0)$ geodesic and intersecting $E$. But by Proposition 3.10, such curves have deformation index $-1$ and so can be excluded if we assume our almost-complex structure is regular.

Lemma 3.15 implies that if we blow-down our exceptional divisor $E$ (so that the area of any degree 0 curve is a multiple of 0.5) the resulting embedded degree 0 planes must actually all have area at least 1, and if the plane originally intersected $E$ then the blow-down must have area at least 2 (that is, it is a multiple of 0.5 and exceeds $A + 0.5$). Hence we have the following corollary to Lemma 3.15.

**Corollary 3.16.** As we stretch the neck along $L$, the limiting building of a sphere in the class $[\mathbb{CP}^1(\infty)] - [E]$ has at most three components in $Z$. If there are three components then the degree 0 components consist of a plane and a cylinder, one of which intersects $E$.

**Proof.** The limit has a single degree 1 component and as our curves have genus 0 at least one degree 0 component must be a plane. As we take a limit of embedded curves, these components are either embedded or are covers of embedded curves, and so the conclusion of Lemma 3.15 applies to any degree 0 planes.

Let us blow-down the component intersecting $E$. Then it has area at least $A$, or at least 2 if it is a degree 0 plane. Further, as they fit together to form a degree 1 cycle in $\mathbb{CP}^2(R)$, the sum of the areas of all components is $R$.

First suppose that a degree 0 planar component intersects $E$. Then after the blow-down it has area at least 2. The other degree 0 components each contribute area at least 0.5 and so as $R < 3$ there can be at most one more, and if there is another degree 0 component it has area 0.5 and by Lemma 3.15 is not a plane. Therefore the other degree 0 component is a cylinder with one end matching the degree 0 plane and the other end matching the degree 1 curve.
Next suppose that a non-planar degree 0 component intersects \( E \). Together with a planar degree 0 component these contribute area at least \( A + 1 \). Any other degree 0 component would contribute at least another 0.5 and this is impossible since \( A + 1.5 > \frac{11}{8} + 1.5 > R \) by conditions (i) and (ii). Again we can conclude that if there is more than one degree 0 component they are a cylinder and a plane.

Finally suppose that the degree 1 component intersects \( E \). Then it’s blow-down has area at least \( A \) and a degree 0 planar component contributes area at least 1. The inequality above implies that there can be no further components so in this case we have a broken curve consisting of two planes and the proof is complete.

It remains for us to check the homology classes of the asymptotic limits of the various components. For this, the following lemma will be applied.

**Lemma 3.17.** There are no broken curves in our finite energy foliation whose degree 0 component intersects \( E \) and is asymptotic to a geodesic in the class \((0, 2)\).

**Proof.** Suppose that such a degree 0 component \( C \) exists, asymptotic to the double cover, say \( 2\gamma \) of a geodesic \( \gamma \). By Proposition 3.10 the plane \( C \) has index 1 and by automatic regularity there will also exist such planes \( C' \) asymptotic to \( 2\gamma' \) for \( \gamma' \) close to \( \gamma \). As \( C \) forms part of a finite energy foliation we have \( C \cdot C' = 0 \).

Now, there also exist degree 0 planes asymptotic to a \((0, 1)\) geodesic and disjoint from \( E \). Indeed, the part \( \{ |z_1|^2 = 1.5 \} \subset \partial P(1.5, 1) \subset E(B, A) \) of the boundary of our polydisk is disjoint from the ball \( B^{1.5}(A) \subset E(B, A) \) and so we may assume the planes \( \{ z_1 = \sqrt{1.5} e^{i\theta} \} \) asymptotic to \( L \) remain complex throughout. So we may define \( D \) to be a double cover of such a disk asymptotic to \( 2\gamma \) and \( D' \) to be a double cover asymptotic to \( 2\gamma' \). We have \( D \cdot D' = 0 \).

Gluing these disks along their boundaries we can construct spheres \((-D) \cup C \) and \((-D') \cup C'\) homologous to \([-E]\) (since \( D \cdot E = 0 \) and \( C \cdot E = 1 \)). Thus we have

\[
-1 = (-D \cup C) \cdot (-D' \cup C') = D \cdot D' - D \cdot C' - C \cdot D' + C \cdot C' = -2D \cdot C'
\]

since the asymptotic behavior of finite energy curves implies \( D \cdot C' = C \cdot D' \). This gives a contradiction (to parity) as required.

We are now ready to analyze the limiting buildings and complete the proof of Theorem 3.14. First suppose that the limiting building has exactly two components in \( Z \).

**Two components in \( Z \).**

We divide this into two cases according to whether the degree 0 or the degree 1 component of our broken curve intersects \( E \). (Recall that since we
are taking limits of curves in the class \([\mathbb{C}P^1(\infty)] - [E]\) and \(E\) is complex, exactly one of the components will intersect \(E\).

First then suppose that the degree 0 component \(C\) of a broken curve \(C \cup C_1\) is disjoint from \(E\), and let the asymptotic geodesic of \(C\) be in the class \((k, l)\) (so the limiting geodesic of \(C_1\) lies in the class \((-k, -l))\). Then by Proposition 3.10 we have

\[
\text{index}(C) = -1 + 2(k + l); \quad \text{index}(C_1) = 5 - 2(k + l).
\]

\[
\text{area}(C) = 1.5k + l; \quad \text{area}(C_1) = R - A - 1.5k - l.
\]

The index formula for \(C\) implies that \(k + l \geq 1\). (Note here that if \(C\) were a multiple cover then this formula would in fact be true for the underlying curve, which means that \(C\) itself would be asymptotic to a geodesic with \(k + l \geq 2\). The component \(C_1\) has intersection number 1 with \(\mathbb{C}P^1(\infty)\) and so is somewhere injective.) Meanwhile the index formula for \(C_1\) implies \(k + l \leq 1\) and hence \(k + l = 1\).

We can now rewrite \(\text{area}(C) = 0.5k + 1\). By Lemma 3.15 this means \(0.5k + 1 > 0.5\) or \(k \geq 0\). On the other hand, we also have \(\text{area}(C_1) = R - A - 0.5k - 1 > 0\) and so \(0.5k < R - A - 1 < 1\) as \(R < 3\) and \(A > 1\).

Hence \(k \leq 1\) and our degree 0 curve is asymptotic to a \((1, 0)\) or a \((0, 1)\) geodesic.

In the second case suppose that the degree 0 component of our broken curve \(C \cup C_1\) intersects \(E\). Then we have

\[
\text{index}(C) = -3 + 2(k + l); \quad \text{index}(C_1) = 5 - 2(k + l).
\]

\[
\text{area}(C) = 1.5k + l - A; \quad \text{area}(C_1) = R - 1.5k - l.
\]

The index formula for \(C\) now gives \(k + l \geq 2\) and that for \(C_1\) gives \(k + l \leq 2\) and so \(k + l = 2\).

Now \(\text{area}(C) = 0.5k + 2 - A > 0.5\) by Lemma 3.15. Therefore \(0.5k > A - 1.5 > -1\) as \(A > 1\). Hence \(k \geq 0\). Next \(\text{area}(C_1) = R - 0.5k - 2 > 0\) and so \(0.5k < R - 2 < 1\) and \(k \leq 1\).

Putting this together, \(k = 0\) or \(k = 1\) and \(C\) is asymptotic to a geodesic in the class \((0, 2)\) or the class \((1, 1)\). As the first possibility has been excluded in Lemma 3.17, the degree 1 component \(C_1\) is asymptotic to a \((-1, -1)\) geodesic and Theorem 3.14 holds in this case.

**Three components in \(Z\).**

By Corollary 3.16 this scenario can occur in two ways, and in either case we will show that the degree 1 plane is asymptotic to a \((-1, -1)\) geodesic.

First suppose that the cylindrical component \(C\) intersects \(E\) and has ends asymptotic to geodesics in the classes \((k_1, l_1)\) and \((k_2, l_2)\). Let \(D_1\) denote the plane of degree 1, which we take to be asymptotic to a geodesic in the class \((-k_1, -l_1)\) and \(D_2\) denote the plane of degree 0, which is then asymptotic
to a geodesic in the class \((-k_2, -l_2)\). The index and area formulas are as follows.

\[
\text{index}(C) = -2 + 2(k_1 + l_1) + 2(k_2 + l_2); \quad \text{area}(C) = 1.5k_1 + l_1 + 1.5k_2 + l_2 - A.
\]
\[
\text{index}(D_1) = 5 - 2(k_1 + l_1); \quad \text{area}(D_1) = R - 1.5k_1 - l_1.
\]
\[
\text{index}(D_2) = -1 - 2(k_2 + l_2); \quad \text{area}(D_2) = -1.5k_2 - l_2.
\]

Regularity for \(D_1\) and \(D_2\) (or their underlying curves in the case of multiple covers) implies \(k_1 + l_1 \leq 2\) and \(k_2 + l_2 \leq -1\). This means that \(\text{index}(C) \leq 0\) and so for a regular almost-complex structure (even in the case that \(C\) were multiply covered) all inequalities must be equalities.

Using these equalities we find \(\text{area}(D_1) = R - 0.5k_1 - 2 > 0\) and so \(k_1 < 2(R - 2)\) which means \(k_1 \leq 1\). Similarly \(\text{area}(D_2) = -0.5k_2 + 1 \geq 1\) by Lemma 3.15 and so \(k_2 \leq 0\). Finally \(\text{area}(C) = 0.5(k_1 + k_2) + 1 - A > 0\) and so \(k_1 + k_2 \geq 2(A - 1) \geq \frac{3}{4}\) by condition (i). Hence \(k_1 + k_2 \geq 1\) and again all inequalities are equalities which implies \(k_1 = 1, k_2 = 0, l_1 = 1, l_2 = -1\) and \(D_1\) is asymptotic to a \((-1, -1)\) geodesic as claimed.

In the second case suppose that the cylindrical component \(C\) is disjoint from \(E\) and has ends asymptotic to geodesics in the classes \((k_1, l_1)\) and \((k_2, l_2)\). Let \(D_1\) denote the plane of degree 1, which we again take to be asymptotic to a geodesic in the class \((-k_1, -l_1)\) and \(D_2\) denote the plane of degree 0, which is then asymptotic to a geodesic in the class \((-k_2, -l_2)\) and now intersects \(E\). The index and area formulas are now as follows.

\[
\text{index}(C) = 2(k_1 + l_1) + 2(k_2 + l_2); \quad \text{area}(C) = 1.5k_1 + l_1 + 1.5k_2 + l_2.
\]
\[
\text{index}(D_1) = 5 - 2(k_1 + l_1); \quad \text{area}(D_1) = R - 1.5k_1 - l_1.
\]
\[
\text{index}(D_2) = -3 - 2(k_2 + l_2); \quad \text{area}(D_2) = -1.5k_2 - l_2 - A.
\]

Regularity for \(D_1\) and \(D_2\) implies \(k_1 + l_1 \leq 2\) and \(k_2 + l_2 \leq -2\). This means that \(\text{index}(C) \leq 0\) and so for a regular almost-complex structure all inequalities are equalities.

Using these, we find \(\text{area}(D_1) = R - 0.5k_1 - 2 > 0\) and so again \(k_1 \leq 1\). Similarly \(\text{area}(D_2) = -0.5k_2 + 2 - A \geq 0.5\) by Lemma 3.15 and so \(k_2 \leq 2(2 - A - 0.5) < 1\), by condition (i), that is \(k_2 \leq 0\). Finally \(\text{area}(C) = 0.5(k_1 + k_2) > 0\) and so \(k_1 + k_2 \geq 1\). So again all inequalities are equalities and we have \(k_1 = 1, k_2 = 0, l_1 = 1, l_2 = -2\) and \(D_1\) is asymptotic to a \((-1, -1)\) geodesic and the proof of Theorem 3.14 is complete.

\[\square\]

If degree 1 planes disjoint from \(E\) and asymptotic to \((-1, -1)\) geodesics exist then Proposition 3.9 is valid in this case.

If the degree 0 broken components are planes disjoint from \(E\) and asymptotic to \((1, 0)\) geodesics, then the degree 1 components are asymptotic to
(-1,0) geodesics and have area $R - A - 1.5$, recalling that this component must intersect an exceptional divisor of area $A$. But by conditions (i) and (ii) we have $R - A - 1.5 < 17/6 - 11/8 - 1.5 < 0$, a contradiction.

Thus we may assume that we have a finite energy foliation of $Z$ exactly as described in [12], that is, the broken curves consist of degree 0 planes asymptotic to geodesics in the class $(0,1)$ and degree 1 planes which intersect $E$ and are asymptotic to geodesics in class $(0,-1)$. These have area $R - 1 - A$.

It is useful to define the map $\pi : Z \to E$ given by projection along the leaves of the finite energy foliation, which all intersect $E$ in a single point (considering broken finite energy curves as representing a single leaf). Then the broken curves project onto a circle $\Gamma \subset E$.

The next step is to consider limits of high degree holomorphic spheres, whose existence is claimed by the following.

**Proposition 3.18.** (see [11] Proposition 2.2, [12], Proposition 2.1) Let $J$ be a regular almost complex structure on $\mathbb{C}P^2(R)\sharp\mathbb{C}P^1(A)$ (so all somewhere injective curves are regular).

Then, there is a co-meagre set $\mathcal{P} \subset X^{2d}$ consisting of $2d$ constraint points so that for each tuple of constraints $p_1, \ldots, p_{2d}$, there is a unique embedded holomorphic sphere $S$ in the class $d[\mathbb{C}P^1(\infty)] - (d - 1)[E]$ passing through the points.

We note that such curves $S$ have intersection number 1 with curves in the foliation of Proposition 3.12. As in [12], we fix $2d$ points on $L$ and take a limit of our high degree curves from Proposition 3.18 as the almost-complex structure is stretched along $L$. The limiting building contains a union $F$ of finite energy planes in $Z$. By positivity of intersection, if $p \in E \setminus \Gamma$ then $\pi^{-1}(p) \cap F$ consists of a single point corresponding to the unique intersection of the fiber curve through $p$ with $F$. As $F$ is a limit of embedded curves, its components are either embedded themselves or multiply covered, and this positivity of intersection excludes multiply covered components which do not cover fibers. In fact we have two possibilities for the curves $F$.

**Case 1.** $F$ consists of a number of curves covering leaves of the foliation together with a single curve $F_0$ asymptotic to geodesics of the form $(0, \pm l)$. Then $\pi(F_0)$ is equal to $E \setminus \{q_i\}$ where the $\{q_i\}$ correspond to the broken leaves asymptotic to the limiting geodesics.

**Case 2.** $F$ consists of a number of curves covering leaves of the foliation together with two curves $F_0$ and $F_1$ asymptotic to geodesics of the form $(1, l_0)$ and $(-1, l_1)$ respectively. In this case $\pi(F_0) \cup \pi(F_1) = E \setminus \Gamma$.

Case 1 was dealt with in [12]. It turns out that the remaining components of $E$ consist of $2d$ planes covering components of broken curves. These components have a total area of at least $2d$, while closed curves in the class $d[\mathbb{C}P^1(\infty)] - (d - 1)[E]$ have area $dR - (d - 1)A$. Therefore we must have $dR - (d - 1)A \geq 2d$ and hence $R \geq 3 - 1/d$ as $A > 1$. As $R < 3$ this is a contradiction if $d$ is sufficiently large.
Eliminating this case, we have now reduced to Case 2 provided \(d\) is chosen large.

By Proposition 3.10 we see that \(F_0\) and \(F_1\) have odd deformation indices. Hence, as they cannot be multiple covers (as they are planes asymptotic to nondivisible geodesics in the classes \((1, l_0)\) or \((-1, l_1)\)) for a generic almost-complex structure we have \(\text{index}(F_0) \geq 1\) and \(\text{index}(F_1) \geq 1\). In fact, to simplify our calculations, we can actually prove the following.

**Lemma 3.19.** \(\text{index}(F_0) = \text{index}(F_1) = 1\).

_Proof._ Besides \(F_0\) and \(F_1\), we claim that in fact all components of \(F\) in \(Z\) have virtual index at least equal to their number of negative ends. For example, if a component \(C\) with \(s\) ends is an \(r\)-times multiple cover of a broken component of degree 1 (which we recall intersects \(E\) and is asymptotic to a \((0, -1)\) geodesic), then Proposition 3.10 gives

\[
\text{index}(C) = s - 2 + 6r - 2r - 2r = s - 2 + 2r \geq s.
\]

Next we recall that our components arise as a limit of holomorphic spheres of constrained index 0 (that is, the moduli space of holomorphic spheres passing through the constraint points has virtual index 0). Therefore the sum of the constrained indices of the components of our holomorphic building, minus the number of matched asymptotic ends, should also be 0. The virtual index of components in \(T^*L\) is nondecreasing under multiple covers (see again [12], Proposition 3.2 and 3.3) and thus the constrained index can be assumed to be nonnegative. Hence our sum can be 0 only if all components in \(Z\) have index exactly equal to their number of ends. In particular \(F_0\) and \(F_1\) have index 1. \(\square\)

Suppose \(F_1\) has degree \(d_1\) and intersection number \(e_1\) with \(E\). Then computing using Proposition 3.10 we find

\[
1 = \text{index}(F_1) = -1 + 6d_1 - 2e_1 - 2 + 2l_1
\]

and so \(3d_1 - e_1 + l_1 = 2\). Therefore

\[
\text{area}(F_1) = Rd_1 - Ae_1 - 1.5 + l_1
\]

\[
= (3d_1 - e_1 + l_1) - (3 - R)d_1 - (A - 1)e_1 - 1.5
\]

\[
= 2 - (3 - R)d_1 - (A - 1)e_1 - 1.5.
\]

Suppose that \(d_1 \geq 3\). Then

\[
\text{area}(F_1) \leq 2 - 9 + 3R - 1.5
\]

which implies \(R > \frac{17}{6}\), in contradiction to condition (\(ii\)).
Next suppose that $d_1 = 1$ and $e_1 \geq 1$. Then
\[
\text{area}(F_1) \leq 2 - 3 + R - A + 1 - 1.5 = R - A - 1.5
\]
which is negative by the bounds on $A$ and $R$ in conditions (i) and (ii), another contradiction.

If $d_1 = 1$ and $e_1 = 0$ then by the index formula $l_1 = -1$ and we have a curve of degree 1 as required for Proposition 3.9. Thus we reduce to the case when $d_1 = 2$. Now if $e_1 \geq 1$ we have
\[
\text{area}(F_1) \leq 2 - 6 + 2R - A + 1 - 1.5 = 2R - A - 4.5.
\]
This means $2R > A + 4.5 > \frac{47}{8}$ by condition (i). As $\frac{47}{8} > \frac{17}{3}$ this contradicts the upper bound on $R$ in condition (ii).

Hence the curve $F_1$ may be assumed to have degree 2 and avoid $E$. Then by the index formula $l_1 = -4$ and we have a degree 2 curve as required, completing the proof of Proposition 3.9.

References


