Math Reasoning, Practice Exam 1.

February 20, 2013

There are 5 questions worth 10 points each. Please write clearly and give complete proofs.

Problem 1**.**

Let $f : \mathbb{R} \to \mathbb{R}$ be a fixed function.

Consider the statement:

There exists an $x_0 \in \mathbb{R}$ such that for all $\epsilon > 0$ there is a corresponding δ for which $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

(a) Let $P(x, x_0, \epsilon)$ be the statement $(|f(x) - f(x_0)| < \epsilon)$ and $Q(x, x_0, \delta)$ be the statement $(|x - x_0| < \delta)$.

Write the statement above in logical notation involving P and Q but no words.

(b) Find the negation of your answer to (a) in logical notation.

(c) Express your answer to (b) in English.

Problem 2.

Show that the following statements are equivalent regardless of the truth values of P and Q.

(a) $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$.

(b) $P \vee Q$ and $(\neg P) \Rightarrow Q$.

Problem 3.
Prove the formula

$$\prod_{i=2}^{n}(1-\frac{(-1)^i}{i}) = \begin{cases} 1/2 & \text{if } n \text{ even} \\ \frac{n+1}{2n} & \text{if } n \text{ odd} \end{cases}$$

Problem 4**.**

Let a sequence a_n be recursively defined by $a_1 = 2$ and $a_n = a_{n-1} + n^2$ for $n \geq 2$.

Show that

$$a_n = 1 + \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}.$$

Problem 5.

Prove that for any pair of integers a and b we have $a^2 - 4b \neq 2$.

(You can freely use the fact that an integer is even, that is, divisible by 2, if and only if its square is even.)