REVIEW FOR EXAM 3

Convergence Tests.

(1) *Divergence Test:* If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(2) *Geometric Series:* $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$, diverges otherwise. $R_m = \frac{r^{m+1}}{1-r}$.

(3) *Telescoping Series:* Cancel terms in the partial sum s_m (possibly after using partial fractions on a_n) before taking limit.

(4) *Integral Test:* If $a_n = f(n)$, where $f(x)$ is continuous, positive, and decreasing, then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \int_1^{\infty} f(x)\,dx$ converges. $R_m \leq \int_m^{\infty} f(x)\,dx$.

(5) *p-series:* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$. $R_m \leq \int_m^{\infty} \frac{1}{x^p}\,dx = \frac{1}{(p-1)m^{p-1}}$.

(6) *Comparison Test:* Compare a series to a simpler series like (1)–(5). $R_m \leq R'_m$.

(7) *Limit Comparison Test:* If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=1}^{\infty} b_n$ converges.

(8) *Alternating Series Test:* $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges if $b_n \searrow 0$. $|R_m| \leq b_{m+1}$.

(9) *Absolute Convergence:* If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

[*Conditional Convergence:* $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.]

(10) *Ratio Test:* $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ (diverges if > 1).

(11) *Root Test:* $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ (diverges if > 1).

Power Series. $c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$. The *radius of convergence*, R, is found using the Ratio/Root Test. The endpoints, $a \pm R$, must be checked separately, giving the full *interval of convergence*. The extreme cases $R = 0$ or $R = \infty$ are possible.
Derivatives and integrals of power series can be calculated term by term, like a polynomial.

Taylor/Maclaurin Series.
The *Taylor series* of $f(x)$ at a (*Maclaurin series* if $a = 0$) is the power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Any standard calculus function $f(x)$ equals its Taylor series.
The *m-th degree Taylor Polynomial* of $f(x)$ is

$$T_m(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{m!}(x-a)^m$$

The *remainder* is $R_m(x) = f(x) - T_m(x)$, so $T_m(x)$ approximates $f(x)$ with error $R_m(x)$. *Taylor's Inequality* gives an estimate: If $|x-a| \leq d$ and $|f^{(m+1)}(x)| \leq K$ then

$$|R_m(x)| \leq \frac{K}{(m+1)!}|x-a|^{m+1} \leq \frac{K}{(m+1)!}d^{m+1}$$

Standard Series. (May substitute any expression for u.)

(1) $\frac{1}{1-u} = 1 + u + u^2 + \cdots = \sum_{n=0}^{\infty} u^n$, $|u| < 1$.

(2) $\ln(1-u) = -u - \frac{1}{2}u^2 - \frac{1}{3}u^3 - \cdots = -\sum_{n=1}^{\infty} \frac{1}{n}u^n$, $-1 \leq u < 1$.

(3) $\tan^{-1}(u) = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{2n+1}$, $-1 < u \leq 1$.

(4) $e^u = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \cdots \sum_{n=0}^{\infty} \frac{1}{n!}u^n$, all u.

(5) $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)!}$, all u.

(6) $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n}}{(2n)!}$, all u.

(7) $(1+u)^k = 1 + ku + \frac{k(k-1)}{2!}u^2 + \frac{k(k-1)(k-2)}{3!}u^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} u^n$, $|u| < 1$.