Symplectic capacities of domains in $\mathbb{C}^2$

R. Hind

October 10, 2005

1 Introduction

In his paper [3] M. Gromov proved his celebrated non-squeezing theorem. We will study domains D in $\mathbb{C}^2$ with standard coordinates (z_1, z_2) and projections π_1 and π_2 onto the z_1 and z_2 planes respectively. The standard symplectic form on $\mathbb{C}^2$ is $\omega = \frac{i}{2}\sum_{j=1}^{2} dz_j \wedge d\overline{z}_j$ and this restricts to a symplectic form on the balls $B(r) = \{|z_1|^2 + |z_2|^2 < r^2\}$. In this notation Gromov's non-squeezing theorem states that if $\text{area}(\pi_1(D)) \leq C$ and there exists a symplectic embedding $B(r) \to D$ then $\pi r^2 \leq C$. Nowadays this can be rephrased as saying that the Gromov width of D is at most C. Of course this is sharp when D is a cylinder $\{|z_1| < r\}$.

For general D it is natural to ask whether we can estimate the Gromov width instead in terms of the cross-sectional areas $\text{area}(D \cap \{z_2 = b\})$. But for any $\epsilon > 0$ there exists a construction of F. Schlenk, [4], of a domain D lying in a cylinder $\{|z_1| < 1\}$ with Gromov width at least $\pi - \epsilon$ but with all cross-sections having area less than ϵ. At least if we drop the condition on the domain lying in the cylinder, the cross-sections can even be arranged to be star-shaped, see [5]. Nevertheless in this note we will obtain such an estimate in terms of the areas of the cross-sections for domains whose cross-sections are all starshaped about the axis $\{z_1 = 0\}$.

Theorem 1 *Let $D \subset \mathbb{C}^2$ be a domain whose cross-sections $D \cap \{z_2 = b\}$ are star-shaped about center $z_1 = 0$. Define $C = \sup_b \text{area}(\{z_2 = b\} \cap D)$. Then if*

$B(r) \to D$ is a symplectic embedding we have $\pi r^2 \leq C$. In other words, D has Gromov width at most C.

In section 2 we will establish an estimate on the Gromov width for such domains D. This is combined with a symplectic embedding construction to obtain our result in section 3.

The author would like to thank Felix Schlenk for patiently answering many questions.

2 Embedding estimate

Here we prove the following theorem.

Theorem 2 *Fix constants $0 < K \leq M$ and $0 < t < 1$. Let $D \subset \mathbb{C}^2$ be a domain of the form $D = \{r < c(\theta, z_2), |z_2| < M\}$ where (r, θ) are polar coordinates in the z_1 plane and $c(\theta, z_2)$ is a real-valued function satisfying $t \leq c(\theta, z_2) \leq 1$ and $|\frac{\partial c}{\partial z_2}| \leq \frac{1}{K}$.*

Define $C = \sup_b \text{area}(\{z_2 = b\} \cap D)$. Then if $B(r) \to D$ is a symplectic embedding of the standard ball of radius r in $\mathbb{C}^2$ we have $\pi r^2 < C + 3\sqrt{\frac{M}{tK^3}}$.

Its key implication for us is the following.

Corollary 3 *Let $D = \{r < c(\theta, z_2), |z_2| < M\} \subset \mathbb{C}^2$ and $C = \sup_c \text{area}(\{z_2 = b\} \cap D)$. For any $L > 0$ the domain D is a symplectic manifold with symplectic form $\omega_L = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + L dz_2 \wedge d\overline{z}_2)$. Let $r > 0$ with $\pi r^2 > C$. Then for all L sufficiently large the symplectic manifold (D, ω_L) does not admit a symplectic embedding of the ball $B(r)$.*

This follows by rescaling. Note above that the volume of (D, ω_L) approaches infinity as $L \to \infty$.

Proof of Theorem 2

We consider the symplectic manifold $S^2 \times \mathbb{C}$ with a standard product symplectic form $\omega = \omega_1 \oplus \omega_2$ and still use coordinates (z_1, z_2), where z_1 now extends from $\mathbb{C}$ to give a coordinate on the $S^2 = \mathbb{C}P^1$ factor. Still π_1 and π_2 denote the projections onto the coordinate planes. Let F be the area of the first factor, we suppose that this is sufficiently large that the complement of $\{z_1 = \infty\}$ can be identified with a neighborhood of $\{|z_1| \leq 1\}$ in $\mathbb{C}^2$, the identification preserving the product complex and symplectic structures. In other words, from now we assume that $D \subset S^2 \times \mathbb{C} \setminus \{z_1 = \infty\}$ and satisfies the conditions on its cross-sections. Let D^c denote the complement of D in $S^2 \times \mathbb{C}$.

Now let $\phi : B(r) \to D$ be a symplectic embedding. Then we consider almost-complex structures J on $S^2 \times \mathbb{C}$ which are tamed by ω and coincide with the standard product structure on D^c. By now it is well-known, see [3], that for all such J the almost-complex manifold $S^2 \times \mathbb{C}$ can be foliated by J-holomorphic spheres. In $\{|z_2| \geq M\}$ the foliation simply consists of the S^2 factors.

Let S denote the image of the holomorphic curve in our foliation passing through $\phi(0)$. By positivity of intersections S intersects $\{z_1 = \infty\}$ in a single point, say $\{z_2 = b\}$. As above we will use polar coordinates (r, θ) in the plane $\{z_2 = b\}$. So we can write $D \cap \{z_2 = b\} = \{r \leq c(\theta, b) := c(\theta)\}$. Let $A = \text{area}(\{z_2 = b\} \cap D)$. We intend to obtain lower bounds for both $\int_{S \cap D^c} \omega_1$ and $\int_{S \cap D^c} \omega_2$.

First of all, we will suppose that $\pi_1(S \cap D^c) = \{r \geq g(\theta)\}$ for a positive function g and that $S \cap D^c$ is a graph $\{z_2 = u(z_1)\}$ over this region. We explain later how essentially the same proof applies to the general case. Recall that our assumptions imply that $t \leq c(\theta), g(\theta) \leq 1$ for all θ. Define $h(\theta) = |g(\theta) - c(\theta)|$.

Define a holomorphic function $f : \{r \leq \frac{1}{g(-\theta)}\} \to \{|z_2| \leq M\}$ by $f(z) = u(\frac{1}{z})$. Then $f(0) = b$ and $|f(z)| \leq M$ for all z. Therefore composing f with a translation we can redefine f as a function $f : \{r \leq \frac{1}{g(-\theta)}\} \to \{|z_2| \leq 2M\}$ with $f(0) = 0$.

As $g(\theta) \leq 1$ for all θ the map f restricts to one from $\{|z| \leq 1\}$ and so by the Schwarz Lemma, if $|z| < 1$ we have $|f'(z)| \leq \frac{2M}{1-|z|}$. On the boundary of the disk, our assumptions on the boundary of D imply that $|f(\frac{1}{g(-\theta)} e^{i\theta})| \geq Kh(\theta)$.

Now we estimate

$$\begin{aligned}
\int_{S\cap D^c}\omega_2 &= \text{area}(\text{image}(f)) \\
&= \int_0^{2\pi} d\theta \int_0^{\frac{1}{g(-\theta)}} r|f'(z)|^2 dr \\
&= \int_0^{2\pi} g(-\theta) d\theta \left(\int_0^{\frac{1}{g(-\theta)}} r|f'(z)|^2 dr\right)\left(\int_0^{\frac{1}{g(-\theta)}} dr\right) \\
&\geq t\int_0^{2\pi} d\theta \left(\int_0^{\frac{1}{g(-\theta)}} r^{\frac{1}{2}}|f'(z)| dr\right)^2 .
\end{aligned}$$

Now

$$\int_0^{\frac{1}{g(-\theta)}} |f'(z)| dr \geq Kh(\theta)$$

and over all such functions $|f'(z)|$ the final integral above is minimized by taking $|f'(z)|$ as large as possible for small values of r. We compute

$$\int_0^y \frac{2M}{1-r} dr = Kh(\theta)$$

when $y = 1 - e^{\frac{-Kh(\theta)}{2M}} < \frac{1}{g(-\theta)}$. Therefore putting $y = x^2$ we have

$$\begin{aligned}
t\int_0^{2\pi} d\theta \left(\int_0^{\frac{1}{g(-\theta)}} r^{\frac{1}{2}}|f'(z)| dr\right)^2 &\geq t\int_0^{2\pi} d\theta \left(\int_0^{x^2} \frac{2M\sqrt{r}}{1-r} dr\right)^2 \\
&= 4M^2 t\int_0^{2\pi} d\theta \left(\left[-2\sqrt{r} + \ln\left(\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right]_0^{x^2}\right)^2 \\
&= 4M^2 t\int_0^{2\pi} d\theta \left(-2x + \ln\left(\frac{1+x}{1-x}\right)\right)^2 \\
&\geq 4M^2 t\int_0^{2\pi} \frac{4x^6}{9} d\theta
\end{aligned}$$

for the final estimate using the fact that $0 < x < 1$.

Now

$$x^2 = 1 - e^{\frac{-Kh(\theta)}{2M}} \geq (1 - e^{-\frac{1}{2}})\frac{Kh(\theta)}{M}$$

since $\frac{Kh(\theta)}{2M} \leq \frac{1}{2}$.

Therefore

$$\int_{S\cap D^c}\omega_2 \geq 4M^2 t\int_0^{2\pi} \frac{4x^6}{9} d\theta$$

$$\geq \quad \frac{16}{9}(1-e^{-\frac{1}{2}})^3 \frac{tK^3}{M} \int_0^{2\pi} h(\theta)^3 d\theta.$$

Next we compute

$$\begin{aligned}
\int_{S\cap D^c} \omega_1 &= F - \frac{1}{2}\int_0^{2\pi} g(\theta)^2 d\theta \\
&= F - A - \frac{1}{2}\int_0^{2\pi} (g(\theta)^2 - c(\theta)^2) d\theta \\
&\geq F - A - \frac{1}{2}\int_0^{2\pi} (g(\theta) - c(\theta))(g(\theta)+c(\theta)) d\theta \\
&\geq F - A - \int_0^{2\pi} h(\theta) d\theta.
\end{aligned}$$

Therefore writing $k = \frac{16}{9}(1-e^{-\frac{1}{2}})^3 \frac{tK^3}{M}$ we have

$$\begin{aligned}
\int_{S\cap D^c} \omega &\geq F - A - \int_0^{2\pi} (h(\theta) - kh(\theta)^3) d\theta \\
&\geq F - A - 2\pi \frac{2}{3\sqrt{3k}} \\
&= F - A - \pi\sqrt{\frac{M}{3(1-e^{-\frac{1}{2}})^3 tK^3}}.
\end{aligned}$$

Thus $S\cap D$ has symplectic area at most $A + \pi\sqrt{\frac{M}{3(1-e^{-\frac{1}{2}})^3 tK^3}} < A + 3\sqrt{\frac{M}{tK^3}}$, since S itself has area F.

We assumed above that $\pi_1(S\cap D^c)$ is starshaped about $z_1 = 0$ and that $S \cap D^c$ is a graph over this region. If the projection $\pi_1 : S \to \pi_1(S\cap D^c)$ is a branched cover then we can define a function f as before simply choosing a suitable branch along the rays $\{\theta = \text{constant}\}$. The proof then applies as before. Now suppose that $\pi_1(S\cap D^c)$ is not starshaped about $z_1 = 0$. Then we find the smallest possible starshaped set $\{r \leq g(\theta)\}$ containing the complement of $\pi_1(S\cap D^c)$. The defining function g will then have discontinuities but this does not affect the proof which again proceeds as before.

Finally we choose a J which coincides with the push forward of the standard complex structure on the ball $B(r)$ under ϕ but remains standard outside D. The part of S intersecting the image of ϕ is now a minimal surface with respect to the standard pushed forward metric on the ball and so must have area at least πr^2, giving our inequality as required.

3 Proof of Theorem 1

For any domain $E \subset \mathbb{C}^2$ we will write $C(E) = \sup_b \text{area}(\{z_2 = b\} \cap E)$. Again we let $C = C(D)$. Arguing by contradiction suppose that $B(r) \to D$ is a symplectic embedding with $\pi r^2 > C + \epsilon$.

Let B be the image of the ball of radius r in D. We will prove Theorem 1 by finding a symplectic embedding of B into (D_1, ω_L) for all sufficiently large L, where D_1 is a domain C^0 close to D and with $C(D_1) < C(D) + \epsilon$. Such embeddings would contradict Corollary 3.

First we choose a lattice of the z_2 plane sufficiently fine that if we denote the gridsquares by G_i then $\sup_i \text{area}(\pi_1(D \cap \pi_2^{-1}(G_i))) < C(D) + \epsilon$. Then we let $D_1 = \bigcup_i \pi_1(D \cap \pi_2^{-1}(G_i)) \times G_i$, suitably smoothed.

Let $\{b_j\}$ be the vertices of our lattice. We make the following simple observation.

Lemma 4 *Suppose that $B \cap \{z_2 = b_j\} = \emptyset$ for all j. Then there exists a symplectic embedding of B into (D_1, ω_L) for all sufficiently large L.*

Proof It suffices to find a diffeomorphism ψ of $\mathbb{C} \setminus \{b_j\}$ which preserves the G_i and such that $\psi^*(L\omega_0) = \omega_0$, letting $\omega_0 = dz \wedge d\overline{z}$ be the standard symplectic form. It is not hard to construct such a map, and the product of this map on the z_2 plane with the identity map on the z_1 plane gives a suitable embedding.

Given Lemma 4, to find our embedding it remains to find a symplectic isotopy of D_1 such that the image of B is disjoint from the planes $C_j = \{z_2 = b_j\}$. Equivalently we will find a symplectic isotopy of the union of the C_j, compactly supported in a neighborhood of B and moving the C_j away from B.

We may assume that the embedding of the ball of radius r extends to a symplectic embedding of a ball of radius s where s is slightly greater than r. Let U be the image of this ball and J_0 the push-forward of the standard complex structure on $\mathbb{C}^2$ to U under the embedding.

Lemma 5 *There exists a C^0 small symplectic isotopy supported near ∂U which moves each C_j into a J_0-holomorphic curve near ∂U.*

Proof Let $(x+iy, u+iv)$ be local coordinates on $\mathbb{C}^2$. Let C be one of our curves. We may assume that in these coordinates near to the origin $C \cap \partial U$ is the curve $\{(x,0,0,0)\}$ and therefore that nearby C is the graph over the (x,y) plane of a function $h(x,y) = (u,v)$. So $u = v = 0$ when $y = 0$.

There exists a constant k such that $|u|$, $|v|$, $|\frac{\partial u}{\partial x}|$ and $|\frac{\partial v}{\partial x}|$ are all bounded by $k|y|$ near $y = 0$.

Now, such a graph is symplectic provided

$$|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}| < 1.$$

We can make C holomorphic near ∂U by replacing h by $(\chi u, \chi v)$ where χ is a function of y, equal to 0 near $y = 0$ and 1 away from a small neighborhood. The resulting graph remains symplectic provided

$$|\chi\frac{\partial u}{\partial x}(\chi' v + \chi\frac{\partial v}{\partial y}) - \chi\frac{\partial v}{\partial x}(\chi' u + \chi\frac{\partial u}{\partial y})| < 1$$

or rewriting

$$|\chi^2(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}) + \chi\chi'(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})| < 1.$$

If we assume that $|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}| < 1 - \delta$ the graph remains symplectic if χ is chosen such that

$$|\chi\chi'(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})| < \delta$$

which is guaranteed if $\chi' < \frac{\delta}{ky^2}$.

Since the integral $\int_0^t \frac{\delta}{ky^2}dy$ diverges a function χ satisfying this condition while being equal to 0 near 0 and 1 away from an arbitrarily small neighborhood does indeed exist as required. The resulting surface is clearly isotopic through symplectic surfaces to the original C.

We now replace the C_j by their images under the isotopy from Lemma 5. We let J be an almost-complex structure on U which is tamed by ω, coincides with J_0 near ∂U, and such that the $C_j \cap U$ are J-holomorphic.

Now (U, J) is an (almost-complex) Stein manifold in the sense that it admits a plurisubharmonic exhaustion function $\phi : U \to [0, R)$. In fact, work of Eliashberg, see [1] and [2], implies that such a plurisubharmonic exhaustion exists with a unique critical point, its minimum. Generically this will be disjoint from the C_j.

Near the boundary we can take ϕ to be the push-forward under the embedding of a function $\frac{|z|^N}{C}$ for some integer $N \geq 2$ (depending perhaps on U) and (any given) constant C. The definition of a plurisubharmonic function states that $\omega_\phi = -dd^c\phi$ is a symplectic form on U which is compatible with J (for a function f we define $d^c f := df \circ J$). We can choose C such that $\omega_\phi|_{\partial U} = \omega|_{\partial U}$ and thus by Moser's lemma the symplectic manifolds (U, ω) and (U, ω_ϕ) are symplectomorphic via a symplectomorphism F fixing the boundary. In fact, adjusting the isotopy provided by Moser's method we may assume that F fixes the C_j (since they are symplectic with respect to both ω and ω_ϕ). Let V denote the image of $U \setminus B$ under F and suppose that $\{\phi \geq R_0\} \subset V$.

It now suffices to find a symplectic isotopy of the C_j in (U, ω_ϕ) moving the surfaces into the region $\{\phi \geq R_0\}$. Then the preimages of these surfaces under F gives a symplectic isotopy moving them away from B as required.

Let Y be the gradient of ϕ with respect to the Kähler metric associated to ϕ. Equivalently Y is defined by $Y \rfloor \omega_\phi = -d^c\phi$. Define $\chi : [0, R) \to [0, 1]$ to have compact support but satisfy $\chi(t) = 1$ for $t \leq R_0$. Then the images of the C_j under the one-parameter group of diffeomorphisms generated by $X = \chi(\phi)Y$ will eventually lie in $\{\phi \geq R_0\}$. Thus we can conclude after checking that they remain symplectic during this isotopy. We recall that the C_j are J-holomorphic and finish with the following lemma.

Lemma 6 *Let G be a diffeomorphism of U generated by the flow of the vectorfield X. Then $G^*\omega_\phi(Z, JZ) > 0$ for all non-zero vectors Z.*

Proof For any function f we compute

$$\begin{aligned}\mathcal{L}_X f(\phi) d^c\phi &= f'(\phi) X \rfloor d\phi \wedge d^c\phi + f(\phi) X \rfloor dd^c\phi + d(f(\phi) X \rfloor d^c\phi) \\ &= (f'(\phi) d\phi(X) + f(\phi)\chi(\phi)) d^c\phi.\end{aligned}$$

Thus $G^* d^c\phi = g(\phi) d^c\phi$ for some function g and

$$G^*\omega_\phi = g(\phi)\omega_\phi - g'(\phi) d\phi \wedge d^c\phi.$$

The function g is certainly positive and so $G^*\omega_\phi$ evaluates positively on the (contact) planes $\{d\phi = d^c\phi = 0\}$. Therefore if $G^*\omega_\phi$ evaluates nonpositively on a J-holomorphic plane then there exists such a plane containing Y. But this is clearly not the case, as $G^*\omega_\phi(Y, JY) = \omega_\phi(G_*Y, G_*JY) = -kd^c\phi(G_*JY)$ for some positive constant k and $-d^c\phi(G_*JY) = -G^*d^c\phi(JY) = g(\phi)d\phi(Y) > 0$.

References

[1] Y. Eliashberg, Filling by holomorphic disks and its applications, *London Math. Society Lecture Notes*, Series 151 (1991), 45-67.

[2] Y. Eliashberg, Symplectic Geometry of plurisubharmonic functions, with notes by Miguel Abreu, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 488, *Gauge theory and symplectic geometry (Montreal, PQ 1995), 49-67, Kluwer Acad. Publ., Dordrecht, 1997.*

[3] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, *Inv. Math.*, 82(1985), 307-347.

[4] F. Schlenk, On a question of Dusa McDuff. *Int. Math. Res. Not.*, 2003, no. 2, 77–107.

[5] F. Schlenk, Embedding problems in symplectic geometry, de Gruyter Expositions in Mathematics, 40, Berlin, 2005.

Richard Hind
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
email: hind.1@nd.edu