REVIEW OF PARAMETRIC EQUATIONS

Parametric equations, $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, describe a curve $\mathcal{C}$ as the set of points $\{(f(t), g(t)) \mid a \leq t \leq b\}$. They also describe the motion of a particle by giving its position at time t as the point $(f(t), g(t))$. To plot $\mathcal{C}$, make a list of points $(f(t_1), g(t_1))$, …, $(f(t_n), g(t_n))$ for a set of values $a = t_1 < t_2 < \cdots < t_n = b$ and connect them in order with smooth arcs.

Calculus with Parametric Curves. The *slope* of the tangent line to $\mathcal{C}$ at the point $(f(c), g(c))$ is $m(c) = \left.\dfrac{dy}{dx}\right|_{t=c} = \left.\dfrac{dy/dt}{dx/dt}\right|_{t=c} = \dfrac{g'(c)}{f'(c)}$. The equation of the tangent line is $y - g(c) = m(c)(x - f(c))$.

The *concavity* of $\mathcal{C}$ at the point $(f(c), g(c))$ is $\dfrac{d^2y}{dx^2} = \left.\dfrac{dm/dt}{dx/dt}\right|_{t=c} = \dfrac{m'(c)}{f'(c)}$.

The *area* under the curve $\mathcal{C}$ is $A = \displaystyle\int y\,dx = \int y\,\frac{dx}{dt}\,dt = \int g(t)\,f'(t)\,dt$. Set the limits of integration so the x values go from left to right.

A piece of arc length is given by $ds = \sqrt{dx^2 + dy^2}$. The *arc length* of the curve $\mathcal{C}$ is

$$L = \int ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

The *surface area* of the surface obtained by revolving $\mathcal{C}$ about the y-axis is

$$S = \int 2\pi\,x\,ds = \int_a^b 2\pi f(t)\sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

If revolved about the x-axis, replace $x = f(t)$ by $y = g(t)$ in this formula.

Polar Coordinates. A point P in the plane can be specified by giving its distance r from the origin O and the angle θ between OP and the x-axis. These *polar coordinates* (r, θ) of P are related to the rectangular coordinates (x, y) of P by the formulas

$$x = r\cos(\theta) \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin(\theta) \qquad \theta = \tan^{-1}(y/x)$$

A *polar curve* $\mathcal{C}$ is defined by a *polar equation*, $r = f(\theta)$, $\alpha \leq \theta \leq \beta$. To plot $\mathcal{C}$, make a list of values $r_1 = f(\theta_1)$, …, $r_n = f(\theta_n)$, for $\theta_1 = \alpha < \theta_2 < \cdots < \theta_n = \beta$, plot the points (r_1, θ_1), …, (r_n, θ_n) (using polar coordinates), and connect them in order with smooth arcs.

The area inside a polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, is $A = \displaystyle\int_\alpha^\beta \frac{1}{2} r^2\,d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\,d\theta$, and the area between two polar curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\alpha \leq \theta \leq \beta$, is

$$A = \int_\alpha^\beta \frac{1}{2}[r_1^2 - r_2^2]\,d\theta = \int_\alpha^\beta \frac{1}{2}[f_1(\theta)^2 - f_2(\theta)^2]\,d\theta$$

A polar equation $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, can be converted to parametric equations by

$$x = r\cos(\theta) = f(\theta)\cos(\theta), \quad y = r\sin(\theta) = f(\theta)\sin(\theta), \quad \alpha \leq \theta \leq \beta$$

The above formulas for parametric equations then apply. The arc length formula simplifies in this case to $L = \displaystyle\int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$.