Answer Key for Exam A

Section 1. Multiple Choice Questions

1. Compute the indefinite integral $\int_0^{\frac{\pi}{2}} x\cos(x)$

   (a) 0

   (b) $\frac{\pi}{2} - 1$

   (c) $\frac{\pi}{2}$

   (d) $\frac{\pi}{4}$

   (e) $\frac{\pi}{2} + 1$

2. Rewrite the following improper integral in terms of limits of proper integrals.

$$\int_{\frac{1}{2}}^{\infty} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x$$

(a) $\lim_{a \to \infty} \int_{\frac{1}{2}}^{a} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x$

(b) $\lim_{a \to 1^-} \int_{\frac{1}{2}}^{a} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x + \lim_{b \to \infty} \int_{1}^{b} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x$

(c) $\lim_{a \to 1^-} \int_{\frac{1}{2}}^{a} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x + \lim_{b \to 2^-} \int_{1}^{b} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x + \lim_{c \to \infty} \int_{2}^{c} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x$

(d) $\lim_{a \to 1^-} \int_{\frac{1}{2}}^{a} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x + \lim_{b \to 1^+} \int_{b}^{2} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x + \lim_{c \to \infty} \int_{2}^{c} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x$

(e) $\lim_{a \to 1^+} \int_{\frac{1}{2}}^{a} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x + \lim_{b \to 1^-} \int_{b}^{2} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x + \lim_{c \to \infty} \int_{2}^{c} \frac{e^{-x^2}}{\ln(x)} \mathrm{d}x$

3. Which of the following statements is **TRUE**?

(a) $\lim_{x \to 0^+} e^{\left(\frac{\ln(x)}{x}\right)} = 1$

(b) $\lim_{x \to 0} e^{\left(\frac{\ln(x^2)}{x}\right)} = \infty$

(c) $\lim_{x \to 0^+} e^{\left(\frac{\ln(x)}{x}\right)} = 0$

(d) $\lim_{x \to 0} e^{\left(\frac{1}{x}\right)} = 0$

(e) $\lim_{x \to 0} e^{\left(\frac{1}{x}\right)} = 1$

4. Solve for A, B, C to finish the partial fraction decomposition.

$$\frac{x^2+1}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

(a) A = 0, B = 2, C = 1

(b) A = 1, B = 1, C= -1

[(c)] A = -1, B = 1, C = 1

(d) A = 1, B = -1, C = 1

(e) A = -1, B = -1, C = -1

5. Evaluate $\lim_{x\to\infty} \frac{\sin(x)}{x}$

(a) 1

(b) 2

(c) $\frac{1}{2}$

[(d)] 0

(e) Diverges

6. Suppose that $4 \cdot {a_n}^2 < b_n < 2^{c_n}$ for every n. Which of the following is **NOT** a possiblity?

(a) $\{a_n\}, \{b_n\}, \{c_n\}$ all diverge

(b) $\lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} b_n = 3, \lim_{n\to\infty} c_n = 3$

(c) $\lim_{n\to\infty} a_n = 1, \lim_{n\to\infty} b_n = 4, \lim_{n\to\infty} c_n = 2$

(d) $\lim_{n\to\infty} a_n = 0, \{b_n\}$ Diverges , $\lim_{n\to\infty} c_n = 0$

$\boxed{\text{(e)}}$ $\lim_{n\to\infty} a_n = 2, \{b_n\}$ Diverges , $\lim_{n\to\infty} c_n = 4$

7. Which is the best trig substitution to help evaluate the following integral?

$$\int \frac{x^2}{\sqrt{1+x^2}} \mathrm{d}x$$

$\boxed{\text{(a)}}$ $x = \tan(u)$

(b) $x = \sin(u)$

(c) $x = \sqrt{\sec(u)}$

(d) $x = \sqrt{\tan(u)}$

(e) $x = \sec(u)$

8. Supposing $\lim_{x \to 3} f(x) = +\infty$ and $\lim_{x \to 3} g(x) = 0$ compute:

$$\lim_{x \to 3} \left(f(x) + \frac{1}{f(x)} \right)^{g(x)}$$

(a) $\ln(3)$

(b) 1

(c) 0

(d) Diverges

[(e)] Insufficent Information to Determine

9. Evaluate $\int_{-1}^{1} \frac{1}{x^2} \mathrm{d}x$

(a) 1

(b) 0

(c) -2

(d) 2

[(e)] Diverges

10. Which of these choices is the result of a useful trig substitution on the following integral

$$\int \frac{e^x}{e^x \cdot \sqrt{e^{2x} - 1}} \mathrm{d}x$$

- $\boxed{\text{(a)}}$ $\displaystyle\int 1du$
- (b) $\displaystyle\int \ln(u)du$
- (c) $\displaystyle\int \sin^2(u)\cos(u)du$
- (d) $\displaystyle\int \frac{\sec(u)}{\tan(u)}du$
- (e) $\displaystyle\int \sec^2(u)du$

11. Using the definitions below compute $\lim_{n\to\infty} a_n \cdot b_n$

$$a_n = \frac{e^n}{1+e^n} + \frac{1}{n+1}$$

$$b_n = \frac{\ln(n+1)}{n} + \frac{n^2}{2n^2+3}$$

(a) 1

(b) 2

[(c)] $\frac{1}{2}$

(d) 0

(e) Diverges

12. Suppose that $\lim_{n\to\infty} a_n = 1$ and $b_n = \ln(|a_{n+1} - a_n|)$. What is $\lim_{n\to\infty} b_n$?

(a) e

(b) 1

(c) 0

[(d)] $-\infty$ (Diverges)

(e) Insufficent Information to Determine

Section 2. Short Answer Questions

13. Compute $\lim_{x\to\infty}(x+e^x)^{e^{-x}}$

Since the form ∞^0 is indeterminate we must transform the expression to find the limit.

$$\lim_{x\to\infty}(x+e^x)^{e^{-x}} = \lim_{x\to\infty} e^{\ln\left((x+e^x)^{e^{-x}}\right)} = \lim_{x\to\infty} e^{\frac{\ln(x+e^x)}{e^x}} = e^{\left(\lim_{x\to\infty}\frac{\ln(x+e^x)}{e^x}\right)}$$

We now try to find the value of the exponent.

$$\lim_{x\to\infty}\frac{\ln(x+e^x)}{e^x} = \frac{\ln(\infty)}{\infty} = \frac{\infty}{\infty} \qquad \text{Indeterminate, use L'Hopital}$$

$$\lim_{x\to\infty}\frac{\frac{1+e^x}{x+e^x}}{e^x} = \lim_{x\to\infty}\frac{1+e^x}{e^x(x+e^x)} = \lim_{x\to\infty}\frac{1}{e^x(x+e^x)} + \frac{e^x}{e^x(x+e^x)} =$$

$$\lim_{x\to\infty}\frac{1}{e^x(x+e^x)} + \frac{1}{x+e^x} = \frac{1}{\infty(\infty+\infty)} + \frac{1}{\infty+\infty} = \frac{1}{\infty} + \frac{1}{\infty} = 0$$

Remembering to plug back into first line.

$$\lim_{x\to\infty}(x+e^x)^{e^{-x}} = e^{\left(\lim_{x\to\infty}\frac{\ln(x+e^x)}{e^x}\right)} = e^0 = \boxed{1}$$

14. Evaluate $\int e^x \sin(x)\mathrm{d}x$

We apply parts twice and then solve for the integral.

$$\int e^x \sin(x)\mathrm{d}x = e^x \sin(x) - \int e^x \cos(x)\mathrm{d}x \qquad \text{(Parts)}$$

$$= e^x \sin(x) - \left(e^x \cos(x) - \int e^x \cdot (-\sin(x))\mathrm{d}x \right) \qquad \text{(Parts)}$$

$$\int e^x \sin(x)\mathrm{d}x = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\mathrm{d}x$$

$$2\int e^x \sin(x)\mathrm{d}x = e^x \sin(x) - e^x \cos(x) + C$$

$$\int e^x \sin(x)\mathrm{d}x = \boxed{\frac{e^x \sin(x) - e^x \cos(x)}{2} + C}$$

15. Evaluate $\int \frac{x+1}{x^3+4x}\mathrm{d}x$

We begin by factoring $x^3+4x = x(x^2+4)$. Now we break up $\frac{x+1}{x^3+4x}$ using partial fractions.

$$\frac{x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \qquad \text{(partial fraction rule}$$

$$= \frac{A(x^2+4)}{x(x^2+4)} + \frac{x(Bx+C)}{x(x^2+4)} = \frac{Ax^2+4A}{x(x^2+4)} + \frac{Bx^2+Cx)}{x(x^2+4)}$$

$$= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}$$

$$\frac{x+1}{x(x^2+4)} = \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}$$

$$x+1 = (A+B)x^2+Cx+4A$$

Giving:

$$\boxed{\begin{array}{lll} C=1 & 4A=1 & A+B=0 \\ C=1 & A=\frac{1}{4} & B=-\frac{1}{4} \end{array}}$$

$$\int \frac{x+1}{x^3+4x}\mathrm{d}x = \int \frac{A}{x} + \frac{Bx+C}{x^2+4}\mathrm{d}x = \int \frac{1}{4x} + \frac{\frac{1}{4}x+1}{x^2+4}\mathrm{d}x$$

$$= \int \frac{1}{4x}\mathrm{d}x + \frac{1}{8}\int \frac{2x}{x^2+4}\mathrm{d}x + \int \frac{1}{x^2+4}\mathrm{d}x$$

$$\boxed{\begin{array}{l} \int \frac{1}{4x} = \frac{1}{4}\ln(x) + C \\ \int \frac{1}{x^2+4}\mathrm{d}x = \frac{1}{2}\arctan(\frac{x}{2}) + C \\ \int \frac{2x}{x^2+4}\mathrm{d}x = \ln(x^2+4) + C \qquad \left(\text{Set } u = x^2+4\right) \end{array}}$$

$$\int \frac{x+1}{x^3+4x}\mathrm{d}x = \boxed{\frac{1}{4}\ln(x) + \frac{1}{8}\ln(x^2+4) + \frac{1}{2}\arctan(\frac{x}{2}) + C}$$

16. Compute the indefinite integral $\int \frac{(\cos(x))^3}{\sqrt{\sin(x)}} \mathrm{d}x$

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\begin{aligned}
\int \frac{(\cos(x))^3}{\sqrt{\sin(x)}} \mathrm{d}x &= \int \frac{\cos^2(x)\cos(x)}{\sqrt{\sin(x)}} \mathrm{d}x \\
&= \int \frac{(1-\sin^2(x))\cos(x)}{\sqrt{\sin(x)}} \mathrm{d}x && \left(\text{As } \sin^2(x)+\cos^2(x)=1 \right) \\
&= \int \frac{1-u^2}{\sqrt{u}} \cos(x)\mathrm{d}x && \left(\text{Set } u=\sin(x) \right) \\
&= \int \frac{1-u^2}{\sqrt{u}} \mathrm{d}u && \left(\mathrm{d}u=\cos(x)\mathrm{d}x \right) \\
&= \int \frac{1}{\sqrt{u}} - \frac{u^2}{\sqrt{u}} \mathrm{d}u \\
&= \int u^{-\frac{1}{2}} - u^{\frac{3}{2}} \mathrm{d}u \\
&= 2u^{\frac{1}{2}} - \frac{2}{5}u^{\frac{5}{2}} + C \\
&= 2\sqrt{u} - \frac{2}{5}\sqrt{u^5} + C \\
&= \boxed{2\sqrt{\sin(x)} - \frac{2}{5}\sqrt{\sin^5(x)} + C} && \left(u=\sin(x) \right)
\end{aligned}
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