

1.(6pts) Let A be an $n \times n$ matrix satisfying $A^T A = I$. Let \mathbf{u}, \mathbf{v} be vectors in \mathbb{R}^n such that $\mathbf{u} \cdot \mathbf{v} = 4$. Find $(A\mathbf{u}) \cdot (A\mathbf{v})$.

- (a) $1/4$ (b) $-1/4$ (c) 0 (d) -4 (•) 4

Solution:

$A^T A = I$ means A is unitary so $(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = 4$.

2.(6pts) Let $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$. Compute $\text{proj}_W \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

- (a) $\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (•) $\frac{1}{2} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ (e) $\frac{1}{2} \begin{bmatrix} 1 \\ 3 \\ -3 \\ -1 \end{bmatrix}$

Solution:

Note that the vectors in W are orthogonal so

$$\begin{aligned} \text{proj}_W \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} &= \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \\ &= \frac{-2}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \frac{0}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{-4}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} \end{aligned}$$

3.(6pts) Classify the differential equation $\frac{dy}{dx} = \frac{\sin(x)y}{\cos(x) + y}$.

- (a) 2nd order (b) autonomous (c) separable (●) exact (e) linear

Solution:

It appears to be neither linear, separable or autonomous. It is first order, not second. We can write it as

$$\sin(x)ydx + (-\cos(x))dy = 0$$

But $\frac{\partial \sin(x)y}{\partial y} = \sin(x)$ and $\frac{\partial -\cos(x)}{\partial x} = \sin(x)$ so it is exact.

4.(6pts) Solve the differential equation $y' + 3\sqrt{t}y = \sqrt{t}$.

- (a) $y = 2t^{3/2} + C$ (b) $y = Ct^{-3/2}$ (c) $y = \frac{1}{3} + C$ (●) $y = \frac{1}{3} + Ce^{-2t\sqrt{t}}$
 (e) $y = C\sqrt{t}e^{2t\sqrt{t}}$

Solution:

Equation is linear 1st order in standard form. $\int 3\sqrt{t}dt = 3\frac{t^{3/2}}{3/2} + C$ so $\mu = e^{2t^{3/2}}$ is a choice of integrating factor. Need to do $\int \sqrt{t}e^{2t^{3/2}}dt$. Substitute $u = 2t^{3/2}$ so $du = 3\sqrt{t}dt$ so

$$\int \sqrt{t}e^{2t^{3/2}}dt = \frac{1}{3} \int e^u du = \frac{e^u}{3} + C = \frac{e^{2t^{3/2}}}{3} + C \text{ and the solution is } y = \frac{\frac{e^{2t^{3/2}}}{3} + C}{e^{2t^{3/2}}}.$$

5.(6pts) Let $\phi(x)$ be a solution to $\frac{dy}{dx} = \frac{1+y^2}{x^2}$ that satisfies $\phi(1) = 0$. Find $\phi(2)$.

- (a) $\frac{1}{1 - \tan(1/2)}$ (•) $\tan(1/2)$ (c) $\frac{1}{1 - \tan^{-1}(2)}$ (d) $\tan^{-1}(2)$
 (e) $\tan(2)$

Solution:

Equation separates as $\frac{dy}{1+y^2} = \frac{dx}{x^2}$ so $\arctan(y) = -x^{-1} + C$. The initial condition is $y(1) = 0$ so $\arctan(0) = -1 + C$ so $C = 1$ and the solution is $\arctan(y) = \frac{x-1}{x}$. Hence $y = \tan\left(\frac{x-1}{x}\right)$ and $y(2) = \tan(1/2)$.

6.(6pts) Find the general solution to $3y'' + y' - 2y = 0$.

- (a) $y = c_1e^{-t} + c_2e^{3t/2}$ (b) $y = c_1e^{-t/3} + c_2e^{t/2}$ (c) $y = c_1e^{t/2} + c_2e^{-3t/2}$
 (d) $y = c_1e^{t/2} + c_2e^{-2t/3}$ (•) $y = c_1e^{-t} + c_2e^{2t/3}$

Solution:

This equation is 2nd order linear with constant coefficients so e^{rt} is a solution whenever $3r^2 + r - 2 = 0$ or $(3r - 2)(r + 1) = 0$ so the roots are -1 and $\frac{2}{3}$. The general solution is $c_1e^{-t} + c_2e^{\frac{2t}{3}}$

7.(6pts) Determine an interval where the solution to the initial value problem is guaranteed to exist.

$$(t^2 - 4)y' = \sqrt{3-t}y + \ln(1+t), \quad y(0) = 0$$

- (a) $-1 < t < 3$ (b) $-1 < t$ (c) $-1 < t < 2$ (d) $t < 3$ (e) $-2 < t$

Solution:

The equation is linear and the standard form is

$$y' + \frac{-\sqrt{3-t}}{t^2-4}y = \frac{\ln(1+t)}{t^2-4}, \quad y(0) = 0$$

The problem asks for the biggest open interval containing 0 over which the two functions of t are continuous. We need $t \leq 3$ for the square root; $t > -1$ for the log function; and $t \neq 2, -2$ for the division by $t^2 - 4$. Hence $-1 < t < 2$.

8.(6pts) Find all the *stable* equilibrium solutions of the autonomous system

$$\frac{dy}{dt} = 3y - 4y^2 + y^3$$

- (a) $y = 1$ (b) $y = 0, y = -4$ (c) $y = 0, y = 3$
 (d) $y = 1, y = 3, y = -4$ (e) $y = 3$

Solution:

The equilibria occur at solutions to $3y - 4y^2 + y^3 = 0$ or $y(y^2 - 4y + 3) = y(y-1)(y-3)$ or $y = 0, 1$ and 3 . For a stable equilibrium at y_0 , $\frac{dy}{dt} > 0$ changes sign from positive to negative as y crosses y_0 .

Crossing 0, $y(y-1)(y-3)$ changes from negative to positive so this equilibrium is unstable. The same thing happens at 3, but crossing 1 two terms are negative for y a bit less than 1, and only one term is negative if y is a bit bigger than 1 so 1 is stable.

9.(6pts) A large tank contains 500 gallons of a water/sugar mixture. Liquid is entering the tank at a rate of 15 gallons/minute and contains 1 pound of sugar per gallon. The mixture is kept well stirred and drains off the tank at a rate of 10 gallons/minute.

If the tank initially has 100 pounds of sugar, determine a differential equation satisfied by $s(t)$, the amount of sugar in pounds in the tank at time t (at least until the tank is full).

$$(a) \frac{ds}{dt} = 30 - \frac{s}{500 + 20t} \quad (\bullet) \frac{ds}{dt} = 15 - \frac{2s}{100 + t} \quad (c) \frac{ds}{dt} = 500 - \frac{s}{20}$$

$$(d) \frac{ds}{dt} = 15 - \frac{s}{50} \quad (e) \frac{ds}{dt} = 15 - \frac{s}{500 + 20t}$$

Solution:

$\frac{ds}{dt}$ measures the change in the amount of sugar. If time is measured from the beginning of the process, $s(0) = 100$. The amount of sugar is changing because of two things. Liquid is entering at a constant rate of 15 gals/min which adds $1 \text{ lbs/gal} \times 15 \text{ gals/min} = 15 \text{ lbs/min}$ of sugar to the tank.

Liquid is draining out at a rate of 10 gals/min so sugar is leaving at a rate of $10 \text{ gals/min} \times s(t)/V(t) \text{ lbs/gal}$ where $V(t) = 500 + 5t$ is the volume of the liquid in gallons. Hence sugar is leaving at a rate of $\frac{10s(t)}{600 + 5t} \text{ lbs/min}$.

$$\text{Hence } \frac{ds}{dt} = 15 - \frac{10s}{500 + 5t} = 15 - \frac{2s}{100 + t}.$$

10.(14pts) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$.

- (a) (10pts) Use the Gram-Schmidt process to find an orthogonal basis for $\text{col}(A)$.
 (b) (4pts) Use the result of (a) to find the Q in the QR -decomposition of A , $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular matrix.
 DO NOT find R .

Solution:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$$\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} =$$

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\text{Hence } Q = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \end{bmatrix}.$$

You were told not to find R but if you had been required to find it, proceed as follows.
 Since $R = Q^T A$,

$$R = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -2 \\ \sqrt{3} & -\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\ 0 & 6 & 6 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

Check

$$\frac{1}{6} \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\ 0 & 6 & 6 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

11.(14pts) If $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 5 \end{bmatrix}$ find the least squares solution to $A\mathbf{x} = \mathbf{b}$.

Solution:

$A^T = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ so $A^T A = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$ and $A^T \mathbf{b} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$. Hence the least squares solution is the vector $\hat{\mathbf{x}}$ which satisfies

$$\begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 & | & 6 \\ 3 & 3 & | & 9 \end{bmatrix} \quad \begin{bmatrix} 6 & 3 & | & 6 \\ 1 & 1 & | & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & | & 3 \\ 6 & 3 & | & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & | & 3 \\ 0 & -3 & | & -12 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & | & 3 \\ 0 & 1 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 4 \end{bmatrix}$$

so $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ is the least squares solution.

12.(14pts) Determine an explicit solution to $(e^x + e^{-y}) dx + e^x dy = 0$ that satisfies $y(0) = 0$.

(a) (7pts) Find an integrating factor.

(b) (7pts) Give an implicit solution to the original initial value problem.

Solution:

$$M = e^x + e^{-y}, N = e^x \text{ so } M_y - N_x = -e^{-y} - e^x \text{ so } \frac{M_y - N_x}{M} = -1 \text{ so } -\frac{d\mu}{dy} = -\mu \text{ or } \mu = e^y.$$

$$\text{Check } (e^{x+y} + 1) dx + e^{x+y} dy = 0 \text{ and } \frac{\partial e^{x+y} + 1}{\partial y} = e^{x+y} = \frac{\partial e^{x+y}}{\partial x} \text{ so } (e^{x+y} + 1) dx + e^{x+y} dy =$$

0 is exact.

$$\frac{\partial \psi}{\partial x} = e^{x+y} + 1 \text{ so } \psi = e^{x+y} + x + g(y).$$

$$\frac{\partial \psi}{\partial y} = e^{x+y} + g'(y) = e^{x+y} \text{ so } g(y) \text{ is a constant and the solutions are the level curves of}$$

$\psi = e^{x+y} + x$. The curve passes through $(0, 0)$ so $e^{x+y} + x = 1$ is the implicit form of the solution.

$$\text{Explicitly, } e^{x+y} = 1 - x, x + y = \ln(1 - x) \text{ so } y = \ln(1 - x) - x.$$