

1.(5pts) Let U be an $m \times n$ matrix with orthonormal columns and let $W = \text{Col } U$. Determine which statement is *not* always true.

(a) $\|U\mathbf{x}\| = 1$ for $\mathbf{x} \neq \mathbf{0}$ in \mathbb{R}^n

(b) $U^T U = I$

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for \mathbf{x}, \mathbf{y} in \mathbb{R}^n

(d) $\text{proj}_W \mathbf{y} = UU^T \mathbf{y}$ for \mathbf{y} in \mathbb{R}^m

(e) The columns of U form a basis for W .

2.(5pts) Find the orthogonal projection of $\mathbf{y} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$ onto the subspace W spanned by

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

(e) $\begin{bmatrix} 7 \\ 7 \\ 17 \end{bmatrix}$

3.(5pts) Use the Gram-Schmidt process to find an orthogonal basis for the span of the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

4.(5pts) Classify the differential equation $y' = \frac{\cos(x) + y \sin(x)}{\ln(x)}, \quad x > 0$.

(a) *linear*

(b) *separable*

(c) *exact*

(d) *autonomous*

(e) *none of the above*

5.(5pts) Solve the differential equation $y' + 2y = 3e^t$ subject to the initial condition $y(1) = 0$.

(a) $y = e^t - e^{3-2t}$ (b) $y = e^{3t} - e^3$ (c) $y = e^{2t} - e^{3-t}$ (d) $y = e^t - 1$ (e) $y = e^t$

6.(5pts) Let $\phi(x)$ be the solution of the differential equation $y' = xy^2$ satisfying $\phi(0) = 2$. Find $\phi(3)$.

(a) $-1/4$ (b) 0 (c) $2/9$ (d) $1/3$ (e) $-2/5$

7.(5pts) Determine the interval where the solution to the initial value problem is guaranteed to exist.

$$\cos(t)y' = y + e^{1/t}, \quad y(\pi) = 0$$

(a) $(\pi/2, 3\pi/2)$ (b) $(0, \infty)$ (c) $(0, 3\pi/2)$ (d) $(-\pi/2, \pi/2)$ (e) (π, ∞)

8.(5pts) Suppose a certain population of red squirrels satisfies the differential equation

$$\frac{dN}{dt} = -0.05 \left(1 - \frac{N}{700}\right) \left(1 - \frac{N}{2000}\right) N$$

If the current population is 600, determine the long-term population of red squirrels predicted by this model.

(a) 0 (b) 700 (c) 2000 (d) $\ln(700)/0.05$ (e) $\ln(2000)/0.05$

9.(5pts) Find the solution to the initial value problem $y'' - 5y' - 6y = 0$, $y(0) = 0$, $y'(0) = 1$.

(a) $y = \frac{1}{7}(e^{6t} - e^{-t})$ (b) $y = e^{-2t} - e^{-3t}$ (c) $y = -\frac{1}{5}(e^{-2t} + e^{-3t})$

(d) $y = \frac{1}{7}(e^t - e^{-6t})$ (e) $y = \frac{1}{5}(e^{3t} + e^{2t})$

10.(5pts) Find a value for r such that $y = t^r$ is a solution of the differential equation $9t^2y'' + 12ty' - 2y = 0$.

(a) $-2/3$ (b) 2 (c) 3 (d) $-1/6$ (e) $3/4$

11.(10pts) Find the least-squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix}$.

12.(10pts) Gases from the ground below a house seep into its 3000 ft³ basement at the rate of 2 ft³ per day. Air leaves the basement at the same rate. The incoming gases contain 5% radon. Assuming the basement is initially clear of radon gas, calculate how many days it will take for the basement to reach the dangerous level of 1% radon gas.

13.(10pts) Solve the initial value problem by first finding an integrating factor.

$$y \tan(x) dx + (y \sec(x) - 1) dy = 0, \quad y(\pi) = 2$$

Solutions

1. Since $\|Uc\mathbf{x}\| = c\|U\mathbf{x}\|$ (a) is not always true.
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2. \mathbf{u}_1 and \mathbf{u}_2 are orthogonal so

$$\begin{aligned} \text{proj}_W \left(\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \right) &= \frac{\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \\ &= \frac{-1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{8}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{6} \left(\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 8 \\ 16 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \end{aligned}$$

3. $\mathbf{v}_1 = \mathbf{x}_1$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

4. Write it as

$$y' + \frac{-\sin(x)}{\ln(x)}y = \frac{\cos(x)}{\ln(x)}$$

shows that it is linear.

- 5.

The equation is linear first order so $\int 2dt = 2t$ so $\mu = e^{2t}$ and we need to do $\int 3e^t e^{2t} dt = \int 3e^{3t} dt = e^{3t} + C$ and the general solution is $y = e^{-2t}(e^{3t} + C)$. Since $y(1) = 0$, $0 = e^{-2}(e^3 + C)$ so $C = -e^3$ and $y = e^{-2t}(e^{3t} - e^3)$.

- 6.

The equation is separable so $\frac{dy}{y^2} = x dx$ or $-\frac{1}{y} = \frac{x^2}{2} + C$. Since $y(0) = 2$, $-1/2 = 0 + C$ so $-\frac{1}{y} = \frac{x^2 - 1}{2}$ or $y = \frac{2}{1 - x^2}$. $\phi(3) = \frac{2}{1 - 3^2} = \frac{2}{-8} = -1/4$

7. The equation is linear and the standard form is

$$y' + \frac{-1}{\cos(t)}y = \frac{e^{1/t}}{\cos(t)}, \quad y(\pi) = 0$$

We are looking for the largest open interval containing π such that $t \neq 0$ and $\cos(t) \neq 0$.

That interval is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

8. The equation is autonomous and the zeros of the function are $N = 0$, $N = 700$ and $N = 2000$. At $N = 600$, $\frac{dN}{dt} < 0$ so N is decreasing. For all N between 0 and 700, $\frac{dN}{dt} < 0$ so N eventually goes to 0.
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9. 2nd order linear with constant coefficients so solve $r^2 - 5r - 6 = (r - 6)(r + 1) = 0$ so the roots are 6 and -1 so the general solution is $y = c_1 e^{-t} + c_2 e^{6t}$ and then $y' = -c_1 e^{-t} + 6c_2 e^{6t}$ so $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = -c_1 + 6c_2$. Hence $c_2 = 1/7$ and $c_1 = -1/7$.
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10. $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$ so

$$9r(r-1)t^r + 12rt^r - 2t^r = (9r^2 - 9r + 12r - 2)t^r$$

which vanishes if and only if $9r^2 + 3r - 2 = (3r-1)(3r+2) = 0$. Hence two values of r work, $r = 1/3$ and $r = -2/3$.

11. Solve $A^T A \mathbf{x} = A^T \mathbf{b}$. $A^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

so solve

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

By inspection $\hat{\mathbf{x}} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

12. Let $R(t)$ denote the total amount of radon in the air in ft^3 . A cubic foot of incoming gas contains 0.05 ft^3 of radon. A cubic foot of outgoing gas contains $\frac{R}{3000} \text{ ft}^3$ of radon.

Hence $0.1 \text{ ft}^3/\text{day}$ is the amount of incoming radon per day and $\frac{2R}{3000} \text{ ft}^3$ is the amount of outgoing radon per day. Hence

$$\frac{dR}{dt} = 0.1 - \frac{R}{1500}$$

This equation is linear and in standard form is

$$R' + \frac{1}{1500}R = 0.1$$

Hence $\mu = e^{t/1500}$ and $\int 0.1e^{-t/1500} dt = 150e^{-t/1500} + C$ so $R = 150 + Ce^{-t/1500}$.

Since the basement is initially clear of radon $R(0) = 0$ and $C = -150$ and so $R = 150(1 - e^{-t/1500})$.

We are looking for the time t for which $R(t) = 0.01 \cdot 3000 = 30$ so $150(1 - e^{-t/1500}) = 30$ or $1 - e^{-t/1500} = 0.2$ or $0.8 = e^{-t/1500}$ or $-t/1500 = \ln(0.8)$ or

$$t = \frac{-\ln(0.8)}{1500} \text{ days}$$

With a calculator this is about 335 days.

13. $M = y \tan(x)$ and $N = y \sec(x) - 1$ so $M_y = \tan(x)$ and $N_x = y \tan(x) \sec(x)$ so $M_y - N_x = \tan(x) - y \tan(x) \sec(x) = \tan(x)(1 - y \sec(x)) = -\tan(x)N$. Hence $\frac{\mu'}{\mu} = -\tan(x)$ or $\ln|\mu| = -\ln|\cos(x)|$ so we may take $\mu = \cos(x)$. The new equation is

$$y \cos(x) \tan(x) dx + \cos(x)(y \sec(x) - 1) dy = 0, \quad y(\pi) = 2$$

or

$$y \sin(x) dx + (y - \cos(x)) dy = 0, \quad y(\pi) = 2$$

$$\frac{\partial \psi}{\partial x} = y \sin(x) \text{ so } \psi = -y \cos(x) + g(y).$$

Then $\frac{\partial \psi}{\partial y} = -\cos(x) + g'(y) = y - \cos(x)$ so $g'(y) = y$ and $\psi = -y \cos(x) + y^2/2$. Since $y(\pi) = 2$, the value of the level curve is $-2 \cos(\pi) + 2^2/2 = 2 + 2 = 4$ so $-y \cos(x) + y^2/2 = 4$ or

$$y^2 - 2y \cos(x) - 8 = 0$$

You can solve for y using the quadratic formula although you were not asked for this. If you do proceed however,

$$y = \frac{2 \cos(x) \pm \sqrt{4 \cos^2(x) + 32}}{2}$$

When $x = \pi$, $y = \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm 6}{2} = 2$ or -4 . Since we are looking for the solution with $y(\pi) = 2$, the *unique* solution is

$$y = \frac{2 \cos(x) + \sqrt{4 \cos^2(x) + 32}}{2}$$
