Record your answers to the multiple choice problems by placing an × through one letter for each problem on this answer sheet. There are 24 multiple choice questions. Each problem counts 6 points and you start with 6 points. Please sign the honor statement if you agree:

“I strictly followed the Notre Dame Honor Code during this test.”

Your Signature ____________________________

1. a b c • e  
2. a b c d •  
3. a b c d •  
4. a b • d e  
5. a b • d e  
6. a b • e  
7. • b c d e  
8. a • c d e  
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10. a b • d e 
11. a b c • d  
12. a b • d e  
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14. • b c d e  
15. a b c • e  
16. a b c • e  
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18. a b c • e  
19. a b • d e  
20. a b c • e  
21. a • c d e  
22. a • c d e  
23. a • c d e  
24. a • c d e
Math 20580

Your Name: ________________________

Final Exam  May 8, 2007  Instructor’s name: ________________________

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1. a b c d e  13. a b c d e
2. a b c d e  14. a b c d e
3. a b c d e  15. a b c d e
4. a b c d e  16. a b c d e
5. a b c d e  17. a b c d e
6. a b c d e  18. a b c d e
7. a b c d e  19. a b c d e
8. a b c d e  20. a b c d e
9. a b c d e  21. a b c d e
10. a b c d e  22. a b c d e
11. a b c d e  23. a b c d e
12. a b c d e  24. a b c d e

1
1. Let $y_1(t)$ and $y_2(t)$ are two fundamental solutions $y'' + y' + \frac{\sin t}{t} y = 0$ with initial conditions $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$. Then the Wronskian $W(t) = [y_1(t)y_2(t) - y_1'(t)y_2'(t)]$ is equal to

(a) $\frac{\sin t}{t}$  
(b) $e^t$  
(c) $\frac{1}{t}$  
(d) $e^{-t}$  
(e) $\sin t$.

2. Let $Y(t) = A_0t^2 + A_1t + A_2$ be a solution to $y'' + 4y = 4t^2$ where $\{A_0, A_1, A_2\}$ are constant numbers. Then $A_2$ is equal to

(a) $-1$  
(b) $4$  
(c) $1$  
(d) $0$  
(e) $-\frac{1}{2}$.
3. The linear system \[
\begin{pmatrix}
1 & 5 & -3 \\
1 & 4 & -1 \\
2 & 7 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
-4 \\
-3 \\
h
\end{pmatrix}
\] has a solution if and only if \( h = \)

(a) 2 (b) 1 (c) 5 (d) 3 (e) \(-5\)

4. Suppose that \( Y(t) = At e^{-t} + B \) is a solution to \( y'' - 3y' - 4y = -5e^{-t} - 4 \), where \{A, B, s\} are constant numbers. Then \( A \) is equal to

(a) \(-\frac{2}{5}\) (b) 4 (c) 1 (d) \(-4\) (e) 1
5. Find the adjoint $\text{adj}(A)$ of $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$.

(a) $\begin{pmatrix} 7 & 4 \\ 2 & 1 \end{pmatrix}$  
(b) $\begin{pmatrix} -7 & -4 \\ -2 & -1 \end{pmatrix}$  
(c) $\begin{pmatrix} 7 & -4 \\ -2 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -2 \\ -4 & 1 \end{pmatrix}$  
(e) $\begin{pmatrix} -7 & 4 \\ 2 & -1 \end{pmatrix}$

6. The reduced row echelon form of $\begin{pmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{pmatrix}$ is equal to

(a) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$  
(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  
(c) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$  
(e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
7. Find the integrating factor $\mu$ for $dx + \left(\frac{x}{y} - \sin y + y^2\right)dy = 0.$

(a) $y$   (b) $\sin y$   (c) 1   (d) $y^2$   (e) $x$

8. Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \text{proj}_V \vec{u}$ where $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}$ and $V = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}\right\}$. Then $x_1$ is equal to

(a) 2   (b) 4   (c) 1   (d) 0   (e) 3
9. Which of the following sets is an orthonormal basis of $R^2$?

(a) $\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix} \}$

(b) $\{ \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \}$

(c) $\{ \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix}, 0 \}$

(d) $\{ \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \}$

(e) $\{ \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$

10. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$. Then $b_{11}$ is equal to

(a) 1  
(b) -2  
(c) 10  
(d) -6  
(e) 5
11. Let \(
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\) be a solution to
\[
\begin{pmatrix}
  1 & -2 & 1 \\
  0 & 2 & -8 \\
  4 & -5 & -9
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  8 \\
  9
\end{pmatrix}.
\]
Then \(x_1\) is equal to
(a) 3  
(b) 16  
(c) 8  
(d) 9  
(e) 29

12. Use the method of reduction of order to find a second solution \(y_2 = v(t)y_1(t)\) of the given differential equation
\[
t^2y'' + 2ty' - 2y = 0 \quad \text{where} \quad y_1(t) = t.
\]
Then \(v(t)\) is equal to
(a) \(\frac{4}{t}\)  
(b) \(t\)  
(c) \(t^{-3}\)  
(d) 1  
(e) \(t^{-2}\)
13. Let $r_1$ and $r_2$ be two roots of the characteristic equation for $y'' + 100y = 0$. Then $r_1$ and $r_2$ are

(a) $\pm 10\sqrt{-1}$
(b) $0, 10$
(c) $-100, 0$
(d) $\pm 10$
(e) $-10 \pm 10\sqrt{-1}$

14. If $\det A = 2$ where $A$ is a $4 \times 4$ matrix, then $\det(-2A)$ is

(a) 32
(b) $-4$
(c) $-32$
(d) 16
(e) $-16$
15. The following two solutions form a fundamental set of solutions of linear homogeneous differential equation $2t^2y'' + 3ty' - y = 0$.

(a) $t^3, t$  (b) $t, t^{-1}$  (c) $t, 1$  (d) $t^{1/2}, t^{-1}$  (e) $t^{1/2}, 0$

16. If $\mathbf{B} = \{(1 \ 0), (1 \ 2)\}$ and $\vec{x} = (1 \ 6)$, then $[\vec{x}]_\mathbf{B}$ is equal to

(a) $(1 \ 6)$  (b) $(1 \ 0)$  (c) $(3 \ 2)$  (d) $(-2 \ 3)$  (e) $(1 \ 2)$
17. The eigenvalues of \( A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \) are

(a) \(-3, -5, -3\)   (b) \(1, -5, 0\)   (c) \(1, 3, 3\)   (d) \(1, 3, 5\)   (e) \(1, -2, -2\)

18. Let \( y(t) \) be the unique solution to the initial value problem \( y'' - y = 0\), \( y(0) = 2\), \( y'(0) = 0\). Then \( y(1) \) is equal to

(a) \(2e\)   (b) \(2\)   (c) \(2e^{-1}\)   (d) \(e + e^{-1}\)   (e) \(2e - 2\)
19. Let $y(t)$ be the unique solution to $y' + \frac{2}{t}y = 4t$ with initial condition $y(1) = 3$. Then $y(2)$ is equal to

(a) 8  (b) $\ln 2 + 2$  (c) $4 + \frac{1}{2}$  (d) $e^4 + 2$  (e) $8 + \frac{1}{4}$

20. Let $Y(t) = v_1(t) \cos 3t + v_2(t) \sin 3t$ be a solution to $y'' + 9y = \frac{1}{\sin 3t}$. Then $v_2(t)$ is equal to

(a) $\frac{1}{\sin 3t}$  (b) $\frac{t}{3}$  (c) $\cos 3t$

(d) $\frac{1}{9} \ln |\sin 3t|$  (e) $\frac{1}{3} \ln |\sin 3t|$
21. If \( y' = 2y^{100}(3 - y) \) and \( y(0) = 5 \), then find \( \lim_{t \to \infty} y(t) \). (Hint: This is an autonomous equation. You can find the answer by studying graphs of the solution).

(a) 2  
(b) 3  
(c) 1  
(d) 0  
(e) 5

22. Let \( y(t) \) be the unique solution to the initial value problem \( y'' + 2y' + y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \). Then \( y(1) \) is equal to

(a) \( e \)  
(b) \( e^{-1} \)  
(c) \( e + e^{-1} \)  
(d) 1  
(e) 0
23. Let \( y(t) \) be the unique solution to the equation \( y' = y^2 \) with \( y(0) = -1 \). Then \( y(1) \) is equal to

(a) 0  (b) \(-\frac{1}{2}\)  (c) \(-4\)  (d) \(-1\)  (e) \(-3\)

24. The determinant of

\[
\begin{pmatrix}
1 & 5 & 0 \\
2 & 4 & 1 \\
0 & 2 & 0
\end{pmatrix}
\]

is equal to

(a) 1  (b) 2  (c) \(-2\)  (d) 0  (e) 5
1. By Abel’s Theorem, $W = ce^{-\int p(t)dt} = ce^{-t}$. Evaluating at 0 shows $W = e^{-t}$.

2. $2A_0 + 4A_2 + 4A_1 t + 4A_0 t^2 = 4t^2$ so $A_2 = 1$, $A_1 = 0$ and $2A_0 + 4A_2 = 0$ so $A_2 = -1/2$.

3. 
\[
\begin{bmatrix}
1 & 5 & -3 & -4 \\
1 & 4 & -1 & -3 \\
2 & 7 & 0 & h
\end{bmatrix} \begin{bmatrix}
1 & 5 & -3 & -4 \\
0 & -1 & 2 & 1 \\
0 & -3 & 6 & h + 8
\end{bmatrix} = \begin{bmatrix}
1 & 5 & -3 & -4 \\
0 & -1 & 2 & 1 \\
0 & 0 & 0 & h + 5
\end{bmatrix}
\]
If this system has a solution $h + 5 = 0$.

4. $Y'' = As(s - 1)t^s e^{-t} - Ast^{s-1}e^{-t} - Ast^{s-1}e^{-t} + At^s e^{-t}$

so $L[y] = (A + 3A - 4A)t^s e^{-t} + (-2As - 3As + 0)t^{s-1}e^{-t} + (s(s - 1)A + 0 + 0)t^{s-2}e^{-t} - 4B = -5Ast^{s-1}e^{-t} + s(s - 1)At^{s-2}e^{-t} - 4B$. Since $L[y] = -5e^{-t} - 4$, $s = 1$, $B = 1$ and $A = 1$

5. $\text{adj}(A) = [a_{ij}]$ where $a_{ij} = (-1)^{i+j}C_{ji}$ and $C_{ki}$ is the determinant of the $k$-th minor:

hence \[
\begin{bmatrix}
7 & -4 \\
-2 & 1
\end{bmatrix}
\]

6. 
\[
\begin{bmatrix}
3 & -1 & 3 \\
6 & 0 & 12 \\
2 & 1 & 7
\end{bmatrix} \begin{bmatrix}
6 & 0 & 12 \\
3 & -1 & 3 \\
2 & 1 & 7
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 2 \\
0 & -1 & -3 \\
0 & 1 & 3
\end{bmatrix} \begin{bmatrix}
1 & 0 & 2 \\
0 & -1 & -3 \\
0 & 1 & 3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{bmatrix}
\]

7. $\frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = \frac{1}{y}$. Hence $M_y - N_x = -\frac{1}{y} = -\frac{1}{y}$. Hence $\mu = e^{\ln y} = y$

8. The vectors in $V$ are an orthonormal pair. Hence $\text{proj}_V \vec{u} = (\vec{u} \cdot \vec{v}_1)v_1 + (\vec{u} \cdot \vec{v}_2)v_2$.

\[
\vec{u} \cdot \vec{v}_1 = \frac{1}{2}(1 + 3 + 1 + 7) = \frac{12}{2} = 6.
\]

\[
\vec{u} \cdot \vec{v}_2 = \frac{1}{2}(1 - 3 - 1 + 7) = \frac{4}{2} = 2.
\]

Hence $\text{proj}_V \vec{u} = \frac{1}{2} \begin{pmatrix}
8 \\
4 \\
8
\end{pmatrix} = \begin{pmatrix}
2 \\
2 \\
4
\end{pmatrix}$

9. (a) is orthogonal but not unit length; (b) is orthonormal; (c) is not linearly independent; (d) is not a spanning set; (e) is not linearly independent

10. 
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
2 & 3 & 2 & 0 & 1 & 0 \\
3 & 8 & 2 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 0 & -1 & 7 & -5 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 0 & -1 & 7 & -5 & 1
\end{bmatrix}
\]
12. $y = tv$, $y' = v + tv'$, $y'' = v' + tv'' = 2v' + tv''$ so $L[tv] = t^3v'' + t^2(2v' + 2v') + t(2v - 2v) = 0$ or $t^3v'' + t^24v' = 0$ or $v'' = -4v'$ or $\frac{dv}{dt} = -4$. Then $|v'| = -4\ln|t| + C$ or $v' = At^{-4}$ and $v = Bt^{-3} + K$. Hence $t^{-3}$ is the answer.

13. $r_1$ and $r_2$ are roots of $r^2 + 100r = 0$ so $r = \pm\sqrt{-100} = \pm10\sqrt{-1}$.

14. $\det(cA) = c^n \det A$ if $A$ is $n \times n$ so $\det(-2A) = (-2)^4 \det A = 32$.

15. We are looking for solutions of the form $t^s$ so $2s(s - 1) + 3s - 1 = 0$ or $2s^2 + s - 1 = 0$. $(2s - 1)(s + 1) = 0$ so $s = -1$ and $s = 1/2$.

16. We are being asked to solve $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. By inspection, second entry is 3 and then first entry is $-2$ so $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Or use Cramer’s rule or row reduction.

17. To set up and solve the cubic equation is hard and takes a long time. The problem is easy because you have a short list of possible answers. The trace of the matrix is the sum of the eigenvalues with multiplicity. The trace is $1 + (-5) + 1 = -3$ and $1, -2, -2$ is the only one of the answers which sums to $-3$.

18. The characteristic equation is $r^2 - 1 = 0$ so $y = ae^t + be^{-t}$ and $y' = ae^t - be^{-t}$. $y(0) = a + b = 2$ and $y'(0) = a - b = 0$. Hence $a = b = 1$ so $y = e^t + e^{-t}$ and $y(1) = e + e^{-1}$.

19. $L[t^r] = rt^{r-1} + 2t^{r-1} = (r + 2)t^{r-1}$. Hence one particular solution is $t^2$ and a solution to the homogeneous system is $t^{-2}$ so the general solution is $y = t^2 + \frac{C}{t^2}$. Hence $y(1) = 1 + C = 3$ so $C = 2$ and $y(2) = 2^2 + \frac{2}{2^2} = 4.5$ or $4 + \frac{1}{2}$.
20. Use Variation of Parameters. The Wronskian is

\[ W(t) = \det \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix} = 3 \]

One easy way to remember the formulas in the book is the following. A particular solution is given by

\[ Y = \det \begin{bmatrix} u_2 & -u_1 \\ y_1 & y_2 \end{bmatrix} = \det \int \frac{g(t) y_1(t)}{W(t)} dt \int \frac{g(t) y_2(t)}{W(t)} dt \]

In this problem we want

\[ \int \frac{g(t) y_1(t)}{W(t)} dt = \frac{1}{3} \int \frac{\cos(3t)}{\sin(3t)} dt = \frac{1}{3} \cdot \frac{1}{3} \cdot \ln |\sin(3t)| \]

21. The solution starts out in the strip above \( y = 3 \) since \( y(0) = 5 \) and hence it stays there. In this strip, \( y \) is decreasing since \( y' < 0 \). Hence the limit is 3.

22. The characteristic equation is \( r^2 + 2r + 1 = 0 \) so \( r = -1 \) is a double root and hence the general solution to the homogeneous equation is \( y = ae^{-t} + bte^{-t}, \ y' = -ae^{-t} + be^{-t} - bte^{-t} = (b - a)e^{-t} - bte^{-t}. \) Hence \( y(0) = a + b = 1 \) and \( y'(0) = b - a = 0 \) so \( a = b = \frac{1}{2}. \)

Hence \( y = e^{-t}\frac{1 + t}{2} \) so \( y(1) = \frac{2}{2e} = \frac{1}{e}. \)

23. The equation is separable so \( \int y^{-2} dy = \int dt \) or \( -y^{-1} = t + C \) or \( y^{-1} = A - t \) or \( y = \frac{1}{A - t}. \ Y(0) = \frac{1}{A} = -1 \) so \( y = \frac{-1}{1 + t}. \) Hence \( y(1) = \frac{-1}{2}. \)

24. Expand along the last column \( \det A = 0 - \det \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix} + 0 = -(-2) = 2. \)