LAGRANGIAN ISOTOPIES AND STABILIZED SYMPLECTIC EMBEDDINGS

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Abstract. We investigate relations between isotopies of Lagrangian tori and stabilized symplectic embeddings. An analogue of a conjecture of McDuff about stabilized embeddings holds for a relation on 4 dimensional ellipsoids involving Lagrangian isotopies, at least if we assume certain Markov triples generate a complete list of the relevant Hamiltonian isotopy classes of monotone Lagrangian tori in the ball. This relation interpolates between the notions of symplectic and stabilized embeddings.

1. Introduction

The symplectic embedding problem is widely open in dimension greater than 4, although there has been progress on the stabilized problem, which studies embeddings of products of 4 dimensional domains with Euclidean space. Adding the Euclidean factor can be shown to introduce more flexibility, and it is natural to ask if the additional flexibility entirely results from known constructions. A conjecture of McDuff [20], see also [17], formalizes this, where we use notation introduced below.

Stabilized Embedding Conjecture. Let $x > \tau^4$ and $n \geq 3$. There exists a symplectic embedding $E(1,x) \times \mathbb{R}^{2n-4} \hookrightarrow B^4(c) \times \mathbb{R}^{2n-4}$ if and only if $c \geq \frac{3x}{x+1}$.

In Definitions 2.1, 2.4, 3.2, 3.4 we give four relations on 4 dimensional subsets of $\mathbb{R}^4$, which we usually assume to be toric, that is, invariant under the diagonal torus action. Roughly speaking, we write $U \hookrightarrow V$ if there exists a Hamiltonian diffeomorphism of $\mathbb{R}^4$ mapping U into V. Write $U \hookrightarrow_S V$ if there exists a so called stabilized embedding $U \times \mathbb{R}^{2n-4} \hookrightarrow V \times \mathbb{R}^{2n-4}$ for some $n \geq 3$.

Denote by $L(r,s)$ the Lagrangian torus equal to the product of a circle of area r in the first $\mathbb{R}^2$ factor and a circle of area s in the second $\mathbb{R}^2$ factor. We write $U \hookrightarrow_L V$ if there exists a smooth family

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of Hamiltonian diffeomorphisms $\phi_{r,s}$ with $\phi_{r,s}(L(r,s)) \subset V$, where the family is parameterized by all (r,s) for which $L(r,s) \subset U$. A slight refinement of this, denoted $U \hookrightarrow_P V$, which requires the images of the $L(r,r)$ to bound cubes, will also be introduced.

The following holds, where the first two statements are immediate consequences of the definitions, and the third statement is Proposition 4.1.

Suppose U is toric and convex, then

$$(U \hookrightarrow V) \implies (U \hookrightarrow_P V).$$

$$(U \hookrightarrow_P V) \implies (U \hookrightarrow_L V).$$

If U is toric then

$$(U \hookrightarrow_L V) \implies (U \hookrightarrow_S V).$$

We collect evidence in Theorems 2.5, 2.6, 2.7 and 3.3 suggesting the relations $U \hookrightarrow_L V$ and $U \hookrightarrow_S V$ are very close, at least when U is an ellipsoid. We also outline an argument that either there are Hamiltonian isotopy classes of monotone Lagrangian tori in the ball which lie in the boundary of cubes and fall outside of the (infinitely many) classes already discovered, or otherwise if $x \geq \tau^4$ then $E(1,x) \hookrightarrow_P B^4(c)$ if and only if $c \geq \frac{3x}{x+1}$.

In Section 2 we briefly introduce symplectic embeddings, then discuss stabilized embeddings and some results in the ellipsoid case. Section 3 talks about Lagrangian isotopies and some results there, highlighting the similarity between the relations $U \hookrightarrow_L V$ and $U \hookrightarrow_S V$. We also discuss the version of McDuff's question for Lagrangian isotopies. In section 4 we prove Proposition 4.1.

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2. The stabilized symplectic embedding problem

A fundamental problem in quantitative symplectic geometry is the embedding problem, one version of which we can state as follows. Let $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ be the standard symplectic vector space with symplectic form $\omega = \frac{i}{2} \sum dz_k \wedge d\overline{z_k}$. A Hamiltonian diffeomorphism is the time 1 flow of a possibly time dependent vector field $X_{H_t} = i \nabla H_t$, where $H : \mathbb{C}^n \times \mathbb{R} \to \mathbb{R}$ is smooth and $H_t(z) = H(z,t)$. The Hamiltonian diffeomorphisms are the diffeomorphisms of $\mathbb{C}^n$ which preserve ω.

Embedding Problem. Given $U, V \subset \mathbb{C}^n$, does there exist a Hamiltonian diffeomorphism ϕ with $\phi(U) \subset V$?

To simplify the discussion we introduce the following notation.

Definition 2.1. *Write $U \hookrightarrow V$ if, for all compact subsets $K \subset U$, there exists a Hamiltonian diffeomorphism ϕ with $\phi(K) \subset V$.*

The relation $U \hookrightarrow V$ is weaker than than the existence of an actual embedding of U into V, but the two notions coincide in many interesting situations, see [19], [22].

By Liouville's Theorem, existence of a symplectic embedding $U \hookrightarrow V$ implies $\mathrm{vol}(U) \leq \mathrm{vol}(V)$. The next embedding obstruction comes from Gromov's nonsqueezing theorem.

We fix some notation. The moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ is given by $\mu(z_1, \ldots, z_n) = (\pi|z_1|^2, \ldots, \pi|z_n|^2)$. A toric domain $X \subset \mathbb{C}^n$ is a subset invariant under the diagonal torus action, or equivalently a domain of the form $X = X_\Omega := \mu^{-1}\Omega$ for $\Omega \subset \mathbb{R}^n_{\geq 0}$.

Let

$$e(a_1, \ldots, a_n) = \{\sum \frac{x_k}{a_k} < 1\} \subset \mathbb{R}^n_{\geq 0}$$

and

$$p(a_1, \ldots, a_n) = \{x_k < a_k \, \forall k\} \subset \mathbb{R}^n_{\geq 0}.$$

It is also convenient to set $b(a) = e(a, \ldots, a)$ and $z(a) = e(a, \infty, \ldots, \infty)$. Given this we can define symplectic ellipsoids and polydisks by

$$E(a_1, \ldots, a_n) = \mu^{-1}e(a_1, \ldots, a_n), \; P(a_1, \ldots, a_n) = \mu^{-1}p(a_1, \ldots, a_n).$$

When describing ellipsoids and polydisks we will always assume $a_1 \leq \cdots \leq a_n$. The ball of capacity a is $B^{2n}(a) = \mu^{-1}b(a)$ and the cylinder of capacity a is $Z^{2n}(a) = \mu^{-1}z(a)$.

Gromov's nonsqueezing theorem is the following.

Theorem 2.2 ([9]). *$B^{2n}(a) \hookrightarrow Z^{2n}(c)$ if and only if $a \leq c$.*

There has been much work studying symplectic embeddings since Gromov's result, but the Embedding Problem remains broadly open, even for toric domains, and even in dimension 4. Nevertheless symplectic capacities give obstructions to embeddings, and some very precise results have been obtained, especially in dimension 4 as a consequence of Embedded Contact Homology (ECH), see [16]. The ECH capacities associate a sequence of nonnegative real numbers $c_0(U), c_1(U), \ldots$ to a $U \subset \mathbb{C}^2$, and turn out to give a complete set of obstructions for embedding ellipsoids into ellipsoids [18], and more generally concave toric domains into convex toric domains, see [3]. It remains a difficult problem to work out exactly when one given domain embeds in another, but the solution for 4 dimensional ellipsoids into balls was worked out by

McDuff and Schlenk even before the development of ECH capacities, although still using holomorphic curves.

Let $\{f_n\}_{n\geq 0}$ be given by $f_0 = 1$ and then follow the odd index terms in the Fibonacci sequence. This is the sequence beginning $1, 1, 2, 5, 13, 34, \ldots$. Then let $a_n = (\frac{f_{n+1}}{f_n})^2$ and $b_n = \frac{f_{n+2}}{f_n}$ for $n \geq 0$. We have $1 = a_0 < b_0 < a_1 < b_1 < a_2 < \ldots$ and $\lim a_n = \lim b_n = \tau^4$ where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Theorem 2.3 ([21])**.**

(i) *Suppose* $a_n \leq x \leq b_n$. *Then* $E(1, x) \hookrightarrow B^4(c)$ *if and only if* $c \geq \frac{x}{\sqrt{a_n}}$.

(ii) *Suppose* $b_n \leq x \leq a_{n+1}$. *Then* $E(1, x) \hookrightarrow B^4(c)$ *if and only if* $c \geq \sqrt{a_{n+1}}$.

In other words a 'Fibonacci staircase' appears in the solution of the embedding problem. We note that $\frac{b_n}{\sqrt{a_n}} = \frac{f_{n+2}}{f_{n+1}} = \frac{3b_n}{b_n+1}$. Hence, when $x < \tau^4$ the graph of minimal ball sizes is piecewise linear with corners alternately sitting on the graphs of $y = \sqrt{x}$ and $y = \frac{3x}{x+1}$. When $x > \tau^4$ we have $\frac{3x}{x+1} < \sqrt{x}$, the volume constraint, and so there are never embeddings $E(1, x) \hookrightarrow B^4(\frac{3x}{x+1})$.

Moving to higher dimensions, we have the general form of the ellipsoid embedding problem.

Ellipsoid Embedding Problem. Given $a_1, \ldots, a_n, b_1, \ldots, b_n$, does there exist a symplectic embedding $E(a_1, \ldots, a_n) \hookrightarrow E(b_1, \ldots, b_n)$?

This is widely open, and there are no conjectures of a complete set of invariants. Indeed, even if we fix $n = 3$ and $b_1 = b_2 = b_3$, that is, study ellipsoid embeddings into the 6 dimensional ball, there is much we do not know, see [1], Figure 1. Note that by Liouville's Theorem existence of an embedding implies $\Pi a_k \leq \Pi b_k$, and, recalling we always list the factors in nondecreasing order, Gromov's theorem implies $a_1 \leq b_1$. In the other direction Guth [10] established the existence of constants $C(n)$ such that $E(a_1, \ldots, a_n) \hookrightarrow C(n) \cdot E(b_1, \ldots, b_n)$ whenever $a_1 \leq b_1$ and $\Pi a_k \leq \Pi b_k$.

Regarding obstructions, most results in higher dimension address the Stabilized Embedding Problem.

Stabilized Embedding Problem. Let $n \geq 3$ and $U, V \subset \mathbb{C}^2$. Does there exist a symplectic embedding $U \times \mathbb{C}^{n-2} \hookrightarrow V \times \mathbb{C}^{n-2}$?

When U and V are ellipsoids, this is the special case of the Ellipsoid Embedding Problem given by setting $a_3 = \cdots = a_n = \infty$, $b_3 = \cdots =$

$b_n = \infty$. The problem remains open in general, although progress has been made.

Again to simplify the discussion we introduce the following.

Definition 2.4. *Write $U \hookrightarrow_S V$ if there exists $n \geq 3$ such that, for all compact subsets $K \subset U \times \mathbb{C}^{n-2}$, there exists a Hamiltonian diffeomorphism ϕ with $\phi(K) \subset V \times \mathbb{C}^{n-2}$.*

We note that a symplectic embedding $U \hookrightarrow V$ is a sufficient condition for a stabilized embedding $U \hookrightarrow_S V$, but since the work of Guth has been known not to be necessary. There are no known examples where there exist embeddings $U \times \mathbb{C}^{n-2} \hookrightarrow V \times \mathbb{C}^{n-2}$ for some $n \geq 3$ but not for all $n \geq 3$.

We close this section with some results, covering ellipsoid embeddings into balls, ellipsoids and polydisks.

Theorem 2.5 ([12], [4], [20], [5])**.**

(i) *Suppose $x \leq \tau^4$. There exists a stabilized embedding $E(1,x) \hookrightarrow_S B^4(c)$ if and only if there exists a 4 dimensional embedding $E(1,x) \hookrightarrow B^4(c)$.*

(ii) *Suppose $x > \tau^4$. There exists a stabilized embedding $E(1,x) \hookrightarrow_S B^4(\frac{3x}{x+1})$. Moreover, there exist sequences $x_n \to \infty$ and $y_n \to (\tau^4)^+$ where these embeddings are optimal, that is, if $x = x_n$ or $x = y_n$ and $c < \frac{3x}{x+1}$ then there do not exist symplectic embeddings $E(1,x) \hookrightarrow_S B^4(c)$.*

The first statement above says that the Fibonacci staircase also exists in the stabilized case. As $\frac{3x}{x+1} < \sqrt{x}$ when $x > \tau^4$, the second statement implies that in this range there is strictly more flexibility in the stabilized case. This statement also led McDuff to conjecture the embeddings $E(1,x) \times \mathbb{C}^{n-2} \hookrightarrow B^4(\frac{3x}{x+1}) \times \mathbb{C}^{n-2}$ are always optimal when $x > \tau^4$. Much more evidence for the Stablilized Embedding Conjecture comes from Siegel's work, see [24].

Theorem 2.6 ([6])**.** *Fix $k \in \mathbb{N}_{\geq 2}$ and $b = ka$. Let $x \in \mathbb{N}$ have the opposite parity to k and satisfy $x \geq k+1$. There exists a stabilized symplectic embedding*

$$E(1,x) \hookrightarrow_S E(a,b)$$

if and only if $a \geq \frac{2x}{x+k-1}$.

Theorem 2.7 ([6])**.** *Let x be an odd integer with $x \geq 2\frac{b}{a} - 1$. There exists a stabilized symplectic embedding*

$$E(1,x) \hookrightarrow_S P(a,b)$$

if and only if $a \geq \frac{2x}{x+2\frac{b}{a}-1}$.

The upper bounds in these theorems come from explicit constructions, roughly following [11]. The lower bounds come from holomorphic curves in ellipsoid cobordisms, and rely specifically on work of Siegel, [24]. More is known about such curves in dimension 4, where for example the techniques of ECH are available, but a key observation is that in favorable circumstances counts of genus 0 curves with one negative end in a 4 dimensional ellipsoid cobordism $V \setminus U$ coincide with counts of analogous moduli spaces in $(V \times \mathbb{C}^{n-2}) \setminus (U \times \mathbb{C}^{n-2})$.

3. Lagrangian isotopies

As well as for open sets, one can also study the Embedding Problem in the case when U is a Lagrangian torus $L(r,s)$. Here we concentrate on dimension 4 and define

$$L(r,s) = \{\pi|z_1|^2 = r,\, \pi|z_2|^2 = s\}.$$

Solutions when the target V is a ball, a polydisk, or an integral ellipsoid were found in [13] and [14] respectively.

One way of stating these results is as a computation of the shape invariant. Given $U \subset \mathbb{C}^2$ we define the (Hamiltonian) shape as

$$\mathrm{Sh}_H(U) = \{(r,s)\,|\,0 < r \leq s,\, L(r,s) \hookrightarrow U\} \subset \mathbb{R}^2_{>0}.$$

In the case when U is toric we have $\mu(U) \cap \{0 < r \leq s\} \subset \mathrm{Sh}_H(U)$ although we do not have equality.

Theorem 3.1 ([13, 14])**.**

(i)

$$\mathrm{Sh}_H(B^4(R)) = \left\{(r,s) \in \mathbb{R}^2_{>0} \,\middle|\, r+s < R \text{ or } r < \frac{R}{3}\right\} \cap \{r \leq s\}.$$

(ii) *Suppose* $\frac{b}{a} \in \mathbb{N}_{\geq 2}$,

$$\mathrm{Sh}_H(E(a,b)) = \left\{(r,s) \in \mathbb{R}^2_{>0} \,\middle|\, \frac{r}{a} + \frac{s}{b} < 1 \text{ or } r < \frac{a}{2}\right\} \cap \{r \leq s\}.$$

(iii)

$$\mathrm{Sh}_H(P(c,d)) = \left\{(r,s) \in \mathbb{R}^2_{>0} \,\middle|\, \begin{array}{c} r < c \\ s < d \end{array} \text{ or } r < \frac{c}{2}\right\} \cap \{r \leq s\}.$$

Clearly, if we have a symplectic embedding $U \hookrightarrow V$ then $\mathrm{Sh}_H(U) \subset \mathrm{Sh}_H(V)$ and so the shape gives a sort of set valued symplectic capacity obstructing embeddings. However it is far from a complete invariant and following [15] we can try to get finer invariants by studying not just the set of Lagrangians in $U \subset \mathbb{C}^2$, but the topology of the space of these Lagrangians (with the natural topology of smooth convergence). Similar considerations also appear in the article of Shelukhin, Tonkonog and Vianna [23]. To this end, we make the following definition.

Definition 3.2. *Write $U \hookrightarrow_L V$ if, for all compact $\mathcal{K} \subset \mu(U)$, there exists a smooth family of Hamiltonian diffeomorphisms $\phi_{r,s}$ for $(r,s) \in \mathcal{K}$, such that $\phi_{r,s}(L(r,s)) \subset V$.*

In this definition, we are assuming not just that the time 1 flows, that is, the Hamiltonian diffeomorphisms, vary smoothly, but also the flows over all t. This means we may assume the $\phi_{r,s}$ are generated by a smooth family of functions $H_{r,s} : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}$. We also allow $r = 0$ or $s = 0$; in this case $L(r,s)$ is a circle or a point.

We have the following.

Isotopy obstruction. Suppose $U = X_\Omega$ is toric and $U \hookrightarrow V$. Then $U \hookrightarrow_L V$.

The following was shown in [15].

Theorem 3.3 ([15])**.** *Let $\Omega = e(1,x)$.*

(i) *Suppose $x > c$. There exists a family of Hamiltonian embeddings $\phi_{r,s} : L(r,s) \hookrightarrow B^4(c)$ for $(r,s) \in \Omega$ with $\phi_{\frac{x}{x+1},\frac{x}{x+1}}$ the inclusion, if and only if $c \geq \frac{3x}{x+1}$;*

(ii) *Let $b = ka$ with $k \in \mathbb{N}_{\geq 2}$ and suppose $x > b$. There exist a family of Hamiltonian embeddings $\phi_{r,s} : L(r,s) \hookrightarrow E(a,b)$ for $(r,s) \in \Omega$ with $\phi_{\frac{x}{x+k-1},\frac{(k-1)x}{x+k-1}}$ the inclusion, if and only if $a \geq \frac{2x}{x+k-1}$;*

(iii) *Suppose $x > b$ and $a > 1$. Set $k = \frac{b}{a} \in \mathbb{R}$. There exist a family of Hamiltonian embeddings $\phi_{r,s} : L(r,s) \hookrightarrow P(a,b)$ for $(r,s) \in \Omega$ with $\phi_{\frac{x}{x+2k-1},\frac{(2k-1)x}{x+2k-1}}$ the inclusion, if and only if $a \geq \frac{2x}{x+2k-1}$.*

We note that the bounds here are identical to those in Theorem 2.5, 2.6 and 2.7. Theorem 3.3 has the advantage that we do not make assumptions about x being an integer, but replaces this with a hypothesis about unknottedness of a Lagrangian torus. By unknottedness, for example in (i), we mean the requirement that $\phi_{\frac{x}{x+1},\frac{x}{x+1}}$ is the inclusion could be replaced by asking that the image of $L(\frac{x}{x+1},\frac{x}{x+1})$ be Hamiltonian isotopic to $L(\frac{x}{x+1},\frac{x}{x+1})$ itself, where the Hamiltonian isotopy has support in the target space $B^4(c)$. As a result of this hypothesis we do not have the same sharp estimates for the general Lagrangian isotopy relation $U \hookrightarrow_L V$.

In the remainder of this section we consider a connection between the unknottedness condition in Theorem 3.3 and the hypothesis $x > \tau^4$

in the Stabilized Embedding Conjecture. We concentrate on case (i), where $V = B^4(c)$. By Theorem 2.3, when $x < \tau^4$ we know there are symplectic embeddings $E(1,x) \hookrightarrow B^4(c)$ with $c < \frac{3x}{x+1}$, indeed this holds whenever $x \neq b_n$. Then Theorem 3.3 implies that the restrictions of these embeddings to the tori $L(\frac{x}{x+1}, \frac{x}{x+1}) \subset \partial E(1,x)$ must be knotted in $B^4(c)$.

The tori arising from ellipsoid embeddings are not completely arbitrary, in particular they all lie in the boundary of polydisks, namely the image of the embedding restricted to $P(r,s)$. Hence it is natural to consider a stronger relation than $U \hookrightarrow_L V$, but which is still implied by $U \hookrightarrow V$ when U is convex. Asking all images of Lagrangian tori to bound polydisks is very restrictive, and eliminates much of the flexibility present for stabilized embeddings, but it is reasonable to ask the images of the monotone tori to bound cubes.

Definition 3.4. *Write $U \hookrightarrow_P V$ if, for all compact $\mathcal{K} \subset \mu(U)$, there exists a smooth family of Hamiltonian diffeomorphisms $\phi_{r,s}$ for $(r,s) \in \mathcal{K}$, such that $\phi_{r,s}(L(r,s)) \subset V$. Further, if $r = s$, then $\phi_{r,r}(P(r,r)) \subset V$.*

It has been known for a while that there are actually infinitely many Hamiltonian isotopy classes of monotone tori in $\mathbb{C}P^2$. This was conjectured by Galkin and Usnich, [8], and proven rigorously in the symplectic category by Galkin and Mikhalkin [7] and Vianna [25]. We recall that a Lagrangian torus L is monotone if the relative area class $[\omega] \in H^2(\mathbb{C}P^2, L)$ is proportional to the relative first Chern class $c_1(T\mathbb{C}P^2, TL)$, where the Chern class is defined with respect to an almost complex structure compatible with the symplectic form, so then TL defines a totally real subbundle. In the case when $\mathbb{C}P^2$ is scaled so that lines have area 3, the monotone condition means that Maslov 2 disks with boundary on our Lagrangian have area 1.

The monotone Lagrangian tori come from monotone torus orbits in weighted projective spaces $\mathbb{C}P^2(a^2, b^2, c^2)$, where (a,b,c) is a Markov triple, that is $a^2+b^2+c^2 = 3abc$. Indeed, these are symplectic toric orbifolds which arise from algebraic degenerations of $\mathbb{C}P^2$, and monotone tori in the orbifold correspond to tori in $\mathbb{C}P^2$ by parallel transport, see for example [7]. However not all of these tori bound cubes, and in fact the only examples come from Markov triples $(1, f_n, f_{n+1})$, where the f_n are the odd index Fibonacci numbers we saw in Theorem 2.3. We denote a representative of the corresponding Hamiltonian isotopy class by L_n. We can, and do, choose L_n such that it bounds a cube lying in the affine part of $\mathbb{C}P^2$, a symplectic ball.

Here is a very brief outline of the obstruction part of the proof of Theorem 3.3 (i). Suppose we have a 1 parameter family of embeddings $L(r,s) \to L_t \subset B^4(c)$ where $r = \frac{x(1-t)}{x+1}$, $s = \frac{x^2t+x}{x+1}$, and $0 \le t \le 1$, coming from $(r,s) \in \partial e(1,x)$. These embeddings give a natural basis for $H_1(L_t, \mathbb{Z})$ where the classes $(1,0)$ and $(0,1)$ bound Maslov 2 disks of area r and s respectively. Arguing by contradiction, if $c < \frac{3x}{x+1}$ our ball $B^4(c)$ can be included in a ball $B^4(\frac{3x}{x+1})$, which itself can be thought of as the affine part of a copy of $\mathbb{C}P^2$, in which lines have area $\frac{3x}{x+1}$ and L_0 becomes monotone. The assumption that L_0 is standard implies that the class $(-1,-1)$ also bounds a Maslov 2 disk, intersecting the line at infinity once. The proof analyses a moduli space of such disks with boundary on the L_t. We note that the area of a disk in the corresponding relative homology class with boundary on L_t is

$$\frac{3x}{x+1} - r - s = \frac{3x}{x+1} - \frac{x(1-t)}{x+1} - \frac{x^2t+x}{x+1}$$

which, at least if we assume $x > 2$, is negative when t approaches 1. We conclude that families of Maslov 2 disks in this relative homology class must degenerate before $t = 1$, and a bubbling analysis gives the contradiction.

Now suppose we have the same 1 parameter family of embeddings $L(r,s) \to L_t \subset B^4(c)$, but with L_0 Hamiltonian isotopic to the torus L_n coming from the Markov triple $(1, f_n, f_{n+1})$. Assuming $c < \frac{3x}{x+1}$, as above we can include $B^4(c)$ in the affine part of a copy of $\mathbb{C}P^2$ with lines of area $\frac{3x}{x+1}$. The moment polytope of the corresponding weighted projective space shows now that either the class $(-f_{n+1}^2, -f_n^2)$ or the class $(-f_n^2, -f_{n+1}^2)$ bounds a Maslov 2 disk in our copy of $\mathbb{C}P^2$. The disk intersects the line at infinity $f_n f_{n+1}$ times. The ambiguity here is coming from the fact that we are not assuming L_0 is isotopic to L_n via a Hamiltonian diffeomorphism which acts in a particular way on homology.

In the first case, the area of a disk in the corresponding class with boundary on L_t will be

$$f_n f_{n+1}\frac{3x}{x+1} - f_{n+1}^2\frac{x(1-t)}{x+1} - f_n^2\frac{x^2t+x}{x+1}.$$

This expression is negative when t is close to 1 unless

$$x \le \frac{3f_{n+1}}{f_n} - 1.$$

In the second case this upper bound could be replaced by $\frac{3f_n}{f_{n+1}} - 1$. The sequence $\frac{f_{n+1}}{f_n}$ is increasing and converges to τ^2 as $n \to \infty$, and we note that $3\tau^2 - 1 = \tau^4$. Hence we arrive at the following extension of Theorem 3.3 (i).

Theorem 3.5. *Let $x \geq \tau^4$ and $\Omega = e(1,x)$. Suppose there exists a family of Hamiltonian embeddings $\phi_{r,s} : L(r,s) \hookrightarrow B^4(c)$ for $(r,s) \in \Omega$ with $\phi_{\frac{x}{x+1},\frac{x}{x+1}}$ Hamiltonian isotopic in $B^4(c)$ to one of the L_n. Then $c \geq \frac{3x}{x+1}$.*

A polydisk analogue of the Stabilized Embedding Conjecture would state that when $x > \tau^4$ we have $E(1,x) \hookrightarrow_P B^4(c)$ if and only if $c \geq \frac{3x}{x+1}$. Our L_n are the monotone tori in the ball which appear in the boundary of cubes and are associated to Markov triples. Rescaling Theorem 3.5, together with Theorem 3.3 we then have the following.

Corollary 3.6. *Either the polydisk analogue of the Stabilized Embedding Conjecture holds, or else there are embeddings of the cube $P(1,1)$ into $B^4(3)$ whose singular Lagrangian boundary is not Hamiltonian isotopic to one of the L_n.*

Remark 3.7. *Let $x < \tau^4$ and $\psi : E(1,x) \hookrightarrow B^4(c) \subset B^4(\frac{3x}{x+1})$ be given by Theorem 2.3. Then when $b_{n-1} < x < b_n$ the torus $\psi(L(\frac{x}{x+1}, \frac{x}{x+1}))$ is Hamiltonian isotopic to L_n, see* [2].

4. Lagrangian isotopies imply stabilized embeddings

Here we prove the following.

Proposition 4.1. *Suppose $U, V \subset \mathbb{C}^2$ with U toric. If $U \hookrightarrow_L V$ then $U \hookrightarrow_S V$.*

Proof. Let $\Omega = \mu(U)$ and $\mathcal{K} \subset \Omega$ compact. We recall $U \hookrightarrow_L V$ means there exists a family of Hamiltonian functions $H_{r,s} : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{R}$ for $(r,s) \in \mathcal{K}$ whose time 1 flows satisfy $\phi_{r,s} L(r,s) \subset V$. We can extend the family arbitrarily to define a smooth family $H_{r,s}$ with $(r,s) \in \mathbb{R}^2$.

Given $S > 0$, we would like to construct a Hamiltonian diffeomorphism mapping

$$\mu^{-1}\mathcal{K} \times B^2(S) \hookrightarrow V \times \mathbb{C}.$$

Let $M > 0$ and consider the Hamiltonian

$$G : \mathbb{C}^3 \to \mathbb{R},\ (z_1, z_2, z_3) \mapsto H_{x_3/M, y_3/M}(z_1, z_2)$$

where we set $z_3 = x_3 + iy_3$. The projection of the Hamiltonian vector field X_G to the (z_1, z_2) plane is equal to $X_{H_{x_3/M,y_3/M}}$ while the projection to the z_3 plane has order $1/M$. Hence, if M is chosen sufficiently large, the time 1 flow maps $L(x_3/M, y_3/M) \times \{z_3 = x_3 + iy_3\}$ into $V \times \mathbb{C}$.

Next we consider the Hamiltonian

$$K : \mathbb{C}^3 \to \mathbb{R},\ (z_1, z_2, z_3) \mapsto M(-y_3\pi|z_1|^2 + x_3\pi|z_2|^2).$$

The corresponding flow X_K preserves $|z_1|$ and $|z_2|$, and its z_3 component is equal to

$$M\pi|z_1|^2 \frac{\partial}{\partial x_3} + M\pi|z_2|^2 \frac{\partial}{\partial y_3}.$$

Suppose then that $(w_1, w_2, w_3) = \phi_K(z_1, z_2, z_3)$, where ϕ_K is the time 1 flow and $(z_1, z_2, z_3) \in \mathcal{K} \times B^2(S)$. Putting $w_3 = u_3 + iv_3$ we see that

$$\pi|z_1|^2 = \frac{u_3}{M} + O(S/M),\ \ \pi|z_2|^2 = \frac{v_3}{M} + O(S/M).$$

In other words, by taking M large with respect to S, we may assume that $\phi_K(\mu^{-1}\mathcal{K} \times B^2(S)) \cap \{z_3 = u_3 + iv_3\}$ lies in an arbitrarily small neighborhood of $L(u_3/M, v_3/M) \times \{z_3 = u_3 + iv_3\}$. Hence we can compose with the map ϕ_G to conclude as required. □

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