Exam 3D solutions

Multiple choice.

(1) Separating variables in $\frac{dy}{dx} = \frac{1-x^2}{y}$ gives $y\,dy = (1-x^2)\,dx$ so $\int y\,dy = \int(1-x^2)\,dx$, $\frac{1}{2}y^2 = x - \frac{1}{3}x^3 + C$ and $y = \pm\sqrt{2x - \frac{2}{3}x^3 + 2C}$. When $x = 0$, $y(0) = 4 = \pm\sqrt{2C}$ so $C = 8$ and the sign is "+". Thus, $\phi(x) = \sqrt{2x - \frac{2}{3}x^3 + 16}$. So $\phi(3) = \sqrt{6 - 18 + 16} = \sqrt{4} = 2$.

(2) $e^{\int -2x\,dx} = e^{-x^2}$

(3) Note $f(y) = y - y^3 = y(1-y)(1+y)$. So the critical points are $y = -1, 0, 1$. For $y < -1$ (e.g. $y = -2$) or $0 < y < 1$ (e.g. $y = \frac{1}{2}$) one has $f(y) > 0$. For $-1 < y < 0$ (e.g. $y = -\frac{1}{2}$) or $y > 1$ (e.g. $y = 2$), one has $f(y) < 0$. The stable equilibria occur at critical points c where $f(y) > 0$ for $y < c$ and $f(y) < 0$ for $y > c$ i.e. at $y = 1$ and $y = -1$.

(4) The solution for the IVP $y' = f(x,y)$, $y'(x_0) = y_0$ will be unique provided f and $\frac{\partial f}{\partial y}$ are defined and continuous on an open rectangle containing (x_0, y_0). For the equation $y' = (y-1)^{1/5}$ with $y(1) = 0$, one has $f(x,y) = (y-1)^{1/5}$, $\frac{\partial f}{\partial y} = \frac{1}{5}(y-1)^{-\frac{4}{5}}$, and $(x_0, y_0) = (1, 0)$, which satisfies these conditions. For the other equations, the partial derivative with respect to y does not exist at (x_0, y_0) so uniqueness is not guaranteed.

(5) The IVP is $y' - \frac{\sqrt{t+4}}{9-t^2}y = \frac{\ln(2-t)}{9-t^2}$ with $y(-2) = 0$. This is $y' + p(y)y = g(t)$ where $p(t) = -\frac{\sqrt{t+4}}{9-t^2}$ and $q(t) = \frac{\ln(2-t)}{9-t^2}$. The solution will exist on any open interval containing -2 on which $p(t)$ and $g(t)$ are defined and continuous i.e. not containing any point t with $t \leq -4$, $t^2 = 9$ (i.e. $t = \pm 3$) or $t \geq 2$. The maximum such interval is $-3 < t < 2$.

(6) A least squares solution is given by solving $A^TAx = A^Tb$. Here, $A^TA = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ Since A^TA is invertible, the unique least squares solution is $x = (A^TA)^{-1}(A^Tb) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{24} \end{bmatrix} \begin{bmatrix} 9 \\ 24 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$.

(7) If $y = e^{-2t} + c$, then $y' = -2e^{-2t}$ and $y' + 2y = 2c$ which is not zero for arbitrary c. The other parts are all true.

(8) The IVP is $y' + \frac{1}{2}y = 3$. An integrating factor is $e^{\int \frac{1}{2}dt} = e^{\frac{1}{2}t}$. Multiplying by $e^{\frac{1}{2}t}$ gives $\frac{d}{dt}(e^{\frac{1}{2}t}y) = 3e^{\frac{1}{2}t}$. Integrating, $e^{\frac{1}{2}t}y = \int 3e^{\frac{1}{2}t}\,dt =$

$6e^{\frac{1}{2}t} + C$ and $y(t) = 6 + Ce^{-\frac{1}{2}t}$. So $1 = y(0) = 6 + C$, $C = -5$ and $y(t) = 6 - 5e^{-\frac{1}{2}t}$.

(9) Substituting the points in the equation of the line gives equations $2 = a_0 - 2a_1$, $3 = a_0$ and $1 = a_0 + 2a_1$. These are inconsistent so we calculate a least squares solution. The system is $Au = b$ where $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$, $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. We calculate $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$, $A^T b = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $u = (A^T A)^{-1}(A^T b) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{4} \end{bmatrix}$. So $a_0 = 2$, $a_1 = -\frac{1}{4}$.

(10) Let $V = 120$ denote the volume, $Q(t)$ denote quantity of salt. Concentration of salt at time t is $Q/120$, so the ODE is $\frac{dQ}{dt} = 60(2t + 4) - \frac{Q}{120}60$ i.e. $\frac{dQ}{dt} + \frac{1}{2}Q = 120(t + 2)$. Mutiplying by integrating factor $e^{\int \frac{1}{2}\, dt} = e^{\frac{1}{2}t}$ gives $\frac{d}{dt}(e^{\frac{1}{2}t}Q) = 120(t + 2)e^{\frac{1}{2}t}$. Integrating, $e^{\frac{1}{2}t}Q = \int 120(t + 2)e^{\frac{1}{2}t}\, dt$. Using integration by parts, the right hand side is $120[(t+2)2e^{\frac{1}{2}t} - \int 2e^{\frac{1}{2}t}\, dt] = 120[2(t+2)e^{\frac{1}{2}t} - 4e^{\frac{1}{2}t}] + C = 240te^{\frac{1}{2}t} + C$. Hence $Q(t) = 240t + Ce^{-\frac{1}{2}t}$. Putting $t = 0$, $0 = Q(0) = 0 + C$ and $C = 0$. So $Q(t) = 240t$.

(11)(a) Separating variables, $-\int \frac{1}{y(y-2)}\, dy = \int dx + c$. The integrand on the left is of the form $\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{(y-2)}$. So $1 = A(y - 2) + By$. Putting $y = 2$ gives $B = \frac{1}{2}$. Putting $y = 0$ gives $A = -\frac{1}{2}$. So $-\int \frac{1}{y(y-2)}\, dy = \int -\frac{1}{2(y-2)} + \frac{1}{2y}\, dy = \frac{1}{2}(\ln|y| - \ln|y-2|) = \frac{1}{2}\ln|\frac{y}{y-2}|$. So the solution is $\frac{1}{2}\ln|\frac{y}{y-2}| = x + c$, $\ln|\frac{y}{y-2}| = 2x + 2c$ or $|\frac{y}{y-2}| = e^{2x+2c}$. So $\frac{y}{y-2} = \pm e^{2c}e^{2x} = Ce^{2x}$ where $C = \pm e^{2c}$ is another constant. That is $\frac{y-2}{y} = Ce^{-2x}$, $1 - \frac{2}{y} = Ce^{-2x}$, $\frac{y}{2} = \frac{1}{1-Ce^{-2x}}$ and $y = \frac{2}{1-Ce^{-2x}}$.

(b) $f(y) = y(2 - y) = 0$ when $y = 0, 2$. So the equilibrium solutions are $y = 0$, $y = 2$. For $y < 0$ or $y > 2$, $f(y) < 0$, while for $0 < y < 2$, $f(y) > 0$. Hence only $y = 2$ is stable.

(c) $y(0) = 1 = \frac{2}{1-Ce^0}$ so $C = -1$ and the solution is $y = \frac{2}{1+e^{-2x}}$.