Name:________________________

Instructor:________________________

**MATH 20580: Introduction to Linear Algebra and Differential Equations**

**Final Exam** *May 13, 2011*

Record your answers to the multiple choice problems by placing an × through one letter for each problem on this page. There are 24 multiple choice questions worth 6 points. You start with 6 points.

**You may not use a calculator.**

**1.** Let $A$ be an $n \times n$ invertible matrix. Determine which statement is *not* always true.

(a) $A$ is diagonalizable. (b) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto. (c) $\det A \neq 0$
(d) The rank of $A$ is equal to $n$. (e) The columns of $A$ are linearly independent.

**2.** Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the linear transformation of polynomials $T[p(t)] = (t^2 + 1)p(t)$. Find the matrix of $T$ with respect to the standard bases $\{1, t, t^2\}$ of $\mathbb{P}_2$ and $\{1, t, t^2, t^3, t^4\}$ of $\mathbb{P}_4$.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

**3.** Find the value of $h$ such that the system of linear equations is inconsistent.

$$\begin{bmatrix} 3 & 2 & 1 & -1 \\ 2 & 4 & 3 & 1 \\ 1 & h & 5 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

(a) 6 (b) 8 (c) 7 (d) 4 (e) $-2$

**4.** Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 & -1 \\ -1 & 1 & 6 & 3 & -1 \\ 0 & 3 & 15 & 6 & -3 \end{bmatrix}$.

Determine which statement is *true*.

(a) $T$ is neither one-to-one nor onto. (b) $T$ is one-to-one. (c) $T$ is onto.
(d) The rank of $A$ is 3. (e) The rows of $A$ are linearly independent.

**5.** Determine which set is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\begin{bmatrix} 2a+b+1 \\ a-2b+2 \end{bmatrix}$ with $a, b \in \mathbb{R}$.
(b) The union of the first and the third quadrant of $\mathbb{R}^2$.
(c) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1^2 + x_2^2 < 1$.
(d) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1^2 - x_2^2 = 0$.
(e) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $3x_1 + 4x_2 = 1$.

**6.** Let $A$ be a $5 \times 7$ matrix with $\operatorname{rank} A = 4$. Find the dimension of $\operatorname{Null} A$.

(a) 3 (b) 1 (c) 2 (d) 4 (e) 5

**7.** Find the determinant of $\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$.

(a) 0 (b) 1 (c) 8 (d) 15 (e) 20

**8.** Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. If $A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, find $b_{13}$.

(a) 8 (b) $-4$ (c) 3 (d) $-2$ (e) $-6$

**9.** Determine which statement is *not* always true.

(a) $\operatorname{Col} A$ is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.
(b) A null space is a vector space.
(c) The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.
(d) $\operatorname{Null} A$ is the kernel of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.
(e) The range of a linear transformation is a vector space.

**10.** Find a basis for the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

(a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ (b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ (c) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ (d) $\{\mathbf{v}_4, \mathbf{v}_5\}$
(e) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

**11.** Find the change-of-coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ to the basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.

(a) $\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 10 \\ 9 & 13 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 7 \\ 5 & 18 \end{bmatrix}$ (e) $\begin{bmatrix} 18 & -7 \\ -5 & 2 \end{bmatrix}$

**12.** Let $A = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $T(\mathbf{x}) = A\mathbf{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

(a) $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}$
(e) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

**13.** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$. Find the least-squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$.

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (d) $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (e) $\frac{1}{2}\begin{bmatrix} 9 \\ -1 \end{bmatrix}$

**14.** Determine which set is an orthogonal basis of $\mathbb{R}^3$.

(a) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$
(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

**15.** Find the distance between $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) 1 (b) $\sqrt{3}$ (c) 2 (d) $\sqrt{2}$ (e) $\sqrt{6}$

**16.** Determine which statement is always *true*.

(a) If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, then $A\hat{\mathbf{x}}$ is the point in $\operatorname{Col} A$ closest to $\mathbf{b}$.
(b) If $\mathbf{x}_1$, $\mathbf{x}_2$ are linearly independent, the Gram-Schmidt process produces an orthogonal set
$\left\{\mathbf{x}_1,\ \dfrac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2\right\}$.
(c) $(\operatorname{Row} A)^{\perp} = \operatorname{Null} A^T$.
(d) A vector $\mathbf{u}$ may be nonzero when $\mathbf{u} \cdot \mathbf{u} = 0$.
(e) If $U$ is an orthogonal matrix, then $U^{-1} = U$.

**17.** Suppose $y(t)$ satisfies $ty' - 3y = 4t$ and $y(1) = 0$. Calculate $y(2)$.

(a) 12 (b) 4 (c) 8 (d) 6 (e) 10

**18.** Find the largest region in the plane where the existence of a unique solution through any point is guaranteed for the differential equation $y' = \sin\left[(xy)^{1/3}\right]$.

(a) $\{(x, y) \mid y \neq 0\}$ (b) $\{(x, y) \mid x \neq 0 \text{ and } y \neq 0\}$
(c) $\{(x, y) \mid x > 0 \text{ and } y > 0, \text{ or } x < 0 \text{ and } y < 0\}$ (d) $\{(x, y) \mid y > 0\}$
(e) *the entire plane*

**19.** The force of air resistance on a skydiver weighing 160 lb with an unopened parachute is proportional to velocity, $-kv$. After falling for some time, the skydiver approaches the limiting velocity of 200 ft/sec. Find the value of $k$ that yields this limiting velocity.

(a) 0.8 (b) 0.16 (c) 0.32 (d) 0.4 (e) 1.25

**20.** The differential equation $(3y^2 - 4x(y^3 + 1))\,dx + xy(2 - 3xy)\,dy = 0$

(a) is exact.
(b) is homogenous.
(c) has an integrating factor that is a function of $x$ alone.
(d) has an integrating factor that is a function of $y$ alone.
(e) *none of the above*

**21.** Find a suitable form for a particular solution $y_p$ of the differential equation $y'' - 4y' + 4y = 4te^{2t} + t\sin(2t)$.

(a) $y_p = t^2(A_0 + A_1t)e^{2t} + (B_0 + B_1t)\sin(2t) + (C_0 + C_1t)\cos(2t)$
(b) $y_p = t(A_0 + A_1t)e^{2t} + t(B_0 + B_1t)\sin(2t) + t(C_0 + C_1t)\cos(2t)$
(c) $y_p = (A_0 + A_1t)e^{2t} + (B_0 + B_1t)\sin(2t) + (C_0 + C_1t)\cos(2t)$
(d) $y_p = Ate^{2t} + Bt\sin(2t) + Ct\cos(2t)$
(e) $y_p = At^2e^{2t} + (B_0 + B_1t)\sin(2t) + (C_0 + C_1t)\cos(2t)$

**22.** Solve the initial value problem $y'' + 2y' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$.

(a) $y = e^{-t}\Big(\cos(\sqrt{3}t) + \dfrac{\sqrt{3}}{3}\sin(\sqrt{3}t)\Big)$ (b) $y = e^{-t}\Big(\cos(\sqrt{3}t) - \dfrac{1}{3}\sin(\sqrt{3}t)\Big)$

(c) $y = e^{-t}\Big(\cos(\sqrt{5}t) + \dfrac{\sqrt{5}}{5}\sin(\sqrt{5}t)\Big)$ (d) $y = e^{-t}\Big(\cos(\sqrt{5}t) - \dfrac{1}{5}\sin(\sqrt{5}t)\Big)$

(e) $y = (1 + 2t)e^{-2t}$.

**23.** Compute the Wronskian of the functions $y_1 = t\cos(t)$ and $y_2 = t\sin(t)$.

(a) $t^2$ (b) $t^2(1 - 2\sin^2(t))$ (c) $t^2(1 - 2\cos^2(t))$ (d) $2t\sin(t)\cos(t)$ (e) 0

**24.** Let $L[y] = y'' + p(t)y' + q(t)$. Suppose $t$ and $t^{-1}$ are solutions to $L[y] = 0$. Calculate a particular solution $y_p = u_1(t)t + u_2(t)t^{-1}$ to the differential equation $L[y] = t^3$ using the method of variation of parameters.

(a) $y_p = \dfrac{1}{24}t^5$ (b) $y_p = \dfrac{1}{8}t^4 - \dfrac{1}{12}t^6$ (c) $y_p = \dfrac{1}{6}t^6$ (d) $y_p = \dfrac{1}{2}(t^4 - t)$ (e) $y_p = \dfrac{1}{20}t^5 - \dfrac{1}{6}t^3$

**1.** Not all $x \times n$ matrices are diagonalizable. The others are all true for *invertible* matrices.

**2.** We need to calculate $T[t^i]$ for $i = 0$, 1 and 2.

$T[1] = t^2 + 1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is first column.

$T[t] = t^3 + t = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is second column.

$T[t^2] = t^4 + t^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is third column.

**3.** We need to row reduce

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 5 \\ 2 & 4 & 3 & 11 & 2 \\ 1 & h & 5 & 3 & 1 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & h & 5 & 3 & 1 \\ 3 & 2 & 1 & -1 & 5 \\ 2 & 4 & 3 & 11 & 2 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & h & 5 & 3 & 1 \\ 0 & 2-3h & -14 & -10 & 2 \\ 0 & 4-2h & -7 & 5 & 0 \end{array}\right]$$

To be inconsistent the last column must be a pivot column so if $2(4-2h) = 2-3h$ this will happen. Hence $8-4h = 2-3h$ or $6 = h$.

**4.** Since $A$ is $4 \times 5$, $T$ can not be one-to-one, but the easiest solution is to row reduce

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 0 & -1 \\ -1 & 1 & 6 & 3 & -1 \\ 0 & 3 & 15 & 6 & -3 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 5 & 2 & -1 \\ 0 & 1 & 5 & 2 & -1 \\ 0 & 3 & 15 & 6 & -3 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 5 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the rows are not independent; the rank is 2; $T$ is neither one-to-one nor onto.

**5.** Since $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ is invertible, $\begin{bmatrix} 2a+b+1 \\ a-2b+2 \end{bmatrix}$ is all of $\mathbf{R}^2$. The vectors $\begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$ is in the first quadrant, $\begin{bmatrix} -1 \\ -0.1 \end{bmatrix}$ is in the third but the sum $\begin{bmatrix} -0.9 \\ 0.9 \end{bmatrix}$ is in the second. The vector $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ satisfies $x_1^2 + x_2^2 < 1$ but $10\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ does not. The vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ both satisfy $x_1^2 - x_2^2 = 0$ but their sum does not. The sum of any two vectors satisfying $3x_1 + 4x_2 = 1$ satisfies $3x_1 + 4x_2 = 2$.

**6.** The rank and the dimension of the null space are related by $\text{rank}A + \dim \text{Null}A =$ number of columns of $A$ so in this case, $\dim \text{Null}A = 7 - 4 = 3$.

**7.** Row reduction is faster than Lagrange expansion.

$$\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{bmatrix}$$

Since the second row is a multiple of the third, the determinant is 0.

**8.** $b_{13}$ is $(-1)^{1+3}\det\begin{bmatrix} 6 & 7 \\ 2 & 1 \end{bmatrix} = -8$ divided by the determinant. The determinant is $3\det\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} - 0\det\begin{bmatrix} 6 & 7 \\ 3 & 4 \end{bmatrix} + 2\det\begin{bmatrix} 6 & 7 \\ 2 & 1 \end{bmatrix} = 3\cdot 5 + 2\cdot(-8) = 15 - 16 = -1$

**9.** Null spaces and ranges are always subspaces. The null space of a matrix is the kernel of its linear transformation. The column space of an $m \times n$ matrix is $\mathbb{R}^m$. The column space of a matrix is the range of its linear transformation and so in general has nothing to do with the set of solutions.

**10.** Row reduce

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 0 & 3 & -7 & -9 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 0 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so the first three original vectors span the entire span.

**11.** The first column is the solution to the equation $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By row reduction, Cramer's rule or inspection, $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The second column is the solution to the equation $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. This time $\mathbf{x} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

**12.** You are being asked for a basis of eigenvectors. The characteristic equation is $\det\begin{bmatrix} 1-\lambda & 0 \\ 4 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)$ so the eigenvalues are 1 and 2. The eigenvector for 1 is a solution of $\begin{bmatrix} 0 & 0 \\ 4 & 1 \end{bmatrix}\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. The eigenvector for 2 is a solution of $\begin{bmatrix} -1 & 0 \\ 4 & 0 \end{bmatrix}\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**13.** $A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $A^TA = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$. $A^T\mathbf{b} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} =$ $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ so we need to solve $\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. By inspection $\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

**14.** Just check which collections are orthogonal. There is only one. The minimum you need to check is to dot the first vector against the second and third and finally the second against the third.

**15.** First compute $\text{proj}_W\mathbf{y}$. Since the two vectors are orhtogonal,

$$\text{proj}_W\mathbf{y} = \frac{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} =$$

$$\frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The distance between $\mathbf{y}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is $\sqrt{(1-1)^2 + (2-2)^2 + (-1-0)^2} = 1$

**16.** The least squares solution is always the point in Col$A$ closest to $\mathbf{b}$. The two vectors $\mathbf{x}_1$ and $\dfrac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2}\mathbf{x}_2$ are NEVER and orthogonal basis. The subspaces $(\text{Row}A)^\perp$ and Null$A$ are equal. If $\mathbf{u} \cdot \mathbf{u} = 0$ then $\mathbf{u}$ is always zero. If $U$ is an ortho*normal* square matrix, then $U^{-1} = U^T$.

**17.** The equation is first order linear and the normal form is $y' + \left(-\dfrac{3}{t}\right) y = 4$. The integrating factor is $\mu = e^{-3\ln|t|} = t^{-3}$. $(t^{-3}y)' = 4t^{-3}$ so $t^{-3}y = -2t^{-2} + C$ and $y = -2t + Ct^3 = -4 + 16 = 12$. If $y(1) = 0$, $-2 + C = 0$ so $y = -2t + 2t^3$.

**18.** $F(x, y) = \sin\left((xy)^{1/3}\right)$ is continuous everywhere.

$$F_y(x, y) = \left(\cos\left((xy)^{1/3}\right)\right)\left(\frac{1}{3}x^{1/3}y^{-2/3}\right)$$

The first factor is defined and continuous everywhere and the second factor is continuous everywhere it is defined, which is for all $y \neq 0$.

**19.** By Newton's $F = ma$, $m\dfrac{dv}{dt} = -kv + 32m$ where $m = \dfrac{160}{32}$. Hence $v' = 32 - \kappa v$ where $\kappa = \dfrac{32k}{160}$. Therefore $v' + \kappa v = 32$, the integrating factor is $e^{\kappa t}$ so $(e^{\kappa t}y)' = 32e^{\kappa t}$ or $e^{\kappa t}y) = \dfrac{32}{\kappa}e^{\kappa t} + C$ or $y = \dfrac{32}{\kappa} + Ce^{-\kappa t}$. As $t \to \infty$, $y$ approaches $\dfrac{32}{\kappa}$ or 200. Hence

$\kappa = \dfrac{32}{200} = \dfrac{4}{25}$ so $\dfrac{32k}{160} = \dfrac{4}{25}$ and $k = \dfrac{4 \cdot 160}{25 \cdot 32} = \dfrac{128 \cdot 5}{32 \cdot 25} = \dfrac{4}{5} = 0.8$. You need to divide 160 by 32 to get the mass (ask your physics professor).

**20.** $M = 3y^2 - 4x(y^3+1)$ and $N = xy(2-3xy)$. $M_y = 6y - 12xy^2$; $N_x = 2y - 6xy^2$ so not exact: $N_x - M_y = 2y - 6xy^2 - (6y - 12xy^2) = -4y + 6xy^2$ It is not homogeneous although this term was only discussed in some unassigned homework. The expression $\dfrac{N_x - M_y}{M} = \dfrac{-4y + 6xy^2}{3y^2 - 4x(y^3+1)}$ does not seem to simplify. The expression $\dfrac{N_x - M_y}{N} = \dfrac{-4y+6xy^2}{xy(2-3xy)} = \dfrac{-2y(2-3xy)}{xy(2-3xy)} = \dfrac{-2y}{xy} = \dfrac{-2}{x}$. Hence there is an integrating factor which is a function of $x$ alone.

**21.** This is an undetermined coefficients problem. The characteristic equation is $r^2 - 4r + 4 = (r-2)^2$ so 2 is root of order 2. For the $4te^{2t}$ term the corresponding term in the solution is $t^2(A_0 + A_1 t)e^{2t}$. For the $t\sin(2t)$ term the corresponding term in the solution is $(B_0 + B_1 t)\sin(2t) + (C_0 + C_1 t)\cos(2t)$.

**22.** The characteristic equation is $r^2 + 2r + 4$ which has roots $\dfrac{-2 \pm \sqrt{4 - 4\cdot 4}}{2} = \dfrac{-2 \pm \sqrt{4-16}}{2} = \dfrac{-2 \pm \sqrt{-12}}{2} = -1 \pm \sqrt{3}i$. Hence a basis for the solution space of the homogeneous equation is $e^{-t}\cos(\sqrt{3}\,t)$, $e^{-t}\sin(\sqrt{3}\,t)$ so $y = Ae^{-t}\cos(\sqrt{3}\,t) + Be^{-t}\sin(\sqrt{3}\,t)$ and $y' = -Ae^{-t}\cos(\sqrt{3}\,t) - \sqrt{3}\,Ae^{-t}\sin(\sqrt{3}\,t) - Be^{-t}\sin(\sqrt{3}\,t) + \sqrt{3}\,Be^{-t}\cos(\sqrt{3}\,t)$.

Hence $y(0) = A = 1$ and $y'(0) = -A + \sqrt{3}\,B = 0$ and $B = \dfrac{\sqrt{3}}{3}$ so

$$Y = e^{-t}\left(\cos(\sqrt{3}\,t) + \frac{\sqrt{3}}{3}\sin(\sqrt{3}\,t)\right)$$

**23.** $\det\begin{bmatrix} t\cos(t) & t\sin(t) \\ \cos(t) - t\sin(t) & \sin(t) + t\cos(t) \end{bmatrix} = \det\begin{bmatrix} t\cos(t) & t\sin(t) \\ -t\sin(t) & t\cos(t) \end{bmatrix}$ Add $\dfrac{-1}{t}$ times the top row to the bottom row. $\det\begin{bmatrix} t\cos(t) & t\sin(t) \\ -t\sin(t) & t\cos(t) \end{bmatrix} = t^2$.

**24.** A particular solution is given by

$$Y = \det\begin{bmatrix} \displaystyle\int \frac{g(t)\,y_1(t)}{W(t)}\,dt & \displaystyle\int \frac{g(t)\,y_2(t)}{W(t)}\,dt \\ y_1 & y_2 \end{bmatrix}$$

where $g(t) = t^3$ and $y_1$, $y_2$ are a basis for the solution space of the associated homogeneous equation. Here $y_1 = t$ and $y_2 = t^{-1}$.

The Wronskian is $W = \det\begin{bmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{bmatrix} = -2t^{-1}$.

Hence $\displaystyle\int \frac{g(t)\,y_1(t)}{W(t)}\,dt = \int \frac{t^3 \cdot t}{-2t^{-1}}dt = -1/2\int t^5 dt = -t^6/12$.

Also $\displaystyle\int \frac{g(t)\, y_2(t)}{W(t)}\ dt = \int \frac{t^3 \cdot t^{-1}}{-2t^{-1}} dt = -1/2 \int t^3 dt = -t^4/8$

$Y = \det \begin{bmatrix} -t^6/12 & -t^4/8 \\ t & t^{-1} \end{bmatrix} = -\dfrac{t^5}{12} + \dfrac{t^5}{8} = \dfrac{1}{24} t^5$